Preserver problems for surjective isometries on Banach spaces of continuous functions

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Introduction

Let $(B_i, \|\cdot\|_i)$ be a Banach space for $i = 1, 2$ and $T : B_1 \to B_2$ a map between B_1 and B_2. The map T is an isometry between B_1 and B_2 if

$$\|T(x) - T(y)\|_2 = \|x - y\|_1 \qquad (x, y \in B_1).$$

By the above equality, we notice that isometries on Banach spaces preserve the metric structure induced by the norm. The main purpose of Preserver problems for surjective isometries on Banach spaces of continuous functions is to characterize the forms of surjective isometries and to clarify how the metric structure affects other structures of Banach spaces.

The Mazur–Ulam theorem [**32**, Theorem 1.3.5], which is one of the most prominent theorem for the study of surjective isometries on Banach spaces, asserts that if $T : B_1 \to B_2$ is a surjective isometry, then $T - T(0)$ is real linear. We note that the map $T - T(0) : B_1 \to B_2$ is also a surjective isometry. We infer from this theorem that the metric structure is closely related to the algebraic structure of Banach spaces. On the other hand, the forms of surjective isometries on Banach spaces cannot be characterized in the Mazur–Ulam theorem.

Let $C(X)$ be a Banach space of all complex-valued continuous functions on a compact Hausdorff space X equipped with the supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ for all $f \in C(X)$. The Banach–Stone theorem is one of the most important theorems in the study of surjective isometries on Banach spaces which consist of continuous functions. This theorem characterizes the forms of surjective complex linear isometries between two continuous function spaces $C(X)$ and $C(Y)$;

Theorem (The Banach–Stone theorem [**17**, Theorem 2.1, p172]). *If $T : C(X) \to C(Y)$ is a surjective complex linear isometry, then there exist a continuous function $\alpha : Y \to \mathbb{T}$ and a homeomorphism $\tau : Y \to X$ such that*

$$T(f)(y) = \alpha(y) f(\tau(y)) \qquad (f \in C(X),\ y \in Y),$$

where we denote by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The form of T in the Banach–Stone theorem is called a weighted composition operator. We note that each surjective complex linear isometry on Banach spaces preserves the metric structure and the algebraic structure of Banach spaces. The Banach–Stone theorem states that every surjective complex linear isometry between two continuous function spaces preserves the topological spaces, that is, the metric structure and the algebraic structure are closely related to the topological structure of continuous function spaces. According to the Mazur–Ulam theorem, every surjective isometry which corresponds 0 to 0 between two Banach spaces is real linear. Hence, it is natural to consider surjective real linear isometries instead of surjective complex linear isometries when we explore the relation between the metric structure and others structures of Banach spaces. By the result of T. Miura [**54**], we obtain the Banach–Stone theorem in the case of surjective real linear isometries;

Theorem (Miura [**54**]). *If $T : C(X) \to C(Y)$ is a surjective real linear isometry, then there exist a continuous function $\alpha : Y \to \mathbb{T}$, a homeomorphism $\tau : Y \to X$, and a closed and open subset $K \subset Y$ such that*

$$T(f)(y) = \begin{cases} \alpha(y)f(\tau(y)) & (y \in K) \\ \\ \alpha(y)\overline{f(\tau(y))} & (y \in Y \setminus K) \end{cases}$$

for all $f \in C(X)$.

This theorem states that the metric structure has a strong influence on the algebraic structure and the topological structure of continuous function spaces. It is well known that surjective isometries on various Banach spaces which consist of continuous functions can be described as a weighted composition operator [**13, 37, 45, 50, 53, 54, 55, 64**]. In addition, the study of surjective isometries on Banach spaces of vector–valued continuous functions has been studied by many mathematicians actively [**5, 6, 39, 40, 46, 49, 51**]. Let $C(X, C(Y))$ be a Banach space of all continuous functions $F : X \to C(Y)$ equipped with the norm $\|F\| = \sup_{x \in X} \|F(x)\|_\infty$ for all $F \in C(X, C(Y))$. We denote $F(x)(y)$ by $F(x, y)$ for $F \in C(X, C(Y))$, $x \in X$, and $y \in Y$. By the Banach–Stone theorem, if $T : C(X, C(Y)) \to C(X, C(Y))$ is a unital surjective complex linear isometry, then the map T induces a homeomorphism $\tau : X \times Y \to X \times Y$ such that $T(F)(x, y) = F(\tau(x, y))$ for all $F \in C(X, C(Y))$, $x \in X$, and $y \in Y$. Using natural projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$, there exist two continuous maps $\tau_1 : X \times Y \to X$ and $\tau_2 : X \times Y \to Y$ so that

$$T(F)(x, y) = F(\tau_1(x, y), \tau_2(x, y)) \qquad (F \in C(X \times Y),\ x \in X,\ y \in Y).$$

Of course, the maps τ_1 and τ_2 depend on two variables $x \in X$ and $y \in Y$. On the other hand, it turns out that the continuous map τ_1 or τ_2 depends on only one of the two variables $x \in X$ and $y \in Y$ in a particular Banach space of vector-valued continuous functions. In addition, the maps τ_1 or τ_2 has a special form. Let $\mathrm{Lip}(X)$ be a Banach space of all Lipschitz functions defined on a compact metric space X and $\mathrm{Lip}(X, C(Y))$ a Banach space of all Lipschitz functions defined on a compact metric space X with values in a continuous function space $C(Y)$. By the result of Hatori and Oi [**39**] which generalize the result of Botelho and Jamison [**6**], if T is a unital surjective complex linear isometry on $\mathrm{Lip}(X, C(Y))$, then the map T induces a continuous map $\tau_1 : X \times Y \to X$ and a homeomorphism $\tau_2 : Y \to Y$ such that $\tau_1(\cdot, y) : X \to X$ is a surjective isometry for each $y \in Y$ and

$$T(F)(x, y) = F(\tau_1(x, y), \tau_2(y)) \qquad (F \in \mathrm{Lip}(X, C(Y)),\ x \in X,\ y \in Y).$$

We note that the map $\tau : X \times Y \to X \times Y$ defined by $\tau(x, y) = (\tau_1(x, y), \tau_2(y))$ is a homeomorphism on $X \times Y$. A composition operator induced by such a homeomorphism is said to be of type BJ in [**39, 40**]. For each $f \in \mathrm{Lip}(X)$ and $g \in C(Y)$, we define a map $f \otimes g : X \to C(Y)$ by $f \otimes g(x) = f(x)g$ for $x \in X$, and then $f \otimes g \in \mathrm{Lip}(X, C(Y))$. Let $1_X : X \to \mathbb{C}$ and $1_Y : Y \to \mathbb{C}$ be constant functions with value 1 on X and Y, respectively. By regarding $f \in \mathrm{Lip}(X)$ as $f \otimes 1_Y$, we see that $\mathrm{Lip}(X, C(Y))$ contains $\mathrm{Lip}(X)$. In the same way, we observe that the continuous function space $C(Y)$ is contained in $\mathrm{Lip}(X, C(Y))$ if we identify g with $1_X \otimes g$ for each $g \in C(Y)$. Entering $F = f \otimes g$ for $f \in \mathrm{Lip}(X)$ and $g \in C(Y)$ into the above equality, we obtain

$$T(f \otimes g)(x, y) = f(\tau_1(x, y))g(\tau_2(y)) \qquad (x \in X, y \in Y).$$

The result of Hatori and Oi indicates that unital surjective isometries on $\mathrm{Lip}(X, C(Y))$ separate $C(Y)$ from $\mathrm{Lip}(X)$, because $C(Y)$ and $\mathrm{Lip}(X)$ are totally different Banach spaces. In [**40**], Hatori and Oi gives a sufficient condition such that surjective complex linear isometries on Banach spaces of vector-valued continuous functions forms a weighted composition operator of type BJ. Koshimizu and Miura [**46**] characterize surjective isometries on the Banach spaces $C^1(I, A)$ of continuously differentiable functions defined on $I = [0, 1]$ with values in a uniform algebra A on a compact Hausdorff space X. Of course, a continuous function space $C(X)$ is a uniform algebra on X. By this result, if T is a surjective complex linear isometry on the Banach space $C^1(I, C(X))$ of continuously differentiable maps defined on $I = [0, 1]$ with values in $C(X)$, then the map T induces a continuous function $\alpha : X \to \mathbb{T}$, a surjective isometry

$\tau_1 : I \to I$ and a homeomorphism $\tau_2 : X \to X$ such that

$$T(F)(s,x) = \alpha(x)F(\tau_1(s), \tau_2(x)) \qquad (F \in C^1(I, C(X)), s \in I, x \in X).$$

If $T : C^1(I, C(X)) \to C^1(I, C(X))$ is a surjective real linear isometry, we infer from the result of Koshimizu and Miura [**46**] that the map T is a weighted composition operator induced by a homeomorphism $\tau(s,x) = (\tau_1(s), \tau_2(x))$ for $(s,x) \in I \times X$, where $\tau_1 : I \to I$ is a surjective isometry and $\tau_2 : X \to X$ is a homeomorphism. This shows that surjective isometries on $C^1(I, C(X))$ distinguish $C^1(I)$ and $C(X)$, because $C^1(I)$ and $C(X)$ have totally different structures from each other as a Banach space. In fact, all the functions of $C^1(I)$ are differentiable on I while continuous functions on X need not be differentiable on I. By these results, we see that surjective isometries distinguish two Banach spaces of continuous functions which have some totally different structures from each other.

In Chapter 1, we consider the Banach spaces $C^1(I, \mathrm{Lip}(I))$ of all continuously differentiable maps with values in Lipschitz algebra and characterize surjective isometries on $C^1(I, \mathrm{Lip}(I))$. By the main result in Chapter 1, if $T : C^1(I, \mathrm{Lip}(I)) \to C^1(I, \mathrm{Lip}(I))$ is a unital surjective isometry, then the map T induces two surjective isometries τ_1 and τ_2 on I such that

$$T(F)(s,x) = F(\tau_1(s), \tau_2(x)) \qquad (F \in C^1(I, \mathrm{Lip}(I)), s, x \in I).$$

If $F = f \otimes g$ is a tensor product defined by $f \otimes g(s,x) = f(s)g(x)$ for $f \in C^1(I)$, $g \in \mathrm{Lip}(I)$, and $s, x \in I$, then $T(f \otimes g)(s,x) = f(\tau_1(s))g(\tau_2(x))$. It is well known that Lipschitz functions on I have derivatives almost everywhere. Hence, $C^1(I)$ and $\mathrm{Lip}(I)$ have differential structures. This results indicates that each surjective isometry $T : C^1(I, \mathrm{Lip}(I)) \to C^1(I, \mathrm{Lip}(I))$ preserves respective two differential structures of $C^1(I)$ and $\mathrm{Lip}(I)$, that is, the metric structure is closely related to the differential structure of Banach spaces.

Let $S(B)$ be the unit sphere of a Banach space of B. In 1987, D. Tingley suggested the following problem which is called Tingley's problem;

Problem (Tingley's Problem [**74**]). *Let B_1 and B_2 be Banach spaces. We denote by $S(B_i)$ the closed unit sphere of B_i for $i = 1, 2$. If $\Delta : S(B_1) \to S(B_2)$ is a surjective isometry, then does there exist a surjective real linear isometry $T : B_1 \to B_2$ such that $T|_{S(B_1)} = \Delta$?*

This problem asserts that surjective isometries on Banach spaces can be determined by the information of the closed unit sphere, that is, the closed unit sphere of Banach spaces contains the essential information of surjective isometries on Banach spaces. This problem has been investigated for several Banach spaces since then. However, the problem is still open even for finite dimensional Banach spaces whose dimension is more than 3. Quite recently, Banakh [**2**] gave an affirmative answer to Tingley's problem for 2-dimensional Banach spaces. Let $C_0(X)$ be a Banach space of all continuous functions defined on a locally compact Hausdorff space X which vanishes at infinity equipped with the supremum norm. Wang [**75**] proves that each surjective isometry $\Delta : S(C_0(X_1)) \to S(C_0(X_2))$ admits an extension to a surjective real linear isometry between $C_0(X_1)$ and $C_0(X_2)$. Hatori, Oi, and Togashi [**38**] prove that Tingley's problem for uniform algebras is affirmative. In the first part of Chapter 2, we consider Tingley's problem for uniformly closed function algebras and prove that each surjective isometry $\Delta : S(A) \to S(B)$ between the unit spheres of two uniformly closed function algebras A and B can be extended to a surjective real linear isometry $T : A \to B$. A uniformly closed function algebra A on X is a uniformly closed and strongly separating subalgebra of $C_0(X)$. We can regard A as a subalgebra of $C(X \cup \{\infty\})$, where $X \cup \{\infty\}$ denotes the one-point compactification of X. Under such identification, A never contains the constant functions. Roughly speaking, a uniformly closed function algebra is a uniform algebra which does not have the unit element. Of course, uniform algebras are examples of uniformly closed function algebras. In the second part of Chapter 2, we give an affirmative answer to Tingley's problem for abelian JB*-triples. According to [**43**, Corollary 1.11], each abelian JB*-triple can be represented as a subspace of a continuous function space $C_0(X)$ as follows;

$$C_0^{\mathbb{T}}(X) = \{f \in C_0(X) \ : \ f(\lambda x) = \lambda f(x) \text{ for every } (\lambda, x) \in \mathbb{T} \times X\},$$

where X is a principal $\mathbb{T}$-bundle. By the results of chapter 2, we see that every surjective isometry Δ between the unit spheres of two uniformly closed function algebras and JB*-triple forms a kind of weighted composition operator. These results in Chapter 2 indicate that the forms of surjective isometries on Banach spaces can be determined by the information of only the closed unit spheres of Banach spaces in the case of uniformly closed function algebras and abelian JB*-triples.

On the other hand, Tingley's problem for $C^1(I, \mathrm{Lip}(I))$ has yet to be solved, because the structure of the unit sphere of $C^1(I, \mathrm{Lip}(I))$ is more complicated than that of Banach space of continuous functions equipped with the supremum norm. Let $\mathrm{Lip}(I)$ be a Banach space of all Lipschitz functions defined on a closed unit interval I equipped with the norm $\|f\|_\sigma = |f(0)| + \|f'\|_{L^\infty}$ for all $f \in \mathrm{Lip}(I)$, where $\|\cdot\|_{L^\infty}$ denotes the essential supremum norm. In order to clue to the solution of Tingley's problem for $C^1(I, \mathrm{Lip}(I))$, we consider surjective isometries $\Delta : S(\mathrm{Lip}(I)) \to S(\mathrm{Lip}(I))$ with respect to the norm $\|\cdot\|_\sigma$ in Chapter 3. At that time, we prove that $\Delta : S(\mathrm{Lip}(I)) \to S(\mathrm{Lip}(I))$ can be represented by a sum of two weighted composition operators and extended to a surjective real linear isometry between the whole spaces with a different technique from the result Wang and Orihara [**77**].

CHAPTER 1

Surjective isometries on the Banach algebra of continuously differentiable maps with values in Lipschitz algebra

Abstract

Let $\mathrm{Lip}(I)$ be the Banach algebra of all Lipschitz functions on the closed unit interval I with the norm $\|f\|_L = \|f\|_\infty + L(f)$ for $f \in \mathrm{Lip}(I)$, where $L(f)$ is the Lipschitz constant of f. We denote by $C^1(I, \mathrm{Lip}(I))$ the Banach algebra of all continuously differentiable maps F from I to $\mathrm{Lip}(I)$ equipped with the norm $\|F\|_\Sigma = \sup_{s\in I} \|F(s)\|_L + \sup_{t\in I} \|D(F)(t)\|_L$ for $F \in C^1(I, \mathrm{Lip}(I))$. In this paper, we prove that if T is a surjective, not necessarily linear, isometry on $C^1(I, \mathrm{Lip}(I))$, then $T - T(0)$ is a weighted composition operator or its complex conjugation. Among other things, any surjective complex linear isometry on $C^1(I, \mathrm{Lip}(I))$ is of the following form: $c_1 F(\tau_1(s), \tau_2(x))$, where c_1 is a complex number of modulus 1, and τ_1 and τ_2 are isometries of I onto itself.

1. Introduction

The study of surjective isometries is one of the main themes in theory of Banach spaces. Let $C(K)$ be the Banach space of all complex-valued continuous functions on a compact Hausdorff space K equipped with the supremum norm $\|f\|_\infty = \sup_{y\in K} |f(y)|$. The Banach–Stone theorem determines the form of surjective complex linear isometries between Banach spaces $C(X)$ and $C(Y)$. This theorem shows that $T : C(X) \to C(Y)$ is a surjective complex linear isometry if and only if there exist $\alpha \in C(Y)$ with $|u| = 1$ on Y and a homeomorphism $\tau : Y \to X$ such that

$$T(f)(y) = \alpha(y) f(\tau(y)) \qquad (f \in C(X),\ y \in Y),$$

that is, T is a weighted composition operator.

Cambern [**13**] extended the result above to the Banach space $C^1(I)$ of all continuously differentiable functions f on the closed unit interval $I = [0, 1]$ equipped with the norm $\|f\| = \max_{s\in I}\{|f(s)| + |f'(s)|\}$. Rao and Roy [**64**] characterized the surjective complex linear isometries on $C^1(I)$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

The above results by Cambern [**13**] and Rao and Roy [**64**] were extended to surjective isometries on vector-valued function spaces. Botelho and Jamison [**5**] gave a characterization of surjective complex linear isometries on the Banach space $C^1(I, E)$ of all continuously differentiable functions F on I with values in a finite dimensional Hilbert space E, equipped with the norm $\|F\| = \max_{s \in I}\{\|F(s)\|_E + \|F'(s)\|_E\}$. Li and Wang [**51**] considered surjective complex linear isometries on the Banach space $C_0^n(\Omega, E)$ of all n-times continuously differentiable functions on an open subset Ω in a p-dimensional Euclidean space $\mathbb{R}^p$ with values in a reflexive, strictly convex Banach space E. Leung, Ng, and Tang [**49**] showed the result by Li and Wang [**51**] for an arbitrary Banach space E: More explicitly, suppose that $T\colon C_0^{n_1}(\Omega_1, E_1) \to C_0^{n_2}(\Omega_2, E_2)$ is a surjective complex linear isometry for an open subset Ω_j in $\mathbb{R}^{p_j}$ and a Banach space E_j with $j = 1, 2$. Then they proved that $p_1 = p_2$, $n_1 = n_2$ and for each $t \in \Omega_2$ there exist a Banach space isomorphism $V(t)\colon E_2 \to E_1$ and a C^{n_1}-diffeomorphism $\tau\colon \Omega_2 \to \Omega_1$ such that $T(F)(t) = V(t)(F(\tau(t)))$ for all $F \in C_0^{n_1}(\Omega_1, E_1)$ and $t \in \Omega_2$.

Let $C^1(I, C(X))$ be the Banach space of all continuously differentiable functions F equipped with the norm $\|F\| = \sup_{s \in I} \|F(s)\|_\infty + \sup_{t \in I} \|F'(t)\|_\infty$. We denote $F(s)(x)$ by $F(s, x)$ for $F \in C^1(I, C(X))$, $s \in I$, and $x \in X$. By the result of Hatori and Oi [**40**, Corollary 18], or Koshimizu and Miura [**46**, Theorem 1], if $T : C^1(I, C(X)) \to C^1(I, C(X))$ is a surjective complex linear isometry, then there exist $\alpha \in C(X)$ with $|u| = 1$ on X, a C^1-diffeomorphism $\tau_1 : I \to I$, and a homeomorphism $\tau_2 : X \to X$ such that

$$T(F)(s, x) = \alpha(x) F(\tau_1(s), \tau_2(x)) \quad (F \in C^1(I, C(X)),\ (s, x) \in I \times X).$$

We note that $\tau_1(s) = s$ or $\tau_1(s) = 1 - s$, because τ_1 is a C^1-diffeomorphism. In particular, if F is a tensor product $f \otimes g$ defined by $(f \otimes g)(s, x) = f(s)g(x)$ for $f \in C^1(I)$, $g \in C(X)$, and $(s, x) \in I \times X$, then $T(f \otimes g)(s, x) = u(x) f(\tau_1(s)) g(\tau_2(x))$. This shows that the surjective complex linear isometry T on $C^1(I, C(X))$ respects $C^1(I)$ and $C(X)$. Such a kind of phenomenon occurs because $C^1(I)$ has a distinct structure, say differential, from $C(X)$. It is well known that the Lipschitz functions on I have derivatives almost everywhere. Thus, the Lipschitz space $\mathrm{Lip}(I)$ has a similar structure to $C^1(I)$.

Now, the following question seems natural:

> Do surjective complex linear isometries on $C^1(I, \mathrm{Lip}(I))$ respect $C^1(I)$ and $\mathrm{Lip}(I)$?

The main result of this paper gives an affirmative answer to the question.

THEOREM 1.1. *Let $T : C^1(I, \mathrm{Lip}(I)) \to C^1(I, \mathrm{Lip}(I))$ be a surjective, not necessarily linear, isometry with respect to the norm $\|\cdot\|_\Sigma$. Then there exist a constant $c_1 \in \mathbb{T}$ and two maps $\tau_1, \tau_2 \in \{id_I, 1_I - id_I\}$ such that*

$$T(F)(s,x) - T(0)(s,x) = c_1 F(\tau_1(s), \tau_2(x)) \qquad (F \in C^1(I, \mathrm{Lip}(I)),\ s, x \in I), \quad \textit{or}$$
$$T(F)(s,x) - T(0)(s,x) = c_1 \overline{F(\tau_1(s), \tau_2(x))} \qquad (F \in C^1(I, \mathrm{Lip}(I)),\ s, x \in I).$$

In the next section, we describe the definition of $C^1(I, \mathrm{Lip}(I))$ and $\|\cdot\|_\Sigma$ in detail. The next result is a direct, but important, consequence of our main theorem.

COROLLARY 1.2. *If $T : C^1(I, \mathrm{Lip}(I)) \to C^1(I, \mathrm{Lip}(I))$ is a surjective complex linear isometry with respect to the norm $\|\cdot\|_\Sigma$, then there exist a constant $c_1 \in \mathbb{T}$ and two maps $\tau_1, \tau_2 \in \{id_I, 1_I - id_I\}$ such that*

$$T(F)(s,x) = c_1 F(\tau_1(s), \tau_2(x)) \quad (F \in C^1(I, \mathrm{Lip}(I)),\ s, x \in I).$$

2. Embedding of $C^1(I, \mathrm{Lip}(I))$ into $C(Z)$

Let I be the closed unit interval $[0,1]$. We denote by $C^1(I)$ the commutative Banach algebra of all complex-valued continuously differentiable functions on I. Let $\mathrm{Lip}(I)$ and $L^\infty(I)$ be the commutative Banach algebra of all complex-valued Lipschitz functions on I and that of all complex-valued essentially bounded Lebesgue measurable functions on I, respectively. It is well known that $g \in C(I)$ is a Lipschitz function on I if and only if the derivative $g'(x)$ exists for almost all $x \in I$ and $g' \in L^\infty(I)$. Denote by $\mathcal{M}$ the maximal ideal space of $L^\infty(I)$. By the Gelfand–Naimark theorem [**17**, VIII. Theorem 2.1], $L^\infty(I)$ is isometrically isomorphic to $C(\mathcal{M})$. We do regard $g' \in L^\infty(I)$ as a continuous function on $\mathcal{M}$ for each $g \in \mathrm{Lip}(I)$. We define the norm $\|g\|_L$ by

$$\|g\|_L = \sup_{x \in I} |g(x)| + \sup_{m \in \mathcal{M}} |g'(m)| \tag{2.1}$$

for $g \in \mathrm{Lip}(I)$.

For each $F \in C(I^2)$ and $s \in I$, we define $F_s : I \to \mathbb{C}$ by $F_s(x) = F(s,x)$ for $x \in I$. Then F_s is continuous on I.

DEFINITION 1.3. Let $C(I, \mathrm{Lip}(I))$ be the algebra of all continuous functions F from I to $\mathrm{Lip}(I)$. We denote $F(s)(x)$ by $F(s,x)$ for $F \in C(I, \mathrm{Lip}(I))$ and $s, x \in I$. Thus, we do regard $F \in C(I, \mathrm{Lip}(I))$ as a continuous function on I^2 such that $F_s \in \mathrm{Lip}(I)$ for each $s \in I$. We define $C^1(I, \mathrm{Lip}(I))$ as the algebra of all $F \in C(I, \mathrm{Lip}(I))$ satisfying the following condition: There exists $G \in C(I, \mathrm{Lip}(I))$ such that

$$\lim_{h \to 0} \left\| \frac{F_{s+h} - F_s}{h} - G_s \right\|_L = 0 \quad (s \in I),$$

when $s = 0, 1$, the limit means the right-hand and left-hand one-sided limit, respectively. Then G is said to be the derivative of F, and we denote it by $D(F)$. By definition, $D(F) \in C(I, \mathrm{Lip}(I))$ for each $F \in C^1(I, \mathrm{Lip}(I))$. Thus, $F_s, D(F)_t \in \mathrm{Lip}(I)$ for each $s, t \in I$. Then the derivatives of F_s and $D(F)_t$ exist: We denote $(F_s)' = F_s'$ and $(D(F)_t)' = D(F)_t'$ for simplicity. As we mentioned above, we do regard F_s' and $D(F)_t'$ as continuous functions on $\mathcal{M}$.

We define the norm $\|\cdot\|_\Sigma$ on $C^1(I, \mathrm{Lip}(I))$ by

$$\|F\|_\Sigma = \sup_{s \in I} \|F_s\|_L + \sup_{t \in I} \|D(F)_t\|_L \qquad (F \in C^1(I, \mathrm{Lip}(I))).$$

Here, we show an outline of proof of Main theorem. From now on, given a normed space E, we will denote by E^*, $(E^*)_1$, and $\mathrm{ext}(E^*)_1$ the dual space of E, the closed unit ball of E^*, and the set of extreme points of $(E^*)_1$, respectively. We assume that T is a surjective isometry on $C^1(I, \mathrm{Lip}(I))$ and T_0 is a mapping, defined by $T_0 = T - T(0)$. Then T_0 is a surjective *real linear* isometry on $C^1(I, \mathrm{Lip}(I))$ by the Mazur–Ulam theorem [**32**, Theorem 1.3.5]. First, we embed $C^1(I, \mathrm{Lip}(I))$ into $C(Z)$ with the supremum norm for some compact Hausdorff space Z; we introduce several variables when we do this. Let A be the isometric image of $C^1(I, \mathrm{Lip}(I))$ in $C(Z)$. We induce a surjective real linear isometry S on A with respect to the supremum norm from T_0 on $C^1(I, \mathrm{Lip}(I))$. Applying the argument in [**64**, Lemma 3.1] with the Arens–Kelley theorem [**32**, p.33], we can characterize $\mathrm{ext}(A^*)_1$. It is well known that one can characterize surjective complex linear isometries on $C(K)$ for a compact Hausdorff space K by the structure of $\mathrm{ext}(C(K)^*)_1$ (see, for example, [**17**, Proof of the Banach–Stone theorem, p.172]). We can characterize the surjective real linear isometry S on A by a similar argument as above; roughly speaking, S is a sum of weighted composition operators. The form of S includes inessential variables, since we introduced several variables to embed $C^1(I, \mathrm{Lip}(I))$ into $C(Z)$. We will cancel inessential variables to determine the surjective isometry T on $C^1(I, \mathrm{Lip}(I))$. The author in present paper refers to [**46, 53, 64**] for this idea.

In the rest of this section, we construct a complex linear isometry U from $C^1(I, \mathrm{Lip}(I))$ into $C(Z)$.

Definition 1.4. We put $X = I^2 \times \mathcal{M} \times \mathbb{T}$, $Y = \mathbb{T} \times X$, and $Z = X \times Y$. We define two operators $\partial_1 : C^1(I, \mathrm{Lip}(I)) \to C(X)$ and $\partial_2 : C^1(I, \mathrm{Lip}(I)) \to C(Y)$ by

$$\partial_1(F)(s,x,m,z) = F_s(x) + F'_s(m)z \qquad ((s,x,m,z) \in X), \tag{2.2}$$

$$\partial_2(F)(\xi,(t,y,n,w)) = \xi D(F)_t(y) + D(F)'_t(n)w \qquad ((\xi,(t,y,n,w)) \in Y) \tag{2.3}$$

for any $F \in C^1(I, \mathrm{Lip}(I))$. By definition, $F'_s, D(F)'_t \in C(\mathcal{M})$ for every $F \in C^1(I, \mathrm{Lip}(I))$ and $s, t \in I$. Hence, ∂_1 and ∂_2 are well defined.

For each $F \in C^1(I, \mathrm{Lip}(I))$, we define $\widetilde{F} : Z \to \mathbb{C}$ by

$$\widetilde{F}(\mathbf{x},\mathbf{y}) = \partial_1(F)(\mathbf{x}) + \partial_2(F)(\mathbf{y}) \quad ((\mathbf{x},\mathbf{y}) \in Z). \tag{2.4}$$

Since $\partial_1(F) \in C(X)$ and $\partial_2(F) \in C(Y)$, we see that $\widetilde{F}$ is a continuous function on Z.

Let $f \in C^1(I)$ and $g \in \mathrm{Lip}(I)$. We define $f \otimes g : I^2 \to \mathbb{C}$ by

$$(f \otimes g)(s,x) = f(s)g(x) \qquad ((s,x) \in I^2).$$

We infer from the definition of $C^1(I, \mathrm{Lip}(I))$ that $f \otimes g \in C^1(I, \mathrm{Lip}(I))$ satisfies

$$(f \otimes g)'_s = f(s)g', \quad D(f \otimes g) = f' \otimes g, \quad \text{and} \quad D(f \otimes g)'_s = f'(s)g' \qquad (s \in I). \tag{2.5}$$

It follows from (2.2),(2.3), and (2.5) that

$$\begin{aligned} \partial_1(f \otimes g)(\mathbf{x}) &= (f \otimes g)_s(x) + (f \otimes g)'_s(m)z = f(s)\big(g(x) + g'(m)z\big), \\ \partial_2(f \otimes g)(\mathbf{y}) &= \xi D(f \otimes g)_t(y) + D(f \otimes g)'_t(n)w = f'(t)\big(\xi g(y) + g'(n)w\big) \end{aligned}$$

for $\mathbf{x} = (s,x,m,z) \in X$ and $\mathbf{y} = (\xi,(t,y,n,w)) \in Y$. Entering the above two equalities into (2.4), we obtain

$$\widetilde{f \otimes g}(\mathbf{z}) = f(s)\big(g(x) + g'(m)z\big) + f'(t)\big(\xi g(y) + g'(n)w\big) \tag{2.6}$$

for any $\mathbf{x} = (s,x,m,z)$ and $\mathbf{y} = (\xi,(t,y,n,w))$ with $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in Z$. We put $\mathbf{1} = 1_I \otimes 1_I$. By (2.6), we observe that $\widetilde{\mathbf{1}}$ is the constant function in A taking only the value 1.

In the next lemma, we shall embed $C^1(I, \mathrm{Lip}(I))$ into $C(Z)$.

Lemma 1.5. *We define a map $U : C^1(I, \mathrm{Lip}(I)) \to C(Z)$ by*

$$U(F) = \widetilde{F} \qquad (F \in C^1(I, \mathrm{Lip}(I))).$$

Then U is a complex linear isometry from $(C^1(I, \mathrm{Lip}(I)), \|\cdot\|_\Sigma)$ into $(C(Z), \|\cdot\|_\infty)$.

Proof. Let ∂_1 and ∂_2 be maps defined by (2.2) and (2.3), respectively. We see that ∂_1 and ∂_2 are both complex linear mappings by definition. Hence, U is a complex linear map by (2.4).

We shall prove that U is an isometry. Fix an arbitrary $F \in C^1(I, \mathrm{Lip}(I))$. We deduce from (2.4) that $\|U(F)\|_\infty = \|\widetilde{F}\|_\infty \leq \|\partial_1(F)\|_\infty + \|\partial_2(F)\|_\infty$. Note that

$$\begin{aligned}\|\partial_1(F)\|_\infty &= \sup_{(s,x,m,z)\in X} |\partial_1(F)(s,x,m,z)| = \sup_{(s,x,m,z)\in X} |F_s(x) + F_s'(m)z| \\ &\leq \sup_{s\in I}\left\{\sup_{x\in I}|F_s(x)| + \sup_{m\in\mathcal{M}}|F_s'(m)|\right\} = \sup_{s\in I}\|F_s\|_L\end{aligned}$$

by (2.1) and (2.2). Hence, $\|\partial_1(F)\|_\infty \leq \sup_{s\in I}\|F_s\|_L$. By the same reasoning, we get $\|\partial_2(F)\|_\infty \leq \sup_{t\in I}\|D(F)_t\|_L$. These inequalities show that

$$\|U(F)\|_\infty \leq \sup_{s\in I}\|F_s\|_L + \sup_{t\in I}\|D(F)_t\|_L = \|F\|_\Sigma,$$

and consequently, $\|U(F)\|_\infty \leq \|F\|_\Sigma$. Now we prove the opposite inequality. Take an arbitrary $\epsilon > 0$. There exists $s_0 \in I$ such that $\sup_{s\in I}\|F_s\|_L - \epsilon/2 < \|F_{s_0}\|_L$. Since F_{s_0} and F_{s_0}' are continuous on I and $\mathcal{M}$, respectively, there are $x_0 \in I$ and $m_0 \in \mathcal{M}$ such that $\sup_{x\in I}|F_{s_0}(x)| = |F_{s_0}(x_0)|$ and $\sup_{m\in\mathcal{M}}|F_{s_0}'(m)| = |F_{s_0}'(m_0)|$. Choose $z_0 \in \mathbb{T}$ so that $|F_{s_0}(x_0) + F_{s_0}'(m_0)z_0| = |F_{s_0}(x_0)| + |F_{s_0}'(m_0)|$. We obtain

$$|F_{s_0}(x_0) + F_{s_0}'(m_0)z_0| = \sup_{x\in I}|F_{s_0}(x)| + \sup_{m\in\mathcal{M}}|F_{s_0}'(m)| = \|F_{s_0}\|_L > \sup_{s\in I}\|F_s\|_L - \frac{\epsilon}{2},$$

and thus, $|F_{s_0}(x_0) + F_{s_0}'(m_0)z_0| > \sup_{s\in I}\|F_s\|_L - \epsilon/2$. We derive from the above arguments that

$$|D(F)_{t_0}(y_0) + D(F)_{t_0}'(n_0)w_0| > \sup_{t\in I}\|D(F)_t\|_L - \frac{\epsilon}{2}$$

for some $(t_0, y_0, n_0, w_0) \in X$. Take $\xi_0 \in \mathbb{T}$ so that

$$\begin{aligned}&\big|F_{s_0}(x_0) + F_{s_0}'(m_0)z_0 + \xi_0\big(D(F)_{t_0}(y_0) + D(F)_{t_0}'(n_0)w_0\big)\big| \\ &\qquad\qquad = |F_{s_0}(x_0) + F_{s_0}'(m_0)z_0| + |D(F)_{t_0}(y_0) + D(F)_{t_0}'(n_0)w_0|.\end{aligned}$$

We infer from the above inequalities with (2.2) and (2.3) that

$$|\partial_1(F)(\mathbf{x}_0) + \partial_2(F)(\mathbf{y}_0)| > \sup_{s\in I}\|F_s\|_L + \sup_{t\in I}\|D(F)_t\|_L - \epsilon = \|F\|_\Sigma - \epsilon,$$

where $\mathbf{x}_0 = (s_0, x_0, m_0, z_0)$ and $\mathbf{y}_0 = (\xi_0, (t_0, y_0, n_0, \xi_0 w_0))$. Hence, $\|\widetilde{F}\|_\infty > \|F\|_\Sigma - \epsilon$. Because $\epsilon > 0$ is arbitrarily chosen, we conclude $\|U(F)\|_\infty = \|\widetilde{F}\|_\infty \geq \|F\|_\Sigma$. Thus, U is a complex linear isometry. $\square$

3. Characterization of induced isometry

Definition 1.6. We define $A = \{\widetilde{F} \in C(Z) : F \in C^1(I, \mathrm{Lip}(I))\}$. By Lemma 1.5, we may and do regard U as a surjective complex linear isometry from $C^1(I, \mathrm{Lip}(I))$ onto the closed linear subspace A of $C(Z)$. We define a mapping $S : A \to A$ by $S = U \circ T_0 \circ U^{-1}$. Then S is a surjective real linear isometry on A, since T_0 is a surjective real linear isometry on $C^1(I, \mathrm{Lip}(I))$.

$$\begin{tikzcd} C^1(I, \mathrm{Lip}(I)) \arrow[r, "T_0"] & C^1(I, \mathrm{Lip}(I)) \arrow[d, "U"] \\ A \arrow[u, "U^{-1}"] \arrow[r, "S"'] & A \end{tikzcd}$$

Because $S \circ U = U \circ T_0$, we obtain

$$S(\widetilde{F}) = \widetilde{T_0(F)} \quad (F \in C^1(I, \mathrm{Lip}(I))). \tag{3.1}$$

Let $\Lambda \in A^*$ with the operator norm $\|\Lambda\|$. We can extend Λ to a bounded linear functional on $C(Z)$ with the same operator norm by the Hahn–Banach theorem [**17**, III. Theorem 6.2]. There exists a regular Borel measure μ on Z such that $\Lambda(\widetilde{F}) = \int_Z \widetilde{F}\, d\mu$ for all $\widetilde{F} \in A$ and that the total variation $\|\mu\|$ of μ satisfies $\|\mu\| = \|\Lambda\|$ by the Riesz representation theorem [**67**, Theorem 2.14]; such a measure μ is called a representing measure for Λ (see [**7**, p.80]). Let $\delta_{\mathbf{z}}$ be a point evaluation at $\mathbf{z} \in Z$ defined by $\delta_{\mathbf{z}}(\widetilde{F}) = \widetilde{F}(\mathbf{z})$ for $\widetilde{F} \in A$. In the next lemma, we prove that every representing measure for $\delta_{\mathbf{z}}$ is the Dirac measure concentrated at $\mathbf{z}$ for any $\mathbf{z} \in Z$.

Lemma 1.7. *Let $\mathbf{x} = (s, x, m, z) \in X$, $\mathbf{y} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z$. If μ is a representing measure for $\delta_{\mathbf{z}}$, then $\mu(\{\mathbf{z}\}) = 1$.*

Proof. First, we see that μ is a probability measure, since $\|\delta_{\mathbf{z}}\| = 1 = \delta_{\mathbf{z}}(\widetilde{\mathbf{1}})$ (see [**7**, p.81]). For simplicity, we shall write $X = (I^2, \mathcal{M}, \mathbb{T})$, $Y = (\mathbb{T}, X)$, $Z = (X, Y)$ and so on. We derive from (2.6) that $\widetilde{f \otimes 1_I}(\mathbf{z}) = f(s) + f'(t)\xi$ for $f \in C^1(I)$, which shows that $f(s) + f'(t)\xi = \delta_{\mathbf{z}}(\widetilde{f \otimes 1_I}) = \int_Z \widetilde{f \otimes 1_I}\, d\mu$ for all $f \in C^1(I)$. We may apply the arguments in [**64**, Proof of Lemma 3.1] to the last equality, and then we observe that μ is concentrated on the set (s, X_1, ξ, t, X_1), where $X_1 = (I, \mathcal{M}, \mathbb{T})$, that is, $\mu((s, X_1, \xi, t, X_1)) = 1$.

Using (2.6), we have $\int_Z \widetilde{1_I \otimes g}\, d\mu = \widetilde{1_I \otimes g}(\mathbf{z}) = g(x) + g'(m)z$ for any $g \in \mathrm{Lip}(I)$. Applying the above argument to $\widetilde{1_I \otimes g}$, we get $\mu((s, x, m, z, \xi, t, X_1)) = 1$.

Finally, we put $f_s = id_I - s1_I$, and then $f_s(s) = 0$ and $f_s' = 1_I$. For each $g \in \mathrm{Lip}(I)$, we get $\int_Z (\widetilde{f_s \otimes g}) d\mu = \widetilde{f_s \otimes g}(\mathbf{z}) = \xi g(y) + g'(n)w$ by (2.6). By the same reasoning as above, we see that $\mu((s, x, m, z, \xi, t, y, n, w)) = \mu(\mathbf{z}) = 1$. $\square$

We denote by $\mathrm{Ch}(A)$ the set of all $\mathbf{z} \in Z$ such that $\delta_{\mathbf{z}} \in \mathrm{ext}(A^*)_1$. The set $\mathrm{Ch}(A)$ is called the Choquet boundary for A. We shall characterize $\mathrm{ext}(A^*)_1$. First, we determine the Choquet boundary for A in the next lemma.

Lemma 1.8. *$\mathrm{Ch}(A) = Z$.*

Proof. It is enough to prove that $Z \subset \mathrm{Ch}(A)$. We take $\mathbf{z} \in Z$ arbitrarily. Assume that $\delta_{\mathbf{z}} = (\Lambda_1 + \Lambda_2)/2$ for some $\Lambda_1, \Lambda_2 \in (A^*)_1$. For each $j \in \{1, 2\}$, there exists a representing measure μ_j such that $\|\mu_j\| = \|\Lambda_j\|$ and $\Lambda_j(\widetilde{F}) = \int_Z \widetilde{F} d\mu_j$ for all $\widetilde{F} \in A$ by the Hahn–Banach theorem and the Riesz representation theorem. We put $\mu = (\mu_1 + \mu_2)/2$. First, we shall prove that μ is the Dirac measure concentrated at $\mathbf{z}$. Since μ_j is a representing measure for Λ_j for $j = 1, 2$, we get

$$\int_Z \widetilde{F} d\mu = \int_Z \widetilde{F} d(\frac{\mu_1 + \mu_2}{2}) = \frac{\Lambda_1(\widetilde{F}) + \Lambda_2(\widetilde{F})}{2} = \delta_{\mathbf{z}}(\widetilde{F}) \qquad (\widetilde{F} \in A).$$

Substituting $\widetilde{F} = \widetilde{\mathbf{1}}$ into the last equality, we obtain $\mu(Z) = \int_Z \widetilde{\mathbf{1}} d\mu = \delta_{\mathbf{z}}(\widetilde{\mathbf{1}}) = 1$, and thus, $\mu(Z) = 1$. Having in mind that $\|\mu_j\| = \|\Lambda_j\| \leq 1$ for $j = 1, 2$, we obtain

$$1 = |\mu(Z)| \leq \|\mu\| \leq \frac{\|\mu_1\| + \|\mu_2\|}{2} \leq 1,$$

which implies that $\|\mu\| = 1 = \|\delta_{\mathbf{z}}\|$. Hence μ is a representing measure for $\delta_{\mathbf{z}}$. By Lemma 1.7, we conclude that μ is the Dirac measure concentrated at $\mathbf{z}$.

If B is any Borel set in Z which contains $\mathbf{z}$, then $(\mu_1(B) + \mu_2(B))/2 = \mu(B) = 1$. Because $\|\mu_j\| \leq 1$, we see that $\mu_j(B) = 1$. This implies that $\mu_j(Z \setminus B) = \mu_j(Z) - \mu_j(B) = 0$. Therefore, μ_j is the Dirac measure concentrated at $\mathbf{z}$ for $j = 1, 2$. For each $\widetilde{F} \in A$ and $j = 1, 2$, we have $\Lambda_j(\widetilde{F}) = \int_Z \widetilde{F} d\mu_j = \widetilde{F}(\mathbf{z}) = \delta_{\mathbf{z}}(\widetilde{F})$, which shows that $\Lambda_1 = \delta_{\mathbf{z}} = \Lambda_2$. This means that $\delta_{\mathbf{z}} \in \mathrm{ext}(A^*)_1$. Since $\mathbf{z} \in Z$ is arbitrary, we conclude that $Z \subset \mathrm{Ch}(A)$. $\square$

By the Arens–Kelley theorem, we have $\mathrm{ext}(A^*)_1 = \{\lambda\delta_{\mathbf{z}} \in (A^*)_1 : \lambda \in \mathbb{T},\ \mathbf{z} \in \mathrm{Ch}(A)\}$. Lemma 1.8 shows that

$$\mathrm{ext}(A^*)_1 = \{\lambda\delta_{\mathbf{z}} \in (A^*)_1 : \lambda \in \mathbb{T},\ \mathbf{z} \in Z\}. \tag{3.2}$$

We shall next verify that $\mathrm{ext}(A^*)_1$ is homeomorphic to compact Hausdorff space $\mathbb{T} \times Z$. Before proving it, we prepare the next lemma.

Lemma 1.9. *For any* $\mathbf{z}_1, \mathbf{z}_2 \in Z = X \times Y$ *with* $\mathbf{z}_1 \neq \mathbf{z}_2$*, there exists* $\widetilde{F} \in A$ *such that* $\widetilde{F}(\mathbf{z}_1) \neq \widetilde{F}(\mathbf{z}_2)$.

Proof. We put $\mathbf{z}_j = (\mathbf{x}_j, \mathbf{y}_j)$ for $\mathbf{x}_j = (s_j, x_j, m_j, z_j) \in X$ and $\mathbf{y}_j = (\xi_j, (t_j, y_j, n_j, w_j)) \in Y$ with $j = 1, 2$. For each $f \in C^1(I)$ and $g \in \mathrm{Lip}(I)$, we have

$$\widetilde{f \otimes 1_I}(\mathbf{z}_j) = f(s_j) + f'(t_j)\xi_j \quad \text{and} \quad \widetilde{1_I \otimes g}(\mathbf{z}_j) = g(x_j) + g'(m_j)z_j$$

by (2.6). If $(s_1, t_1) \neq (s_2, t_2)$, we can choose $f_0 \in C^1(I)$ such that $\widetilde{f_0 \otimes 1_I}(\mathbf{z}_1) = 0$ and $\widetilde{f_0 \otimes 1_I}(\mathbf{z}_2) = 1$. If $(s_1, t_1) = (s_2, t_2)$ and $\xi_1 \neq \xi_2$, then $\widetilde{f_1 \otimes 1_I}(\mathbf{z}_j) = \xi_j$ for some $f_1 \in C^1(I)$. Thus, there exists $\widetilde{F} \in A$ so that $\widetilde{F}(\mathbf{z}_1) \neq \widetilde{F}(\mathbf{z}_2)$, provided that $(s_1, t_1, \xi_1) \neq (s_2, t_2, \xi_2)$.

By a quite similar argument, we can find $g_0 \in \mathrm{Lip}(I)$ such that $\widetilde{1_I \otimes g_0}(\mathbf{z}_1) \neq \widetilde{1_I \otimes g_0}(\mathbf{z}_2)$ if $(x_1, m_1, z_1) \neq (x_2, m_2, z_2)$.

Finally, we consider the case in which $\mathbf{x}_1 = \mathbf{x}_2$, $(\xi_1, t_1) = (\xi_2, t_2)$ and $(y_1, n_1, w_1) \neq (y_2, n_2, w_2)$. Setting $f_2 = id_I - s_1 1_I$, we get $f_2(s_1) = 0$ and $f_2' = 1_I$. We derive from (2.6) that $\widetilde{f_2 \otimes g}(\mathbf{z}_j) = \xi_1 g(y_j) + g'(n_j)w_j$ for all $g \in \mathrm{Lip}(I)$. Applying the above argument to $\widetilde{f_2 \otimes g}$, we see that $\widetilde{f_2 \otimes g_1}(\mathbf{z}_1) \neq \widetilde{f_2 \otimes g_1}(\mathbf{z}_2)$ for some $g_1 \in \mathrm{Lip}(I)$. The proof is complete. □

Now we are in a position to show that $\mathrm{ext}(A^*)_1$ is homeomorphic to $\mathbb{T} \times Z$.

Lemma 1.10. *We define a map* $\mathbf{h} : \mathbb{T} \times Z \to \mathrm{ext}(A^*)_1$ *by*

$$\mathbf{h}(\lambda, \mathbf{z}) = \lambda \delta_{\mathbf{z}} \qquad ((\lambda, \mathbf{z}) \in \mathbb{T} \times Z).$$

Then the mapping $\mathbf{h}$ *is a homeomorphism from* $\mathbb{T} \times Z$ *with the product topology onto* $\mathrm{ext}(A^*)_1$ *with the relative weak***-topology.*

Proof. By the definition of $\mathbf{h}$, we can write (3.2) as $\mathrm{ext}(A^*)_1 = \mathbf{h}(\mathbb{T} \times Z)$, which implies that $\mathbf{h}$ is surjective. Now we shall show that $\mathbf{h}$ is injective. Suppose that $\mathbf{h}(\lambda_1, \mathbf{z}_1) = \mathbf{h}(\lambda_2, \mathbf{z}_2)$ for $\lambda_1, \lambda_2 \in \mathbb{T}$ and $\mathbf{z}_1, \mathbf{z}_2 \in Z$, that is, $\lambda_1 \delta_{\mathbf{z}_1} = \lambda_2 \delta_{\mathbf{z}_2}$. Since $\widetilde{\mathbf{1}} = \widetilde{1_I \otimes 1_I}$ is the constant function in A, we have $\lambda_1 = \lambda_1 \delta_{\mathbf{z}_1}(\widetilde{\mathbf{1}}) = \lambda_2 \delta_{\mathbf{z}_2}(\widetilde{\mathbf{1}}) = \lambda_2$, and thus, $\lambda_1 = \lambda_2$. This implies that $\delta_{\mathbf{z}_1} = \delta_{\mathbf{z}_2}$. Because A separates the points of Z by Lemma 1.9, we obtain $\mathbf{z}_1 = \mathbf{z}_2$. Hence, $(\lambda_1, \mathbf{z}_1) = (\lambda_2, \mathbf{z}_2)$, which shows that $\mathbf{h}$ is injective. By the definition of the weak*-topology, we observe that $\mathbf{h}$ is a continuous map from the compact space $\mathbb{T} \times Z$ onto the Hausdorff space $\mathrm{ext}(A^*)_1$. Therefore, the map $\mathbf{h}$ is a homeomorphism, as is claimed. □

Because S is not necessarily complex linear, the adjoint operator $S^* : A^* \to A^*$ is not well defined. In place of S^*, we define $S_* : A^* \to A^*$ by

$$S_*(\Lambda)(\widetilde{F}) = \mathrm{Re}\big(\Lambda(S(\widetilde{F}))\big) - i\mathrm{Re}\big(\Lambda(S(i\widetilde{F}))\big) \quad (\Lambda \in A^*,\ \widetilde{F} \in A). \tag{3.3}$$

It is well known that S_* is a surjective real linear isometry (see [**67**, Proposition 5.17] and [**55**]). We see that S_* preserves $\mathrm{ext}(A^*)_1$, that is, $S_*(\mathrm{ext}(A^*)_1) = \mathrm{ext}(A^*)_1$.

So as to characterize S_* on $\mathrm{ext}(A^*)_1$, we define two maps using $\mathbf{h}$ as follows.

Definition 1.11. Let $p_1 : \mathbb{T} \times Z \to \mathbb{T}$ and $p_2 : \mathbb{T} \times Z \to Z$ be the natural projections. We define two maps $\alpha : \mathbb{T} \times Z \to \mathbb{T}$ and $\Phi : \mathbb{T} \times Z \to Z$ by $\alpha = p_1 \circ \mathbf{h}^{-1} \circ S_*|_{\mathrm{ext}(A^*)_1} \circ \mathbf{h}$, $\Phi = p_2 \circ \mathbf{h}^{-1} \circ S_*|_{\mathrm{ext}(A^*)_1} \circ \mathbf{h}$. We note that $\mathbf{h} : \mathbb{T} \times Z \to \mathrm{ext}(A^*)_1$ is a homeomorphism by Lemma 1.10 and $S_*(\mathrm{ext}(A^*)_1) = \mathrm{ext}(A^*)_1$. Thus, α and Φ are well defined.

We put $(\lambda_1, \mathbf{z}_1) = (\mathbf{h}^{-1} \circ S_*|_{\mathrm{ext}(A^*)_1} \circ \mathbf{h})(\lambda, \mathbf{z})$ for $(\lambda, \mathbf{z}) \in Z$. By Definition 1.11, we get $\lambda_1 = \alpha(\lambda, \mathbf{z})$ and $\mathbf{z}_1 = \Phi(\lambda, \mathbf{z})$. This shows that $(\mathbf{h}^{-1} \circ S_*|_{\mathrm{ext}(A^*)_1} \circ \mathbf{h})(\lambda, \mathbf{z}) = \big(\alpha(\lambda, \mathbf{z}), \Phi(\lambda, \mathbf{z})\big)$, which is equivalent to $S_*(\mathbf{h}(\lambda, \mathbf{z})) = \mathbf{h}\big(\alpha(\lambda, \mathbf{z}), \Phi(\lambda, \mathbf{z})\big)$ for all $(\lambda, \mathbf{z}) \in \mathbb{T} \times Z$. By the definition of $\mathbf{h}$, we have

$$S_*\big(\lambda\delta_{\mathbf{z}}\big) = \alpha(\lambda, \mathbf{z})\delta_{\Phi(\lambda,\mathbf{z})} \qquad ((\lambda, \mathbf{z}) \in \mathbb{T} \times Z). \tag{3.4}$$

Because $S_*|_{\mathrm{ext}(A^*)_1} : \mathrm{ext}(A^*)_1 \to \mathrm{ext}(A^*)_1$ is a bijective continuous map, it follows that $\mathbf{h}^{-1} \circ S_*|_{\mathrm{ext}(A^*)_1} \circ \mathbf{h}$ is a homeomorphism from $\mathbb{T} \times Z$ onto itself. Hence, we notice that α and Φ are surjective continuous maps.

The following lemma states that $\Phi(\lambda, z)$ is closely connected with $\Phi(1, z)$ and $\Phi(i, z)$, which is a key result to investigate the map Φ.

Lemma 1.12. *There exists a continuous function $\varepsilon_0 : Z \to \{\pm 1\}$ such that*

$$\lambda^{\varepsilon_0(\mathbf{z})}\delta_{\Phi(\lambda,\mathbf{z})} = a\delta_{\Phi(1,\mathbf{z})} + i\varepsilon_0(\mathbf{z})b\delta_{\Phi(i,\mathbf{z})}$$

for all $\mathbf{z} \in Z$ and $\lambda = a + ib \in \mathbb{T}$ with $a, b \in \mathbb{R}$.

Proof. Fix an arbitrary $\mathbf{z} \in Z$. Since S_* is real linear, we derive from (3.4) that

$$\begin{aligned} \alpha(\lambda, \mathbf{z})\delta_{\Phi(\lambda,\mathbf{z})} &= S_*(\lambda\delta_{\mathbf{z}}) = S_*\big((a + ib)\delta_{\mathbf{z}}\big) \\ &= aS_*(\delta_{\mathbf{z}}) + bS_*(i\delta_{\mathbf{z}}) = a\alpha(1, \mathbf{z})\delta_{\Phi(1,\mathbf{z})} + b\alpha(i, \mathbf{z})\delta_{\Phi(i,\mathbf{z})}, \end{aligned}$$

and thus,

$$\alpha(\lambda, \mathbf{z})\delta_{\Phi(\lambda,\mathbf{z})} = a\alpha(1, \mathbf{z})\delta_{\Phi(1,\mathbf{z})} + b\alpha(i, \mathbf{z})\delta_{\Phi(i,\mathbf{z})} \tag{3.5}$$

for every $\lambda = a + ib \in \mathbb{T}$ with $a, b \in \mathbb{R}$. Evaluating the last equality at the constant function $\widetilde{\mathbf{1}} = \widetilde{1_I \otimes 1_I}$, we get

$$\alpha(\lambda, \mathbf{z}) = a\alpha(1, \mathbf{z}) + b\alpha(i, \mathbf{z}). \tag{3.6}$$

Since $\alpha(\lambda, \mathbf{z}) \in \mathbb{T}$, we have $1 = |a\alpha(1, \mathbf{z}) + b\alpha(i, \mathbf{z})| = |a + b\overline{\alpha(1, \mathbf{z})}\alpha(i, \mathbf{z})|$ for all $a, b \in \mathbb{R}$ with $a + ib \in \mathbb{T}$. Entering $a = b = 1/\sqrt{2}$ into the last equalities, we obtain $\sqrt{2} = |1 + \overline{\alpha(1, \mathbf{z})}\alpha(i, \mathbf{z})|$. It follows from $\alpha(1, \mathbf{z}), \alpha(i, \mathbf{z}) \in \mathbb{T}$ that $\overline{\alpha(1, \mathbf{z})}\alpha(i, \mathbf{z})$ is i or $-i$. We define

$$\Omega_0 = \{\mathbf{z} \in Z : \overline{\alpha(1, \mathbf{z})}\alpha(i, \mathbf{z}) = i\}.$$

From the above argument, we see that $Z \setminus \Omega_0 = \{\mathbf{z} \in Z : \overline{\alpha(1, \mathbf{z})}\alpha(i, \mathbf{z}) = -i\}$. Because $\alpha(1, \cdot)$, $\alpha(i, \cdot) : Z \to \mathbb{T}$ are continuous on Z, both Ω_0 and $Z \setminus \Omega_0$ are closed in Z. Hence Ω_0 is a closed and open subset of Z. Next we define a function $\varepsilon_0 : Z \to \{\pm 1\}$ by

$$\varepsilon_0(\mathbf{z}) = \begin{cases} 1 & (\mathbf{z} \in \Omega_0) \\ -1 & (\mathbf{z} \in Z \setminus \Omega_0). \end{cases}$$

The function ε_0 is continuous on Z, because Ω_0 is closed and open in Z. By the definition of ε_0, we get

$$\alpha(i, \mathbf{z}) = i\varepsilon_0(\mathbf{z})\alpha(1, \mathbf{z}) \qquad (\mathbf{z} \in Z). \tag{3.7}$$

Equalities (3.6) and (3.7) yield

$$\alpha(\lambda, \mathbf{z}) = a\alpha(1, \mathbf{z}) + b\alpha(i, \mathbf{z}) = (a + i\varepsilon_0(\mathbf{z})b)\alpha(1, \mathbf{z})$$

for all $\mathbf{z} \in Z$. Since $\varepsilon_0(\mathbf{z}) \in \{\pm 1\}$, we can write $a + i\varepsilon_0(\mathbf{z})b = \lambda^{\varepsilon_0(\mathbf{z})}$ for $\lambda = a + ib \in \mathbb{T}$. This implies that $\alpha(\lambda, \mathbf{z}) = \lambda^{\varepsilon_0(\mathbf{z})}\alpha(1, \mathbf{z})$. Having in mind that $\alpha(1, \mathbf{z}) \in \mathbb{T}$, we deduce from (3.5) and (3.7) that $\lambda^{\varepsilon_0(\mathbf{z})}\delta_{\Phi(\lambda, \mathbf{z})} = a\delta_{\Phi(1, \mathbf{z})} + i\varepsilon_0(\mathbf{z})b\delta_{\Phi(i, \mathbf{z})}$ for every $\mathbf{z} \in Z$. □

For simplicity of notation, we shall write $\alpha(1, \mathbf{z}) = \alpha(\mathbf{z})$ for $\mathbf{z} \in Z$. By (3.7), we have

$$\alpha(i, \mathbf{z}) = i\varepsilon_0(\mathbf{z})\alpha(\mathbf{z}) \qquad (\mathbf{z} \in Z). \tag{3.8}$$

Our next aim is to show that $\Phi(i, \mathbf{z}) = \Phi(1, \mathbf{z})$ or $-\Phi(1, \mathbf{z})$ for each $\mathbf{z} \in Z$. In order to prove it, we define nine maps using projections.

Definition 1.13. Let q_X and q_Y be the projections from $Z = X \times Y$ onto X and Y, respectively. We define $\phi \colon \mathbb{T} \times Z \to X$ and $\psi \colon \mathbb{T} \times Z \to Y$ by $\phi = q_X \circ \Phi$ and $\psi = q_Y \circ \Phi$, where $\Phi \colon \mathbb{T} \times Z \to Z$ is the surjective continuous map as in Definition 1.11. Then

$$\Phi(\boldsymbol{\zeta}) = (\phi(\boldsymbol{\zeta}), \psi(\boldsymbol{\zeta})) \quad (\boldsymbol{\zeta} \in \mathbb{T} \times Z).$$

Since ϕ is the map from $\mathbb{T} \times Z$ onto $X = I^2 \times \mathcal{M} \times \mathbb{T}$, there exist well defined maps $\phi_1, \phi_2 \colon \mathbb{T} \times Z \to I$, $\phi_3 \colon \mathbb{T} \times Z \to \mathcal{M}$, and $\phi_4 \colon \mathbb{T} \times Z \to \mathbb{T}$ such that

$$\phi(\boldsymbol{\zeta}) = (\phi_1(\boldsymbol{\zeta}), \phi_2(\boldsymbol{\zeta}), \phi_3(\boldsymbol{\zeta}), \phi_4(\boldsymbol{\zeta})) \in X \qquad (\boldsymbol{\zeta} \in \mathbb{T} \times Z). \tag{3.9}$$

Moreover, we can define $\psi_1, \psi_2 \colon \mathbb{T} \times Z \to I$, $\psi_3 \colon \mathbb{T} \times Z \to \mathcal{M}$, and $\psi_0, \psi_4 \colon \mathbb{T} \times Z \to \mathbb{T}$ by

$$\psi(\boldsymbol{\zeta}) = (\psi_0(\boldsymbol{\zeta}), (\psi_1(\boldsymbol{\zeta}), \psi_2(\boldsymbol{\zeta}), \psi_3(\boldsymbol{\zeta}), \psi_4(\boldsymbol{\zeta}))) \in Y \qquad (\boldsymbol{\zeta} \in \mathbb{T} \times Z) \tag{3.10}$$

because ψ is the map from $\mathbb{T} \times Z$ onto $Y = \mathbb{T} \times X$. For simplicity of notation, we also write

$$\phi(\boldsymbol{\zeta}) = (\phi_j(\boldsymbol{\zeta}))_{1 \le j \le 4} \quad \text{and} \quad \psi(\boldsymbol{\zeta}) = (\psi_k(\boldsymbol{\zeta}))_{0 \le k \le 4}.$$

If we enter $\mathbf{x} = \phi(\boldsymbol{\zeta})$ and $\mathbf{y} = \psi(\boldsymbol{\zeta})$ into (2.6), we get

$$\begin{aligned} \widetilde{f \otimes g}(\Phi(\boldsymbol{\zeta})) = f(\phi_1(\boldsymbol{\zeta}))\big(g(\phi_2(\boldsymbol{\zeta})) + g'(\phi_3(\boldsymbol{\zeta}))\phi_4(\boldsymbol{\zeta})\big) \\ + f'(\psi_1(\boldsymbol{\zeta}))\big(\psi_0(\boldsymbol{\zeta})g(\psi_2(\boldsymbol{\zeta})) + g'(\psi_3(\boldsymbol{\zeta}))\psi_4(\boldsymbol{\zeta})\big). \end{aligned} \tag{3.11}$$

In particular, we obtain

$$\widetilde{f \otimes 1_I}(\Phi(\boldsymbol{\zeta})) = f(\phi_1(\boldsymbol{\zeta})) + f'(\psi_1(\boldsymbol{\zeta}))\psi_0(\boldsymbol{\zeta}), \tag{3.12}$$

$$\widetilde{1_I \otimes g}(\Phi(\boldsymbol{\zeta})) = g(\phi_2(\boldsymbol{\zeta})) + g'(\phi_3(\boldsymbol{\zeta}))\phi_4(\boldsymbol{\zeta}) \tag{3.13}$$

for $f \in C^1(I)$, $g \in \mathrm{Lip}(I)$, and $\boldsymbol{\zeta} \in \mathbb{T} \times Z$.

In the rest of this section, we will investigate the maps ϕ and ψ.

Lemma 1.14. *The maps ϕ_j for $j = 1, 2, 3$ and ψ_1 are independent from the variable $\lambda \in \mathbb{T}$, that is,*

$$\phi_j(\lambda, \mathbf{z}) = \phi_j(1, \mathbf{z}) \quad \textit{and} \quad \psi_1(\lambda, \mathbf{z}) = \psi_1(1, \mathbf{z}) \qquad (j \in \{1, 2, 3\},\ \lambda \in \mathbb{T},\ \mathbf{z} \in Z).$$

Proof. Take an arbitrary $\mathbf{z} \in Z$. We shall prove that $\phi_1(\lambda, \mathbf{z}) = \phi_1(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$. Suppose that $\phi_1(\lambda_0, \mathbf{z}) \notin \{\phi_1(1, \mathbf{z}), \phi_1(i, \mathbf{z})\}$ for some $\lambda_0 = a_0 + ib_0 \in \mathbb{T}$ with $a_0, b_0 \in \mathbb{R}$. Then there exists $f_1 \in C^1(I)$ such that

$$f_1(\phi_1(\lambda_0, \mathbf{z})) = 1,\ f_1(\phi_1(1, \mathbf{z})) = f_1(\phi_1(i, \mathbf{z})) = 0, \text{ and } f_1'(\psi_1(\mu, \mathbf{z})) = 0 \quad (\mu = \lambda_0, 1, i).$$

By (3.12), we have $\widetilde{f_1 \otimes 1_I}(\Phi(\lambda_0, \mathbf{z})) = 1$ and $\widetilde{f_1 \otimes 1_I}(\Phi(1, \mathbf{z})) = \widetilde{f_1 \otimes 1_I}(\Phi(i, \mathbf{z})) = 0$. By Lemma 1.12, we get $\lambda_0^{\varepsilon_0(\mathbf{z})}\delta_{\Phi(\lambda_0, \mathbf{z})} = a_0\delta_{\Phi(1, \mathbf{z})} + i\varepsilon_0(\mathbf{z})b_0\delta_{\Phi(i, \mathbf{z})}$. Evaluating the last equality at $\widetilde{f_1 \otimes 1_I}$, we obtain $\lambda_0^{\varepsilon_0(\mathbf{z})} = 0$. This contradicts $\lambda_0 \in \mathbb{T}$. We thus conclude that $\phi_1(\lambda, \mathbf{z}) \in \{\phi_1(1, \mathbf{z}), \phi_1(i, \mathbf{z})\}$ for all $\lambda \in \mathbb{T}$. The function $\phi_1(\cdot, \mathbf{z})$, which maps $\lambda \in \mathbb{T}$ to $\phi_1(\lambda, \mathbf{z})$, is continuous on the connected set $\mathbb{T}$. Hence, the image $\phi_1(\mathbb{T}, \mathbf{z})$ of $\mathbb{T}$ is connected as well. Because $\phi_1(\mathbb{T}, \mathbf{z}) \subset \{\phi_1(1, \mathbf{z}), \phi_1(i, \mathbf{z})\}$, we see that $\phi_1(\mathbb{T}, \mathbf{z})$ is a one point set, and consequently, $\phi_1(\lambda, \mathbf{z}) = \phi_1(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$.

We now prove that $\psi_1(\lambda, \mathbf{z}) = \psi_1(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$. If we assume that $\psi_1(\lambda, \mathbf{z}) \notin \{\psi_1(1, \mathbf{z}), \psi_1(i, \mathbf{z})\}$ for some $\lambda \in \mathbb{T}$, we can choose $f_2 \in C^1(I)$ such that

$$f_2'(\psi_1(\lambda, \mathbf{z})) = 1, \quad f_2'(\psi_1(1, \mathbf{z})) = f_2'(\psi_1(i, \mathbf{z})) = 0, \quad \text{and} \quad f_2(\phi_1(\lambda, \mathbf{z})) = 0.$$

Here, we note that $f_2(\phi_1(1, \mathbf{z})) = f_2(\phi_1(i, \mathbf{z})) = 0$, since $\phi_1(\lambda, \mathbf{z}) = \phi_1(1, \mathbf{z}) = \phi_1(i, \mathbf{z})$. Applying Lemma 1.12 with (3.12) to $\widetilde{f_2 \otimes 1_I}$, we will lead a contradiction by a quite similar argument as above. Therefore, we conclude that $\psi_1(\mathbb{T}, \mathbf{z})$ is a connected set, which is contained in $\{\psi_1(1, \mathbf{z}), \psi_1(i, \mathbf{z})\}$. This shows that $\psi_1(\lambda, \mathbf{z}) = \psi_1(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$.

If we consider $g \in \mathrm{Lip}(I)$ with (3.13) instead of $f_1 \in C^1(I)$ with (3.12) in the above arguments, we can prove that $\phi_2(\lambda, \mathbf{z}) = \phi_2(1, \mathbf{z})$ and $\phi_3(\lambda, \mathbf{z}) = \phi_3(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$. $\square$

Lemma 1.15. *For each $\lambda \in \mathbb{T}$ and $\mathbf{z} \in Z$, the following identities hold:*

$$\psi_2(\lambda, \mathbf{z}) = \psi_2(1, \mathbf{z}) \quad \textit{and} \quad \psi_3(\lambda, \mathbf{z}) = \psi_3(1, \mathbf{z}).$$

Proof. Fix $\mathbf{z} \in Z$ arbitrarily. We first notice that $\phi_1(\lambda, \mathbf{z}) = \phi_1(1, \mathbf{z})$ and $\psi_1(\lambda, \mathbf{z}) = \psi_1(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$ by Lemma 1.14. Choose $f_1 \in C^1(I)$ so that $f_1(\phi_1(1, \mathbf{z})) = 0$ and $f_1'(\psi_1(1, \mathbf{z})) = 1$. Equality (3.11) yields

$$\widetilde{f_1 \otimes g}(\Phi(\lambda, \mathbf{z})) = \psi_0(\lambda, \mathbf{z}) g(\psi_2(\lambda, \mathbf{z})) + g'(\psi_3(\lambda, \mathbf{z}))\psi_4(\lambda, \mathbf{z})$$

for all $\lambda \in \mathbb{T}$ and $g \in \mathrm{Lip}(I)$. Since $\psi_0(\lambda, \mathbf{z}) \in \mathbb{T}$, we may apply the same argument as in Proof of Lemma 1.14. Then we obtain $\psi_2(\lambda, \mathbf{z}) = \psi_2(1, \mathbf{z})$ and $\psi_3(\lambda, \mathbf{z}) = \psi_3(1, \mathbf{z})$ for all $\lambda \in \mathbb{T}$. $\square$

For simplicity, we will write

$$\phi_j(\lambda, \mathbf{z}) = \phi_j(\mathbf{z}) \quad \text{and} \quad \psi_j(\lambda, \mathbf{z}) = \psi_j(\mathbf{z}) \qquad (j = 1, 2, 3)$$

for $\lambda \in \mathbb{T}$ and $\mathbf{z} \in Z$. They are reasonable from Lemmas 1.14 and 1.15.

Next, we show that $\phi_4(i, \mathbf{z}) = \phi_4(1, \mathbf{z})$ or $\phi_4(i, \mathbf{z}) = -\phi_4(1, \mathbf{z})$ for each $\mathbf{z} \in Z$.

Lemma 1.16. *There exists a continuous function $\varepsilon_1 : Z \to \{\pm 1\}$ such that*

$$\phi_4(i, \mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(1, \mathbf{z}) \qquad (\mathbf{z} \in Z).$$

Proof. Take arbitrary $\mathbf{z}_0 \in Z$. We put $g_1 = id_I - \phi_2(\mathbf{z}_0)1_I \in \mathrm{Lip}(I)$, and then $g_1(\phi_2(\mathbf{z}_0)) = 0$ and $g_1' = 1_I$. According to (3.13), we get $\widetilde{1_I \otimes g_1}(\Phi(\mu, \mathbf{z}_0)) = \phi_4(\mu, \mathbf{z}_0)$ for all $\mu \in \mathbb{T}$. If we enter $\lambda = (1-i)/\sqrt{2} \in \mathbb{T}$ in Lemma 1.12, then we obtain $\sqrt{2}\lambda^{\varepsilon_0(\mathbf{z}_0)}\delta_{\Phi(\lambda,\mathbf{z}_0)} = \delta_{\Phi(1,\mathbf{z}_0)} - i\varepsilon_0(\mathbf{z}_0)\delta_{\Phi(i,\mathbf{z}_0)}$. Evaluating the last equality at $\widetilde{1_I \otimes g_1}$, we have $\sqrt{2}\lambda^{\varepsilon_0(\mathbf{z}_0)}\phi_4(\lambda,\mathbf{z}_0) = \phi_4(1,\mathbf{z}_0) - i\varepsilon_0(\mathbf{z}_0)\phi_4(i,\mathbf{z}_0)$. We derive from the moduli of the last equality that

$$\sqrt{2} = |\phi_4(1,\mathbf{z}_0) - i\varepsilon_0(\mathbf{z}_0)\phi_4(i,\mathbf{z}_0)| = |1 - i\varepsilon_0(\mathbf{z}_0)\phi_4(i,\mathbf{z}_0)\overline{\phi_4(1,\mathbf{z}_0)}|,$$

where we have used $\phi_4(1,\mathbf{z}_0) \in \mathbb{T}$. This implies that $i\varepsilon_0(\mathbf{z}_0)\phi_4(i,\mathbf{z}_0)\overline{\phi_4(1,\mathbf{z}_0)}$ is i or $-i$, that is, $\varepsilon_0(\mathbf{z}_0)\phi_4(i,\mathbf{z}_0)\overline{\phi_4(1,\mathbf{z}_0)}$ is 1 or -1. We define

$$\Omega_1 = \{\mathbf{z} \in Z : \varepsilon_0(\mathbf{z})\phi_4(i,\mathbf{z})\overline{\phi_4(1,\mathbf{z})} = 1\}.$$

Then we see that $Z \setminus \Omega_1 = \{\mathbf{z} \in Z : \varepsilon_0(\mathbf{z})\phi_4(i,\mathbf{z})\overline{\phi_4(1,\mathbf{z})} = -1\}$ from the above argument. Therefore, Ω_1 and $Z \setminus \Omega_1$ are both closed subsets of Z by the continuity of ε_0 and ϕ_4. Hence, Ω_1 is a closed and open subset of Z. Now we define a function $\varepsilon_1 \colon Z \to \{\pm 1\}$ by

$$\varepsilon_1(\mathbf{z}) = \begin{cases} 1 & (\mathbf{z} \in \Omega_1) \\ -1 & (\mathbf{z} \in Z \setminus \Omega_1). \end{cases}$$

Since Ω_1 is a closed and open set, we observe that ε_1 is a continuous function on Z. By the definition of ε_1, we conclude that $\phi_4(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(1,\mathbf{z})$ for all $\mathbf{z} \in Z$. $\square$

In the next lemma, we shall prove that a similar result to Lemma 1.16 holds for ψ_0 and ψ_4.

Lemma 1.17. *There exist continuous functions $\varepsilon_2, \varepsilon_3 \colon Z \to \{\pm 1\}$ such that*

$$\psi_0(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_2(\mathbf{z})\psi_0(1,\mathbf{z}) \quad \textit{and} \quad \psi_4(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_3(\mathbf{z})\psi_4(1,\mathbf{z}) \qquad (\mathbf{z} \in Z).$$

Proof. Take $\mathbf{z}_0 \in Z$ arbitrarily, and set $f_1 = id_I - \phi_1(\mathbf{z}_0)1_I \in C^1(I)$ and $g_1 = id_I - \psi_2(\mathbf{z}_0)1_I \in \mathrm{Lip}(I)$. Then $f_1(\phi_1(\mathbf{z}_0)) = g_1(\psi_2(\mathbf{z}_0)) = 0$ and $f_1' = g_1' = 1_I$. According to (3.11) and (3.12), we have

$$\widetilde{f_1 \otimes 1_I}(\Phi(\lambda,\mathbf{z}_0)) = \psi_0(\lambda,\mathbf{z}_0) \quad \text{and} \quad \widetilde{f_1 \otimes g_1}(\Phi(\lambda,\mathbf{z}_0)) = \psi_4(\lambda,\mathbf{z}_0)$$

for all $\lambda \in \mathbb{T}$. By the same argument as in Proof of Lemma 1.16, applied to the last two equalities, there exist continuous functions $\varepsilon_2, \varepsilon_3 \colon Z \to \{\pm 1\}$ such that $\psi_0(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_2(\mathbf{z})\psi_0(1,\mathbf{z})$ and $\psi_4(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_3(\mathbf{z})\psi_4(1,\mathbf{z})$ for all $\mathbf{z} \in Z$. $\square$

For simplicity of notation, we shall write $\phi_4(1,\mathbf{z}) = \phi_4(\mathbf{z})$, $\psi_0(1,\mathbf{z}) = \psi_0(\mathbf{z})$, and $\psi_4(1,\mathbf{z}) = \psi_4(\mathbf{z})$ for $\mathbf{z} \in Z$. Lemmas 1.16 and 1.17 show that

$$(3.14)\quad \phi_4(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(\mathbf{z}),$$
$$\psi_0(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_2(\mathbf{z})\psi_0(\mathbf{z}), \quad \text{and} \quad \psi_4(i,\mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_3(\mathbf{z})\psi_4(\mathbf{z})$$

for all $\mathbf{z} \in Z$.

We are now in a position to determine the form of S. In order to represent S simply, we introduce some symbols.

Definition 1.18. For each $a, b \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$, we define $[a+ib]^{\varepsilon} = a + \varepsilon ib$. In particular, $[z]^1 = z$ and $[z]^{-1} = \overline{z}$ for any $z \in \mathbb{C}$. For each $F \in C^1(I, \mathrm{Lip}(I))$, we define $\Delta_1(F)$ and $\Delta_1'(F)$ by

$$(3.15)\quad \Delta_1(F)(\mathbf{z}) = [\alpha(\mathbf{z})F_{\phi_1(\mathbf{z})}(\phi_2(\mathbf{z}))]^{\varepsilon_0(\mathbf{z})}$$
$$\text{and} \quad \Delta_1'(F)(\mathbf{z}) = [\alpha(\mathbf{z})F'_{\phi_1(\mathbf{z})}(\phi_3(\mathbf{z})) \cdot \phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}$$

for all $\mathbf{z} \in Z$. In the same way, we define $\Delta_2(F)$ and $\Delta_2'(F)$ by

$$(3.16)\quad \Delta_2(F)(\mathbf{z}) = [\alpha(\mathbf{z})\psi_0(\mathbf{z})D(F)_{\psi_1(\mathbf{z})}(\psi_2(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})}$$
$$\text{and} \quad \Delta_2'(F)(\mathbf{z}) = [\alpha(\mathbf{z})D(F)'_{\psi_1(\mathbf{z})}(\psi_3(\mathbf{z})) \cdot \psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})}$$

for $\mathbf{z} \in Z$. In particular, if we enter $F = f \otimes 1_I$ into (3.15) and (3.16) for $f \in C^1(I)$, then we derive from (2.5) that

$$(3.17)\quad \Delta_1(f \otimes 1_I)(\mathbf{z}) = [\alpha(\mathbf{z})f(\phi_1(\mathbf{z}))]^{\varepsilon_0(\mathbf{z})}, \quad \Delta_1'(f \otimes 1_I)(\mathbf{z}) = 0,$$
$$\Delta_2'(f \otimes 1_I)(\mathbf{z}) = 0, \quad \text{and} \quad \Delta_2(f \otimes 1_I)(\mathbf{z}) = [\alpha(\mathbf{z})\psi_0(\mathbf{z})f'(\psi_1(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})}$$

for all $\mathbf{z} \in Z$. By the same reasoning, we obtain

$$(3.18)\quad \Delta_1(1_I \otimes g)(\mathbf{z}) = [\alpha(\mathbf{z})g(\phi_2(\mathbf{z}))]^{\varepsilon_0(\mathbf{z})}, \ \Delta_1'(1_I \otimes g)(\mathbf{z}) = [\alpha(\mathbf{z})g'(\phi_3(\mathbf{z})) \cdot \phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})},$$
$$\text{and} \quad \Delta_2(1_I \otimes g)(\mathbf{z}) = \Delta_2'(1_I \otimes g)(\mathbf{z}) = 0$$

for $g \in \mathrm{Lip}(I)$ and $\mathbf{z} \in Z$.

First, we show that $S(\widetilde{F})(\mathbf{z})$ can be expressed as the sum of ∂_1 and ∂_2.

Lemma 1.19. *Let $F \in C^1(I, \mathrm{Lip}(I))$ and $\mathbf{z} \in Z$. Then $S(\widetilde{F})(\mathbf{z})$ is the sum of the followings:*

$$\mathrm{Re}\big(\alpha(\mathbf{z})\partial_1(F)(\phi(1,\mathbf{z}))\big) + i\varepsilon_0(\mathbf{z})\mathrm{Im}\,\big(\alpha(\mathbf{z})\partial_1(F)(\phi(i,\mathbf{z}))\big), \tag{1}$$

$$\mathrm{Re}\big(\alpha(\mathbf{z})\partial_2(F)(\psi(1,\mathbf{z}))\big) + i\varepsilon_0(\mathbf{z})\mathrm{Im}\,\big(\alpha(\mathbf{z})\partial_2(F)(\psi(i,\mathbf{z}))\big). \tag{2}$$

Proof. Let $F \in C^1(I, \mathrm{Lip}(I))$ and $(\lambda, \mathbf{z}) \in \mathbb{T} \times Z$. Then we can write

$$S(\widetilde{F})(\mathbf{z}) = \mathrm{Re}(S(\widetilde{F})(\mathbf{z})) + i\mathrm{Im}\,(S(\widetilde{F})(\mathbf{z})) = \mathrm{Re}(S(\widetilde{F})(\mathbf{z})) - i\,\mathrm{Re}(iS(\widetilde{F})(\mathbf{z})).$$

On the one hand, we note that

$$\mathrm{Re}\big(\lambda S(\widetilde{F})(\mathbf{z})\big) = \mathrm{Re}\big(\lambda\delta_{\mathbf{z}}(S(\widetilde{F}))\big) = \mathrm{Re}\big(S_*(\lambda\delta_{\mathbf{z}})(\widetilde{F})\big)$$

for $\lambda \in \mathbb{T}$ by (3.3). Entering the last equality into the above equality, we have

$$S(\widetilde{F})(\mathbf{z}) = \mathrm{Re}\big(S_*(\delta_{\mathbf{z}})(\widetilde{F})\big) - i\,\mathrm{Re}\big(S_*(i\delta_{\mathbf{z}})(\widetilde{F})\big). \tag{3.19}$$

On the other hand, $S_*(\lambda\delta_{\mathbf{z}}) = \alpha(\lambda, \mathbf{z})\delta_{\Phi(\lambda,\mathbf{z})}$, by (3.4). Hence,

$$\mathrm{Re}\big(S_*(\delta_{\mathbf{z}})(\widetilde{F})\big) = \mathrm{Re}\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(1,\mathbf{z}))\big),$$

where we have used $\alpha(1, \mathbf{z}) = \alpha(\mathbf{z})$. Since $\alpha(i, \mathbf{z}) = i\varepsilon_0(\mathbf{z})\alpha(\mathbf{z})$ by (3.8), we obtain

$$\mathrm{Re}\big(S_*(i\delta_{\mathbf{z}})(\widetilde{F})\big) = \mathrm{Re}\big(\alpha(i,\mathbf{z})\widetilde{F}(\Phi(i,\mathbf{z}))\big) = -\varepsilon_0(\mathbf{z})\mathrm{Im}\,\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(i,\mathbf{z}))\big).$$

We enter these two equalities into (3.19), and then

$$S(\widetilde{F})(\mathbf{z}) = \mathrm{Re}\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(1,\mathbf{z}))\big) + i\varepsilon_0(\mathbf{z})\mathrm{Im}\,\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(i,\mathbf{z}))\big). \tag{3.20}$$

Letting $\mathbf{x} = \phi(\lambda, \mathbf{z})$ and $\mathbf{y} = \psi(\lambda, \mathbf{z})$ in (2.4), we get

$$\widetilde{F}(\Phi(\lambda,\mathbf{z})) = \partial_1(F)(\phi(\lambda,\mathbf{z})) + \partial_2(F)(\psi(\lambda,\mathbf{z})).$$

Substituting the last equality into (3.20), we can rewrite the real part and the imaginary part of $S(\widetilde{F})(\mathbf{z})$ as

$$\mathrm{Re}\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(1,\mathbf{z})\big) = \mathrm{Re}\big(\alpha(\mathbf{z})\partial_1(F)(\phi(1,\mathbf{z}))\big) + \mathrm{Re}\big(\alpha(\mathbf{z})\partial_2(F)(\psi(1,\mathbf{z}))\big),$$

$$i\varepsilon_0(\mathbf{z})\mathrm{Im}\,\big(\alpha(\mathbf{z})\widetilde{F}(\Phi(i,\mathbf{z})\big) = i\varepsilon_0(\mathbf{z})\big(\mathrm{Im}\,\big(\alpha(\mathbf{z})\partial_1(F)(\phi(i,\mathbf{z}))\big) + \mathrm{Im}\,\big(\alpha(\mathbf{z})\partial_2(F)(\psi(i,\mathbf{z}))\big)\big).$$

Adding the last two identities, we can express $S(\widetilde{F})(\mathbf{z})$ as the desired conclusion from (3.20).

$\square$

In the next lemma, we characterize the form of S using the symbols in Definition 1.18.

LEMMA 1.20. *For each $F \in C^1(I, \mathrm{Lip}(I))$, we have*

$$S(\widetilde{F})(\mathbf{z}) = \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z}) + \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z}) \qquad (\mathbf{z} \in Z).$$

PROOF. We take $F \in C^1(I, \mathrm{Lip}(I))$ arbitrarily and let $\mathbf{z} \in Z$. Now, we shall prove that (1) in Lemma 1.19 is written as $\Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z})$. If we apply $\phi(1, \mathbf{z}) = (\phi_j(\mathbf{z}))_{1 \le j \le 4}$ to (2.2), we get

$$\partial_1(F)(\phi(1, \mathbf{z})) = F_{\phi_1(\mathbf{z})}(\phi_2(\mathbf{z})) + F'_{\phi_1(\mathbf{z})}(\phi_3(\mathbf{z})) \cdot \phi_4(\mathbf{z}), \tag{3.21}$$

where we have used the notation $\phi_j(1, \mathbf{z}) = \phi_j(\mathbf{z})$ for $1 \le j \le 4$. Since $\phi_4(i, \mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(\mathbf{z})$ by (3.14), we have

$$\phi(i, \mathbf{z}) = (\phi_1(\mathbf{z}), \phi_2(\mathbf{z}), \phi_3(\mathbf{z}), \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(\mathbf{z})).$$

Entering the last equality into (2.2), we obtain

$$\partial_1(F)(\phi(i, \mathbf{z})) = F_{\phi_1(\mathbf{z})}(\phi_2(\mathbf{z})) + F'_{\phi_1(\mathbf{z})}(\phi_3(\mathbf{z})) \cdot \varepsilon_0(\mathbf{z})\varepsilon_1(\mathbf{z})\phi_4(\mathbf{z}). \tag{3.22}$$

It follows from (3.21) and (3.22) that (1) is written as

$$[\alpha(\mathbf{z})F_{\phi_1(\mathbf{z})}(\phi_2(\mathbf{z}))]^{\varepsilon_0(\mathbf{z})} + [\alpha(\mathbf{z})F'_{\phi_1(\mathbf{z})}(\phi_3(\mathbf{z})) \cdot \phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})} = \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z})$$

by the definition of $\Delta_1(F)$ and $\Delta_1'(F)$.

We next prove that (2) in Lemma 1.19 is written as $\Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z})$ by a similar argument as above. Applying $\psi(1, \mathbf{z}) = (\psi_k(\mathbf{z}))_{0 \le k \le 4}$ to (2.3), we get

$$\partial_2(F)(\psi(1, \mathbf{z})) = \psi_0(\mathbf{z})D(F)_{\psi_1(\mathbf{z})}(\psi_2(\mathbf{z})) + D(F)'_{\psi_1(\mathbf{z})}(\psi_3(\mathbf{z})) \cdot \psi_4(\mathbf{z}). \tag{3.23}$$

Note that $\psi_k(i, \mathbf{z}) = \psi_k(\mathbf{z})$ for each k with $1 \le k \le 3$, $\psi_0(i, \mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_2(\mathbf{z})\psi_0(\mathbf{z})$, and $\psi_4(i, \mathbf{z}) = \varepsilon_0(\mathbf{z})\varepsilon_3(\mathbf{z})\psi_4(\mathbf{z})$ by Lemmas 1.14, 1.15 and 1.17. Equality (2.3) shows that

$$\begin{aligned} \partial_2(F)(\psi(i, \mathbf{z})) = \varepsilon_0(\mathbf{z})\varepsilon_2(\mathbf{z})\psi_0(\mathbf{z})D(F)_{\psi_1(\mathbf{z})}(\psi_2(\mathbf{z})) \\ + D(F)'_{\psi_1(\mathbf{z})}(\psi_3(\mathbf{z})) \cdot \varepsilon_0(\mathbf{z})\varepsilon_3(\mathbf{z})\psi_4(\mathbf{z}). \end{aligned} \tag{3.24}$$

We derive from (3.23) and (3.24) that (2) is written as

$$\begin{aligned} [\alpha(\mathbf{z})\psi_0(\mathbf{z})D(F)_{\psi_1(\mathbf{z})}(\psi_2(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})} + [\alpha(\mathbf{z})D(F)'_{\psi_1(\mathbf{z})}(\psi_3(\mathbf{z})) \cdot \psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})} \\ = \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z}) \end{aligned}$$

by the definition of $\Delta_2(F)$ and $\Delta_2'(F)$.

Finally, since $S(\widetilde{F})(\mathbf{z})$ is the sum of (1) and (2) by Lemma 1.19, we see that

$$S(\widetilde{F})(\mathbf{z}) = \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z}) + \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z}).$$

The proof is complete. $\square$

We derive from (2.4) and (3.1) that

$$S(\widetilde{F})(\mathbf{z}) = \widetilde{T_0(F)}(\mathbf{z}) = \partial_1(T_0(F))(\mathbf{x}) + \partial_2(T_0(F))(\mathbf{y})$$

for $F \in C^1(I, \mathrm{Lip}(I))$ and $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z = X \times Y$. Combining the above identity with Lemma 1.20, we obtain

$$\begin{aligned}(3.25) \qquad \partial_1(T_0(F))(\mathbf{x}) + \partial_2(T_0(F))(\mathbf{y}) &= \widetilde{T_0(F)}(\mathbf{z}) \\ &= \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z}) + \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z})\end{aligned}$$

for any $F \in C^1(I, \mathrm{Lip}(I))$ and $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z$.

4. The form of T_0

We characterize the surjective real linear isometry T_0 on $C^1(I, \mathrm{Lip}(I))$ in this section. First, we investigate the forms of $T_0(\mathbf{1})$, $T_0(1_I \otimes id_I)$, and $T_0(id_I \otimes 1_I)$ in order to determine the form of T_0.

For each $F \in C^1(I, \mathrm{Lip}(I))$, we put $G = T_0(F)$. We derive from (3.25) that

$$\partial_1(G)(\mathbf{x}) + \partial_2(G)(\mathbf{y}) = \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z}) + \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z}).$$

By (2.2) and (2.3), we can rewrite the above equality as

$$\begin{aligned}(4.1) \quad G_s(x) + G_s'(m)z + \xi D(G)_t(y) + D(G)_t'(n)w& \\ = \widetilde{T_0(F)}(\mathbf{z}) = \Delta_1(F)(\mathbf{z}) + \Delta_1'(F)(\mathbf{z}) &+ \Delta_2(F)(\mathbf{z}) + \Delta_2'(F)(\mathbf{z})\end{aligned}$$

for any $G = T(F)$ with $F \in C^1(I, \mathrm{Lip}(I))$, $\mathbf{x} = (s, x, m, z) \in X$, $\mathbf{y} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. The equality (4.1) is crucial to investigate the form of T_0 throughout this section.

The following proposition plays a fundamental role in this section. It is easy to prove it, and we thus omit it.

Proposition 1.21. *Let $l \in \mathbb{N}$ with $l \geq 2$ and $a_j \in \mathbb{C}$ with $j = 0, \cdots, l$. Suppose that $\left|a_0 + \sum_{j=1}^{l} a_j z_j\right| = 1$ for every $z_j \in \mathbb{T}$. There exists $j_0 \in \{0, \cdots, l\}$ such that $|a_{j_0}| = 1$ and $a_j = 0$ for every $j \in \{0, 1, \cdots, l\} \setminus \{j_0\}$.*

Next, we prove that $T_0(\mathbf{1})$ is a constant on I^2. Before proving it, we prepare the following notation.

Definition 1.22. Let $F \in C(I, \mathrm{Lip}(I))$ and $x \in I$. We define a continuous function F^x on I by $F^x(s) = F(s,x)$ for each $s \in I$.

Remark 1.23. Let $F \in C^1(I, \mathrm{Lip}(I))$. Then $D(F)$ is an element of $C(I, \mathrm{Lip}(I))$ with $\lim_{h\to 0} \|(F_{s+h} - F_s)/h - D(F)_s\|_L = 0$ by Definition 1.3. Because $|g(x)| \le \|g\|_L$ for $g \in \mathrm{Lip}(I)$ by (2.1), it follows that

$$\begin{aligned}\lim_{h\to 0}\left|\frac{F^x(s+h)-F^x(s)}{h} - D(F)^x(s)\right| &= \lim_{h\to 0}\left|\frac{F_{s+h}(x)-F_s(x)}{h} - D(F)_s(x)\right| \\ &\le \lim_{h\to 0}\left\|\frac{F_{s+h}-F_s}{h} - D(F)_s\right\|_L = 0\end{aligned}$$

for all $x, s \in I$. Hence, $(F^x)' = D(F)^x$ and $F^x \in C^1(I)$ for $F \in C^1(I, \mathrm{Lip}(I))$ and $x \in I$.

The next proposition gives a sufficient condition in order that $F \in C^1(I, \mathrm{Lip}(I))$ be a constant on I^2.

Proposition 1.24. *Suppose that $F \in C^1(I, \mathrm{Lip}(I))$ satisfies $(F^x)' = 0$ on I for any $x \in I$ and $(F_{s_1})' = 0$ on $\mathcal{M}$ for some $s_1 \in I$. Then F is a constant on I^2*

Proof. Take arbitrary $(s,x) \in I^2$. By assumption, we see that F^x and F_{s_1} are constant on I. We thus obtain $F(s,x) = F^x(s) = F^x(s_1) = F(s_1,x) = F_{s_1}(x) = F_{s_1}(0)$, which shows that F is a constant on I^2. □

Lemma 1.25. *Let $\lambda \in \{1, i\}$ and $\varepsilon_0 : Z \to \{\pm 1\}$ be the function from Lemma 1.12.*

(i) There exists $c_\lambda \in \mathbb{T}$ such that $T_0(\lambda\mathbf{1}) = c_\lambda \mathbf{1}$.

(ii) For all $\mathbf{z} \in Z$, $c_1 = [\alpha(\mathbf{z})]^{\varepsilon_0(\mathbf{z})}$. In particular, both α and ε_0 are constants with $\alpha \in \mathbb{T}$ and $\varepsilon_0 \in \{\pm 1\}$.

Proof. Let $\lambda \in \{1, i\}$ and we put $G_\lambda = T_0(\lambda\mathbf{1}) \in C^1(I, \mathrm{Lip}(I))$. Because $(\lambda\mathbf{1})'_{\phi_1(\mathbf{z})} = D(\lambda\mathbf{1}) = 0$ by (2.5), it follows from (3.15) and (3.16) that

$$\Delta_1(\lambda\mathbf{1})(\mathbf{z}) = [\alpha(\mathbf{z})\lambda]^{\varepsilon_0(\mathbf{z})} \quad \text{and} \quad \Delta_1'(\lambda\mathbf{1})(\mathbf{z}) = \Delta_2(\lambda\mathbf{1})(\mathbf{z}) = \Delta_2'(\lambda\mathbf{1})(\mathbf{z}) = 0$$

for $\mathbf{z} \in Z$. Applying (4.1) to $G = G_\lambda$, we deduce from the above equalities that

$$(G_\lambda)_s(x) + (G_\lambda)_s'(m)z + \xi D(G_\lambda)_t(y) + D(G_\lambda)_t'(n)w = [\alpha(\mathbf{z})\lambda]^{\varepsilon_0(\mathbf{z})} \tag{4.2}$$

for every $\mathbf{x} = (s,x,m,z) \in X$ and $\mathbf{y} = (\xi, (t,y,n,w)) \in Y$ with $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in Z$. We note that $[\alpha(\mathbf{z})\lambda]^{\varepsilon_0(\mathbf{z})} \in \mathbb{T}$ by Definition 1.11.

We shall prove (i). First, we show that $|G_\lambda| = 1$ on I^2. Fixing $(s,x) \in I^2$, we take $(t,y) \in I^2$ and $m,n \in \mathcal{M}$ arbitrarily. We derive from (4.2) that

$$|(G_\lambda)_s(x) + (G_\lambda)'_s(m)z + \xi D(G_\lambda)_t(y) + D(G_\lambda)'_t(n)w| = 1$$

for any $z, \xi, w \in \mathbb{T}$. Applying Proposition 1.21 to the last equality, we obtain $|(G_\lambda)_s(x)|$ is 0 or 1. Since $(s,x) \in I^2$ is chosen arbitrarily, the image $|G_\lambda|(I^2)$ of I^2 is contained in $\{0,1\}$. By the continuity of G_λ on the connected set I^2, we get $|G_\lambda| = 0$ on I^2 or $|G_\lambda| = 1$ on I^2. Because T_0 is injective with $T_0(0) = 0$, we obtain $G_\lambda = T_0(\lambda \mathbf{1}) \neq 0$. Hence,

$$|(G_\lambda)_s(x)| = 1. \tag{4.3}$$

Next, we prove that G_λ satisfies the assumptions of Proposition 1.24. We derive from (4.2) that

$$|(G_\lambda)_s(x) + (G_\lambda)'_s(m)z + \xi D(G_\lambda)_t(y) + D(G_\lambda)'_t(n)w| = 1$$

for $z, \xi, w \in \mathbb{T}$. If we apply Proposition 1.21 to the last equality, then we obtain $(G_\lambda)'_s(m) = D(G_\lambda)_t(y) = 0$, because $(G_\lambda)_s(x) \neq 0$ by (4.3). In particular, $D(G_\lambda)^y(t) = D(G_\lambda)_t(y) = 0$. Since $m \in \mathcal{M}$ and $(t,y) \in I^2$ are arbitrarily chosen, we have $(G_\lambda)'_s = 0$ on $\mathcal{M}$ and $D(G_\lambda)^y = 0$ on I for all $y \in I$. Having in mind that $((G_\lambda)^y)' = D(G_\lambda)^y$ by Remark 1.23, we have $((G_\lambda)^y)' = 0$ on I for any $y \in I$. Therefore, G_λ satisfies the assumptions of Proposition 1.24. We may apply Proposition 1.24 to get that G_λ is a constant on I^2, and hence, there exists $c_\lambda \in \mathbb{T}$ such that $G_\lambda = c_\lambda \mathbf{1}$. We have proved (i).

Now, we shall prove (ii). Because $G_\lambda = c_\lambda \mathbf{1}$, it follows from (4.2) that

$$c_\lambda = (G_\lambda)_s(x) = [\alpha(\mathbf{z})\lambda]^{\varepsilon_0(\mathbf{z})}$$

for each $\mathbf{x} = (s,x,m,z) \in X$ and $\mathbf{y} = (\xi,(t,y,n,w)) \in Y$ with $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in Z$, which implies that $c_1 = [\alpha(\mathbf{z})]^{\varepsilon_0(\mathbf{z})}$ and $c_i = [\alpha(\mathbf{z})i]^{\varepsilon_0(\mathbf{z})}$ for $\mathbf{z} \in Z$. Then we get

$$c_i = i\varepsilon_0(\mathbf{z})[\alpha(\mathbf{z})]^{\varepsilon_0(\mathbf{z})} = i\varepsilon_0(\mathbf{z})c_1$$

for all $\mathbf{z} \in Z$. This shows that ε_0 is a constant on Z, and hence $\varepsilon_0 = 1$ on Z or $\varepsilon_0 = -1$ on Z. It follows that $\alpha(\mathbf{z}) = [c_1]^{\varepsilon_0(\mathbf{z})}$ for all $\mathbf{z} \in Z$, and thus, α is a constant on Z. $\square$

By Lemma 1.25 (ii), we may and do write $\alpha(\mathbf{z}) = \alpha$ and $\varepsilon_0(\mathbf{z}) = \varepsilon_0$ for $\mathbf{z} \in Z$. Since $c_1 = [\alpha]^{\varepsilon_0}$ for $\mathbf{z} \in Z$, we deduce from (3.17) and (3.18) that

$$(4.4) \quad \Delta_1(f\otimes 1_I)(\mathbf{z}) = c_1[f(\phi_1(\mathbf{z}))]^{\varepsilon_0}, \quad \Delta_1'(f\otimes 1_I)(\mathbf{z}) = \Delta_2'(f\otimes 1_I)(\mathbf{z}) = 0,$$
$$\Delta_2(f\otimes 1_I)(\mathbf{z}) = [\alpha\psi_0(\mathbf{z})f'(\psi_1(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})},$$

$$(4.5) \quad \Delta_1(1_I\otimes g)(\mathbf{z}) = c_1[g(\phi_2(\mathbf{z}))]^{\varepsilon_0}, \quad \Delta_1'(1_I\otimes g)(\mathbf{z}) = [\alpha g'(\phi_3(\mathbf{z}))\cdot\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})},$$
$$\text{and} \quad \Delta_2(1_I\otimes g)(\mathbf{z}) = \Delta_2'(1_I\otimes g)(\mathbf{z}) = 0$$

for $f \in C^1(I)$ $g \in \mathrm{Lip}(I)$, and $\mathbf{z} \in Z$.

We put

$$G_f = T_0(f\otimes 1_I) \quad \text{and} \quad H_g = T_0(1_I\otimes g)$$

for $f \in C^1(I)$ and $g \in \mathrm{Lip}(I)$. We note that (4.1) is valid for $G = G_f$ and $G = H_g$. Entering (4.4) into (4.1), we obtain

$$(4.6) \quad (G_f)_s(x) + (G_f)_s'(m)z + \xi D(G_f)_t(y) + D(G_f)_t'(n)w$$
$$= \widetilde{G_f}(\mathbf{z}) = c_1[f(\phi_1(\mathbf{z}))]^{\varepsilon_0} + [\alpha\psi_0(\mathbf{z})f'(\psi_1(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})}$$

for $f \in C^1(I)$, $\mathbf{x} = (s,x,m,z) \in X$ and $\mathbf{y} = (\xi,(t,y,n,w)) \in Y$ with $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in Z$.

In the same way, if we substitute (4.5) into (4.1), then

$$(4.7) \quad (H_g)_s(x) + (H_g)_s'(m)z + \xi D(H_g)_t(y) + D(H_g)_t'(n)w$$
$$= \widetilde{H_g}(\mathbf{z}) = c_1[g(\phi_2(\mathbf{z}))]^{\varepsilon_0} + [\alpha g'(\phi_3(\mathbf{z}))\cdot\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}$$

for $g \in \mathrm{Lip}(I)$, $\mathbf{x} = (s,x,m,z) \in X$ and $\mathbf{y} = (\xi,(t,y,n,w)) \in Y$ with $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in Z$.

Before we investigate $T_0(id_I \otimes 1_I)$ and $T_0(1_I \otimes id_I)$, we shall prove that ϕ_j and ψ_j are invariant with respect to $z, \xi, w \in \mathbb{T}$ for each $j \in \{1,2,3\}$. The author in present paper refers to [**46**, Lemmas 4.1 and 4.2].

Lemma 1.26. *Let $(s,x),(t,y) \in I^2$ and $m,n \in \mathcal{M}$. Set $\mathbf{x}_z = (s,x,m,z) \in X$, $\mathbf{y}_{\xi,w} = (\xi,(t,y,n,w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z,\mathbf{y}_{\xi,w}) \in Z$ for each $z,\xi,w \in \mathbb{T}$. Then the values $\phi_1(\mathbf{z}_{z,\xi,w})$ and $\phi_2(\mathbf{z}_{z,\xi,w})$ are independent from $z,\xi,w \in \mathbb{T}$.*

Proof. Setting $\boldsymbol{w}_0 = \mathbf{z}_{1,1,1}$, $\boldsymbol{w}_1 = \mathbf{z}_{-1,1,1}$, $\boldsymbol{w}_2 = \mathbf{z}_{1,-1,1}$, and $\boldsymbol{w}_3 = \mathbf{z}_{1,1,-1}$, we shall prove that

$$\phi_1(\mathbf{z}_{z,\xi,w}) \in \{\phi_1(\boldsymbol{w}_j) : 0 \le j \le 3\}$$

for all $z, \xi, w \in \mathbb{T}$. Suppose, on the contrary, that

$$\phi_1(\boldsymbol{w}) \notin \{\phi_1(\boldsymbol{w}_j) : 0 \leq j \leq 3\}$$

for some $\boldsymbol{w} = \mathbf{z}_{z,\xi,w}$ with $z, \xi, w \in \mathbb{T}$. We can find $f \in C^1(I)$ such that

$$f(\phi_1(\boldsymbol{w})) = 1, \quad f(\phi_1(\boldsymbol{w}_j)) = 0, \quad \text{and} \quad f'(\psi_1(\boldsymbol{w})) = f'(\psi_1(\boldsymbol{w}_j)) = 0$$

for $j = 0, 1, 2, 3$. Set $G_f = T_0(f \otimes 1_I) \in C^1(I, \mathrm{Lip}(I))$. We derive from (4.6) with $\mathbf{z} = \boldsymbol{w}, \boldsymbol{w_j}$ that $\widetilde{G_f}(\boldsymbol{w}) = c_1$ and $\widetilde{G_f}(\boldsymbol{w}_j) = 0$ for $j = 0, 1, 2, 3$. By (4.6), we obtain

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{G_f}(\boldsymbol{w}_0) \\ \widetilde{G_f}(\boldsymbol{w}_1) \\ \widetilde{G_f}(\boldsymbol{w}_2) \\ \widetilde{G_f}(\boldsymbol{w}_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} (G_f)_s(x) \\ (G_f)'_s(m) \\ D(G_f)_t(y) \\ D(G_f)'_t(n) \end{pmatrix}.$$

Then we have $0 = \widetilde{G_f}(\boldsymbol{w}_0) - \widetilde{G_f}(\boldsymbol{w}_1) = 2(G_f)'_s(m)$, and hence $(G_f)'_s(m) = 0$. By the same argument, we see that $(G_f)_s(x) = D(G_f)_t(y) = D(G_f)'_t(n) = 0$. Entering these equalities into (4.6), we get

$$\widetilde{G_f}(\boldsymbol{w}) = (G_f)_s(x) + (G_f)'_s(m)z + \xi D(G_f)_t(y) + D(G_f)'_t(n)w = 0,$$

and thus, $\widetilde{G_f}(\boldsymbol{w}) = 0$. This contradicts $\widetilde{G_f}(\boldsymbol{w}) = c_1 \in \mathbb{T}$. Therefore, we have proved that $\phi_1(\mathbf{z}_{z,\xi,w}) \in \{\phi_1(\boldsymbol{w}_j) : 0 \leq j \leq 3\}$ for all $z, \xi, w \in \mathbb{T}$.

Since ϕ_1 is continuous, the image of $\mathbb{T}^3$ under the map $(z, \xi, w) \mapsto \phi_1(\mathbf{z}_{z,\xi,w})$ is connected. This implies that $\phi_1(\mathbf{z}_{z,\xi,w}) = \phi_1(\boldsymbol{w}_0)$ for all $z, \xi, w \in \mathbb{T}$, and consequently, the value $\phi_1(\mathbf{z}_{z,\xi,w})$ is independent from $z, \xi, w \in \mathbb{T}$.

The same arguments, applied to $1_I \otimes g$ for a suitable $g \in \mathrm{Lip}(I)$ instead of $f \otimes 1_I$, show that $\phi_2(\mathbf{z}_{z,\xi,w}) = \phi_2(\boldsymbol{w}_0)$ for all $z, \xi, w \in \mathbb{T}$. $\square$

LEMMA 1.27. *Let $(s, x), (t, y) \in I^2$ and $m, n \in \mathcal{M}$. Set $\mathbf{x}_z = (s, x, m, z) \in X$, $\mathbf{y}_{\xi,w} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z, \mathbf{y}_{\xi,w}) \in Z$ for each $z, \xi, w \in \mathbb{T}$. The values $\phi_3(\mathbf{z}_{z,\xi,w})$, $\psi_1(\mathbf{z}_{z,\xi,w})$, $\psi_2(\mathbf{z}_{z,\xi,w})$, and $\psi_3(\mathbf{z}_{z,\xi,w})$ are independent from $z, \xi, w \in \mathbb{T}$.*

PROOF. The same arguments as in Proof of Lemma 1.26 are applied to show the result. We thus omit its proof. $\square$

Our next aim is to determine the forms of $T_0(1_I \otimes id_I)$ and $T_0(id_I \otimes 1_I)$. First, we investigate $T_0(1_I \otimes id_I)$ in the following seven lemmas. The author refers to [**64**, p186–p187] for the main idea.

LEMMA 1.28. *Let $H_{id} = \overline{c_1}\, T_0(1_I \otimes id_I) \in C^1(I, \mathrm{Lip}(I))$. There exist $\mathbf{z}_1, \mathbf{z}_2 \in Z$ such that $\widetilde{H_{id}}(\mathbf{z}_1) = 2$ and $\widetilde{H_{id}}(\mathbf{z}_2) = -1$.*

PROOF. By Lemma 1.5, $\|\widetilde{H_{id}}\|_\infty = \|H_{id}\|_\Sigma$. Since T_0 is a real linear isometry, we have $\|\widetilde{H_{id}}\|_\infty = \|\overline{c_1}\, T_0(1_I \otimes id_I)\|_\Sigma = \|1_I \otimes id_I\|_\Sigma = 2$, and hence $\|\widetilde{H_{id}}\|_\infty = 2$. Then there exists $\mathbf{z}_1 \in Z$ such that $|\widetilde{H_{id}}(\mathbf{z}_1)| = 2$. Applying (4.7) to $H_g = c_1 H_{id} = T_0(1_I \otimes id_I)$, we obtain $\widetilde{c_1 H_{id}}(\mathbf{z}) = c_1 \phi_2(\mathbf{z}) + [\alpha(\mathbf{z})\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}$. That is,

$$\widetilde{H_{id}}(\mathbf{z}) = \phi_2(\mathbf{z}) + \overline{c_1}[\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})} \in I + \mathbb{T} \tag{4.8}$$

for any $\mathbf{z} \in Z$, where we have used $\phi_2(\mathbf{z}) \in I$. Thus, we see that $\widetilde{H_{id}}(\mathbf{z}_1) = 2$.

Setting $H = T_0(1_I \otimes (id_I - 1_I))$, we obtain

$$\|\widetilde{H}\|_\infty = \|T_0(1_I \otimes (id_I - 1_I))\|_\Sigma = \|1_I \otimes (id_I - 1_I)\|_\Sigma = 2,$$

that is, $\|\widetilde{H}\|_\infty = 2$, because T_0 is a norm preserving map. Then there exists $\mathbf{z}_2 \in Z$ such that $|\widetilde{H}(\mathbf{z}_2)| = 2$. By the real linearity of T_0, we get $H = T_0(1_I \otimes id_I) - T_0(1_I \otimes 1_I) = c_1 H_{id} - T_0(\mathbf{1})$. Since $T_0(\mathbf{1}) = c_1 \mathbf{1}$ by Lemma 1.25, we derive from (4.8) that

$$\overline{c_1}\, \widetilde{H}(\mathbf{z}_2) = \overline{c_1}\left(c_1 \widetilde{H_{id}}(\mathbf{z}_2) - \widetilde{T_0(\mathbf{1})}(\mathbf{z}_2)\right) = \widetilde{H_{id}}(\mathbf{z}_2) - 1 \in [-1, 0] + \mathbb{T}.$$

We deduce from $|\widetilde{H}(\mathbf{z}_2)| = 2$ that $\overline{c_1}\, \widetilde{H}(\mathbf{z}_2) = -2$. This implies that

$$\widetilde{H_{id}}(\mathbf{z}_2) = \overline{c_1}\, \widetilde{H}(\mathbf{z}_2) + 1 = -1.$$

The proof is complete. ☐

From Lemmas 1.29 through 1.35, we assume that $H_{id} = \overline{c_1} T_0(1_I \otimes id_I)$ as in Lemma 1.28.

LEMMA 1.29. *We put $a = \sup_{s \in I} \|(H_{id})'_s\|_\infty$ and $b = \sup_{t \in I}(\|D(H_{id})_t\|_\infty + \|D(H_{id})'_t\|_\infty)$. Then $a + b \leq 1$.*

PROOF. By the choice of a and b, there exist $s_0, t_0 \in I$ such that $\|(H_{id})'_{s_0}\|_\infty = a$ and $\|D(H_{id})_{t_0}\|_\infty + \|D(H_{id})'_{t_0}\|_\infty = b$. By Definition 1.3, we note that $(H_{id})'_{s_0}, D(H_{id})'_{t_0} \in C(\mathcal{M})$ and $D(H_{id})_{t_0} \in \mathrm{Lip}(I)$. Thus, there exist $m_0, n_0 \in \mathcal{M}$ and $y_0 \in I$ such that $|(H_{id})'_{s_0}(m_0)| = a$ and $|D(H_{id})_{t_0}(y_0)| + |D(H_{id})'_{t_0}(n_0)| = b$. It follows from the last two equalities that

$$|(H_{id})'_{s_0}(m_0)| + |D(H_{id})_{t_0}(y_0)| + |D(H_{id})'_{t_0}(n_0)| = a + b. \tag{4.9}$$

Now we set $\varepsilon = \operatorname{Im}\big((H_{id})_{s_0}(1)\big)/\big|\operatorname{Im}\big((H_{id})_{s_0}(1)\big)\big|$ if $\operatorname{Im}\big((H_{id})_{s_0}(1)\big) \neq 0$, and $\varepsilon = 1$ if $\operatorname{Im}\big((H_{id})_{s_0}(1)\big) = 0$. We can choose $z_0, \xi_0, w_0 \in \mathbb{T}$ so that

$$(H_{id})'_{s_0}(m_0)z_0 = i\varepsilon|(H_{id})'_{s_0}(m_0)|, \quad \xi_0 D(H_{id})_{t_0}(y_0) = i\varepsilon|D(H_{id})_{t_0}(y_0)|,$$
$$\text{and} \quad D(H_{id})'_{t_0}(n_0)w_0 = i\varepsilon|D(H_{id})'_{t_0}(n_0)|.$$

Having in mind that $i\varepsilon(|(H_{id})'_{s_0}(m_0)| + |D(H_{id})_{t_0}(y_0)| + |D(H_{id})'_{t_0}(n_0)|) = i\varepsilon(a+b)$ by (4.9), we deduce from the above equalities that

$$(H_{id})'_{s_0}(m_0)z_0 + \xi_0 D(H_{id})_{t_0}(y_0) + D(H_{id})'_{t_0}(n_0)w_0 = i\varepsilon(a+b). \tag{4.10}$$

We note that (4.7) is valid for $H_g = c_1 H_{id} = T_0(1_I \otimes id_I)$. We derive from (4.7), multiplied by $\overline{c_1}$, that

$$(H_{id})_{s_0}(1) + (H_{id})'_{s_0}(m_0)z_0 + \xi_0 D(H_{id})_{t_0}(y_0) + D(H_{id})'_{t_0}(n_0)w_0$$
$$= \phi_2(\mathbf{z}_0) + \overline{c_1}\,[\alpha\phi_4(\mathbf{z}_0)]^{\varepsilon_1(\mathbf{z}_0)}$$

for $\mathbf{x}_0 = (s_0, 1, m_0, z_0) \in X$ and $\mathbf{y}_0 = (\xi_0, (t_0, y_0, n_0, w_0)) \in Y$ with $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{y}_0)$, where we have used that $\phi_2(\mathbf{z}_0) \in I$. Entering (4.10) into the above equality, we get

$$(H_{id})_{s_0}(1) + i\varepsilon(a+b) = \phi_2(\mathbf{z}_0) + \overline{c_1}\,[\alpha\phi_4(\mathbf{z}_0)]^{\varepsilon_1(\mathbf{z}_0)} \in I + \mathbb{T}.$$

Having in mind that $\operatorname{Im}\big((H_{id})_{s_0}(1)\big) = \varepsilon\big|\operatorname{Im}\big((H_{id})_{s_0}(1)\big)\big|$, we take the imaginary part of the above equality, and then

$$\varepsilon\big|\operatorname{Im}\big((H_{id})_{s_0}(1)\big)\big| + \varepsilon(a+b) = \operatorname{Im}\big(\overline{c_1}\,[\alpha\phi_4(\mathbf{z}_0)]^{\varepsilon_1(\mathbf{z}_0)}\big).$$

Hence, we obtain $a + b \leq \big|\operatorname{Im}\big(\overline{c_1}\,[\alpha\phi_4(\mathbf{z}_0)]^{\varepsilon_1(\mathbf{z}_0)}\big)\big| \leq 1$. $\square$

Remark 1.30. Let $F \in C^1(I, \operatorname{Lip}(I))$ and $x_0 \in I$. Because $(F^{x_0})' = D(F)^{x_0}$ by Remark 1.23, it follows that

$$\|(F^{x_0})'\|_\infty = \|D(F)^{x_0}\|_\infty = \sup_{t\in I}|D(F)(t, x_0)| \leq \sup_{t\in I}(\sup_{x\in I}|D(F)(t,x)|) = \sup_{t\in I}\|D(F)_t\|_\infty,$$

and hence $\|(F^{x_0})'\|_\infty \leq \sup_{t\in I}\|D(F)_t\|_\infty$.

Lemma 1.31. *Let $\mathbf{z}_1, \mathbf{z}_2 \in Z$ be from Lemma 1.28 and a, b be from Lemma 1.29. We put $\mathbf{z}_j = (\mathbf{x}_j, \mathbf{y}_j)$, $\mathbf{x}_j = (s_j, x_j, m_j, z_j) \in X$, and $\mathbf{y}_j = (\xi_j, (t_j, y_j, n_j, w_j)) \in Y$ for $j = 1, 2$. Then $s_1, x_1 \in \{0, 1\}$, $s_2 = 1 - s_1$, $x_2 = 1 - x_1$, and*

$$|(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| = \|(H_{id})_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty = a + b = 1.$$

PROOF. By (2.4) with (2.2) and (2.3), we have

$$(H_{id})_{s_j}(x_j) + (H_{id})'_{s_j}(m_j)z_j + \xi_j D(H_{id})_{t_j}(y_j) + D(H_{id})'_{t_j}(n_j)w_j = \widetilde{H_{id}}(\mathbf{z}_j)$$

for $j = 1, 2$. Putting $d_j = (H_{id})'_{s_j}(m_j)z_j + \xi_j D(H_{id})_{t_j}(y_j) + D(H_{id})'_{t_j}(n_j)w_j$, we derive from the above equality that

$$(H_{id})_{s_j}(x_j) + d_j = \widetilde{H_{id}}(\mathbf{z}_j). \tag{4.11}$$

Since $\widetilde{H_{id}}(\mathbf{z}_1) = 2$ and $\widetilde{H_{id}}(\mathbf{z}_2) = -1$ by Lemma 1.28, it follows from (4.11) that $3 = |\widetilde{H_{id}}(\mathbf{z}_1) - \widetilde{H_{id}}(\mathbf{z}_2)| \leq |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| + |d_1| + |d_2|$, and hence,

$$3 \leq |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| + |d_1| + |d_2|. \tag{4.12}$$

Here, we note that $|d_j| \leq |(H_{id})'_{s_j}(m_j)| + |D(H_{id})_{t_j}(y_j)| + |D(H_{id})'_{t_j}(n_j)| \leq a + b$, and thus, $|d_j| \leq a + b$ by the choice of a and b. Having in mind that $a + b \leq 1$ by Lemma 1.29, it follows that

$$|d_j| \leq 1 \qquad (j = 1, 2). \tag{4.13}$$

We deduce from (4.12) that $3 \leq |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| + 2$, that is,

$$1 \leq |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)|. \tag{4.14}$$

On the other hand, by the mean value theorem, we get

$$\begin{aligned} |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| &\leq |(H_{id})_{s_1}(x_1) - (H_{id})_{s_1}(x_2)| + |(H_{id})_{s_1}(x_2) - (H_{id})_{s_2}(x_2)| \\ &\leq \|(H_{id})'_{s_1}\|_\infty \, |x_1 - x_2| + \|({H_{id}}^{x_2})'\|_\infty \, |s_1 - s_2| \\ &\leq \|(H_{id})'_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty, \end{aligned}$$

where we have used that $s_1, s_2, x_1, x_2 \in I$. Hence,

$$\begin{aligned} |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| &\leq \|(H_{id})'_{s_1}\|_\infty \, |x_1 - x_2| + \|({H_{id}}^{x_2})'\|_\infty \, |s_1 - s_2| \\ &\leq \|(H_{id})'_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty. \end{aligned} \tag{4.15}$$

Here, we note that $\|(H_{id})'_{s_1}\|_\infty \leq a$ and $\|({H_{id}}^{x_2})'\|_\infty \leq \sup_{t \in I} \|D(H_{id})_t\|_\infty \leq b$ by the choice of a and b with Remark 1.30. That is, $\|(H_{id})'_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty \leq a + b$. Having in mind that $a + b \leq 1$ by Lemma 1.29, we derive from (4.14) and (4.15) that

$$\begin{aligned} |(H_{id})_{s_1}(x_1) - (H_{id})_{s_2}(x_2)| &= \|(H_{id})'_{s_1}\|_\infty \, |x_1 - x_2| + \|({H_{id}}^{x_2})'\|_\infty \, |s_1 - s_2| \\ &= \|(H_{id})'_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty = a + b = 1, \end{aligned}$$

and hence, $|(H_{id})_{s_1}(x_1)-(H_{id})_{s_2}(x_2)| = \|(H_{id})'_{s_1}\|_\infty+\|({H_{id}}^{x_2})'\|_\infty = a+b = 1$. Because $s_j, x_j \in I$ for $j = 1, 2$, the identity $\|(H_{id})'_{s_1}\|_\infty\,|x_1-x_2|+\|({H_{id}}^{x_2})'\|_\infty\,|s_1-s_2| = \|(H_{id})'_{s_1}\|_\infty+\|({H_{id}}^{x_2})'\|_\infty$ shows that $|s_1 - s_2| = |x_1 - x_2| = 1$. This implies that $s_1, x_1 \in \{0, 1\}$ and $s_2 = 1 - s_1$, $x_2 = 1 - x_1$. $\square$

LEMMA 1.32. *Let $s_j, x_j \in I$ be from Lemma 1.31 for $j = 1, 2$. Then $(H_{id})_{s_1}(x_1) = 1$ and $(H_{id})_{s_2}(x_2) = 0$.*

PROOF. First, we show that $(H_{id})_{s_1}(x_1) = 1$. Having in mind that $({H_{id}}^{x_2})' = D(H_{id})^{x_2}$ by Remark 1.23, we deduce from Lemma 1.31 that

$$\|(H_{id})'_{s_1}\|_\infty + \|D(H_{id})^{x_2}\|_\infty = \|(H_{id})'_{s_1}\|_\infty + \|({H_{id}}^{x_2})'\|_\infty = 1.$$

Because $(H_{id})'_{s_1} \in C(\mathcal{M})$ and $D(H_{id})^{x_2} \in C(I)$, there exist $m_1 \in \mathcal{M}$ and $t_1 \in I$ such that $|(H_{id})'_{s_1}(m_1)| = \|(H_{id})'_{s_1}\|_\infty$ and $|D(H_{id})^{x_2}(t_1)| = \|D(H_{id})^{x_2}\|_\infty$. We derive from the above equalities that

$$|(H_{id})'_{s_1}(m_1)| + |D(H_{id})^{x_2}(t_1)| = 1. \tag{4.16}$$

By (2.1), we obtain

$$|(H_{id})_{s_1}(x_1)| + |(H_{id})'_{s_1}(m_1)| \le \|(H_{id})_{s_1}\|_L \quad \text{and}$$
$$|D(H_{id})^{x_2}(t_1)| = |D(H_{id})_{t_1}(x_2)| \le \|D(H_{id})_{t_1}\|_L.$$

We deduce from the above inequalities that

$$|(H_{id})_{s_1}(x_1)| + |(H_{id})'_{s_1}(m_1)| + |D(H_{id})^{x_2}(t_1)| \le \|(H_{id})_{s_1}\|_L + \|D(H_{id})_{t_1}\|_L.$$

Combining (4.16) with the last inequality, we have

$$|(H_{id})_{s_1}(x_1)| + 1 \le \|(H_{id})_{s_1}\|_L + \|D(H_{id})_{t_1}\|_L.$$

By the definition of $\|\cdot\|_\Sigma$, we get $\|(H_{id})_{s_1}\|_L + \|D(H_{id})_{t_1}\|_L \le \|H_{id}\|_\Sigma$. It follows from the above inequality that $|(H_{id})_{s_1}(x_1)| + 1 \le \|H_{id}\|_\Sigma$. Because T_0 is a real linear isometry, we have $\|H_{id}\|_\Sigma = \|\overline{c_1}T_0(1_I \otimes id_I)\|_\Sigma = \|1_I \otimes id_I\|_\Sigma = 2$. This shows $|(H_{id})_{s_1}(x_1)| + 1 \le 2$, and hence, $|(H_{id})_{s_1}(x_1)| \le 1$.

Let $\mathbf{z}_j \in Z$ be from Lemma 1.28 and d_j be as in the proof of Lemma 1.31 for $j = 1, 2$. Since $\widetilde{H_{id}}(\mathbf{z}_1) = 2$ by Lemma 1.28, it follows from (4.11) that $(H_{id})_{s_1}(x_1) + d_1 = 2$. This implies that $|(H_{id})_{s_1}(x_1) - 2| \le 1$ by (4.13). We infer from $|(H_{id})_{s_1}(x_1)| \le 1$ and $|(H_{id})_{s_1}(x_1) - 2| \le 1$ that $(H_{id})_{s_1}(x_1) = 1$.

Next, we prove that $(H_{id})_{s_2}(x_2) = 0$. Because $\widetilde{H_{id}}(\mathbf{z}_2) = -1$, we derive from (4.11) and (4.13) that $|(H_{id})_{s_2}(x_2) + 1| \leq 1$. By Lemma 1.31 with $(H_{id})_{s_1}(x_1) = 1$, we obtain

$$|(H_{id})_{s_2}(x_2) - 1| = |(H_{id})_{s_2}(x_2) - (H_{id})_{s_1}(x_1)| = 1,$$

and thus $|(H_{id})_{s_2}(x_2) - 1| = 1$. It follows from $|(H_{id})_{s_2}(x_2) + 1| \leq 1$ that $(H_{id})_{s_2}(x_2) = 0$. □

LEMMA 1.33. *Let $s_2, x_2 \in I$ be from Lemma 1.31. For any $s, x \in I$,*

$$(H_{id})_s(x) = a|x_2 - x| + b|s_2 - s|.$$

PROOF. Take $s, x \in I$ arbitrarily. Because $(H_{id})_s(x) = (H_{id})^x(s)$, we obtain

$$\begin{aligned}|(H_{id})_s(x) - (H_{id})_{s_j}(x_j)| &\leq |(H_{id})_s(x) - (H_{id})_s(x_j)| + |(H_{id})_s(x_j) - (H_{id})_{s_j}(x_j)| \\ &\leq \|(H_{id})_s'\|_\infty \, |x_j - x| + \|({H_{id}}^{x_j})'\|_\infty \, |s_j - s|\end{aligned}$$

by the mean value theorem, and hence,

$$|(H_{id})_s(x) - (H_{id})_{s_j}(x_j)| \leq \|(H_{id})_s'\|_\infty \, |x_j - x| + \|({H_{id}}^{x_j})'\|_\infty \, |s_j - s|.$$

We get $\|(H_{id})_s'\|_\infty \leq a$ and $\|({H_{id}}^{x_j})'\|_\infty \leq \sup_{t \in I} \|D(H_{id})_t\|_\infty \leq b$ by the choice of a and b with Remark 1.30. We derive from the above inequalities that

$$|(H_{id})_s(x) - (H_{id})_{s_j}(x_j)| \leq a|x_j - x| + b|s_j - s|$$

for $j = 1, 2$. By Lemma 1.32, we get $(H_{id})_{s_1}(x_1) = 1$ and $(H_{id})_{s_2}(x_2) = 0$. Letting $j = 1, 2$ in the above inequality, we have the following identities:

$$|(H_{id})_s(x) - 1| \leq a|x_1 - x| + b|s_1 - s|, \tag{4.17}$$

$$|(H_{id})_s(x)| \leq a|x_2 - x| + b|s_2 - s|. \tag{4.18}$$

We put $r = a|x_2 - x| + b|s_2 - s|$, and then $|(H_{id})_s(x)| \leq r$ by (4.18). Because $x_1 \in \{0, 1\}$ and $x_2 = 1 - x_1$ by Lemma 1.31, it follows that $|x_1 - x| + |x_2 - x| = 1$. By the same reasoning, $|s_1 - s| + |s_2 - s| = 1$. Adding the right sides of (4.17) and (4.18), we get

$$(a|x_1 - x| + b|s_1 - s|) + r = a(|x_1 - x| + |x_2 - x|) + b(|s_1 - s| + |s_2 - s|) = a + b,$$

and then $(a|x_1 - x| + b|s_1 - s|) + r = a + b$. We note that $a + b = 1$ by Lemma 1.31. Hence, we obtain $a|x_1 - x| + b|s_1 - s| = 1 - r$. We infer from (4.17) that $|(H_{id})_s(x) - 1| \leq 1 - r$. We deduce from $|(H_{id})_s(x)| \leq r$ that $(H_{id})_s(x) = r = a|x_2 - x| + b|s_2 - s|$. □

LEMMA 1.34. *There exists $\tau_2 \in \{id_I, 1_I - id_I\}$ such that $H_{id} = 1_I \otimes \tau_2$ or $H_{id} = \tau_2 \otimes 1_I$.*

Proof. Let $s_2, x_2 \in \{0,1\}$ be from Lemma 1.31. By Lemma 1.33, we obtain

$$(H_{id})_s(x) = a|x_2 - x| + b|s_2 - s| \tag{4.19}$$

for any $s, x \in I$. Because $x_2 \in \{0,1\}$, there exists $\delta_1 \in \{\pm 1\}$ such that

$$(H_{id})'_s(m) = \delta_1 a \quad (s \in I, m \in \mathcal{M}). \tag{4.20}$$

By Definition 1.3, we see that

$$D(H_{id})_t(y) = \delta_2 b \quad (t, y \in I) \tag{4.21}$$

for some $\delta_2 \in \{\pm 1\}$. This shows that $D(H_{id})'_t = 0$ on $\mathcal{M}$ for any $t \in I$. Let $m \in \mathcal{M}$ and set $\mathbf{x}_z = (s_2, x_2, m, \delta_1 z) \in X$, $\mathbf{y}_\xi = (\delta_2 \xi, (s_2, x_2, m, 1)) \in Y$, and $\mathbf{z}_{z,\xi} = (\mathbf{x}_z, \mathbf{y}_\xi) \in Z$ for each $z, \xi \in \mathbb{T}$. Applying (4.7) to $H_g = T_0(1_I \otimes id_I) = c_1 H_{id}$, we get

$$\begin{aligned}(c_1 H_{id})_{s_2}(x_2) + (c_1 H_{id})'_{s_2}(m)\delta_1 z + \delta_2 \xi D(c_1 H_{id})_{s_2}(x_2) + D(c_1 H_{id})'_{s_2}(m) \\ = c_1 \phi_2(\mathbf{z}_{z,\xi}) + [\alpha \phi_4(\mathbf{z}_{z,\xi})]^{\varepsilon_1(\mathbf{z}_{z,\xi})}.\end{aligned} \tag{4.22}$$

Note that $(H_{id})_{s_2}(x_2) = 0$ by (4.19) and $D(H_{id})'_{s_2}(m) = 0$. Substituting (4.20) and (4.21) into (4.22), we obtain

$$c_1 a z + \xi c_1 b = c_1 \phi_2(\mathbf{z}_{z,\xi}) + [\alpha \phi_4(\mathbf{z}_{z,\xi})]^{\varepsilon_1(\mathbf{z}_{z,\xi})} \tag{4.23}$$

for any $z, \xi \in \mathbb{T}$. We shall write $\mathbf{z}_{i,i} = \mathbf{z}_{(i)}$ in short. Letting $z = \xi = i$ in (4.23), we obtain

$$i = \phi_2(\mathbf{z}_{(i)}) + \overline{c_1}\,[\alpha \phi_4(\mathbf{z}_{(i)})]^{\varepsilon_1(\mathbf{z}_{(i)})} \in I + \mathbb{T},$$

since $a + b = 1$ by Lemma 1.31. If we take the imaginary part of the above equality, we get $1 = \operatorname{Im}(\overline{c_1}\,[\alpha \phi_4(\mathbf{z}_{(i)})]^{\varepsilon_1(\mathbf{z}_{(i)})})$. Consequently $\overline{c_1}[\alpha \phi_4(\mathbf{z}_{(i)})]^{\varepsilon_1(\mathbf{z}_{(i)})} = i$, and then $\phi_2(\mathbf{z}_{(i)}) = 0$. Because $\phi_2(\mathbf{z}_{z,\xi}) = \phi_2(\mathbf{z}_{(i)}) = 0$ by Lemma 1.26, we derive from (4.23) that

$$c_1(az + b\xi) = [\alpha \phi_4(\mathbf{z}_{z,\xi})]^{\varepsilon_1(\mathbf{z}_{z,\xi})},$$

which shows that $|az + b\xi| = 1$ for all $z, \xi \in \mathbb{T}$. We conclude that $a = 0$ or $b = 0$ by Proposition 1.21.

It follows from $a + b = 1$ that $(H_{id})_s(x) = |s_2 - s|$ or $(H_{id})_s(x) = |x_2 - x|$ by (4.19). If $(H_{id})_s(x) = |s_2 - s|$, we obtain $(H_{id})_s(x) = (\tau_2 \otimes 1_I)_s(x)$ for some $\tau_2 \in \{id_I, 1_I - id_I\}$ because $s_2 \in \{0,1\}$. By the same reasoning, there exists $\tau_2 \in \{id_I, 1_I - id_I\}$ such that $(H_{id})_s(x) = (1_I \otimes \tau_2)_s(x)$ provided that $(H_{id})_s(x) = |x_2 - x|$. $\square$

Next, we determine the form of $T_0(1_I \otimes id_I)$. Note that τ_2 in Lemma 1.34 satisfies that $\tau_2' = 1_I$ or -1_I, since $\tau_2 \in \{id_I, 1_I - id_I\}$.

LEMMA 1.35. *There exists $\tau_2 \in \{id_I, 1_I - id_I\}$ such that $H_{id} = 1_I \otimes \tau_2$.*

PROOF. Let $(s,x),(t,y) \in I^2$ and $m,n \in \mathcal{M}$, and set $\mathbf{x}_z = (s,x,m,z) \in X$, $\mathbf{y}_{\xi,w} = (\xi,(t,y,n,w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z, \mathbf{y}_{\xi,w}) \in Z$ for each $z,\xi,w \in \mathbb{T}$. By Lemma 1.34, there exists $\tau_2 \in \{id_I, 1_I - id_I\}$ such that $H_{id} = 1_I \otimes \tau_2$ or $H_{id} = \tau_2 \otimes 1_I$. Arguing by contradiction, we suppose that $H_{id} = \tau_2 \otimes 1_I$, and then, $(c_1 H_{id})_s(x) = c_1\tau_2(s)$. It follows from (2.5) that

$$(c_1H_{id})_s'(m) = D(c_1H_{id})_t'(n) = 0 \quad \text{and} \quad D(c_1H_{id})_t(y) = c_1\tau_2'(t).$$

Since (4.7) is valid for $c_1H_{id} = T_0(1_I \otimes id_I)$, we get

$$c_1\tau_2(s) + \xi c_1\tau_2'(t) = c_1\phi_2(\mathbf{z}_{z,\xi,w}) + [\alpha\phi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_1(\mathbf{z}_{z,\xi,w})} \tag{4.24}$$

for all $z,\xi,w \in \mathbb{T}$. Noting that $\tau_2' = 1_I$ or -1_I, we set $\xi = \tau_2'(t)i$ and $z = w = i$ in the last equality. Then we obtain

$$\tau_2(s) + i = \phi_2(\mathbf{z}_{i,\tau_2'(t)i,i}) + \overline{c_1}\,[\alpha\phi_4(\mathbf{z}_{i,\tau_2'(t)i,i})]^{\varepsilon_1(\mathbf{z}_{i,\tau_2'(t)i,i})} \in I + \mathbb{T}.$$

Taking the imaginary part of the last equality, we get $1 = \mathrm{Im}\left(\overline{c_1}\,[\alpha\phi_4(\mathbf{z}_{i,\tau_2'(t)i,i})]^{\varepsilon_1(\mathbf{z}_{i,\tau_2'(t)i,i})}\right)$, where we have used $\tau_2(s) \in I$. Consequently, we have $i = \overline{c_1}\,[\alpha\phi_4(\mathbf{z}_{i,\tau_2'(t)i,i})]^{\varepsilon_1(\mathbf{z}_{i,\tau_2'(t)i,i})}$. We deduce from the above equality that $\tau_2(s) = \phi_2(\mathbf{z}_{i,\tau_2'(t)i,i}) = \phi_2(\mathbf{z}_{z,\xi,w})$ for any $z,\xi,w \in \mathbb{T}$ by Lemma 1.26. It follows from (4.24) that

$$\tau_2(s) = \phi_2(\mathbf{z}_{z,\xi,w}) \quad \text{and} \quad \xi c_1\tau_2'(t) = [\alpha\phi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_1(\mathbf{z}_{z,\xi,w})} \tag{4.25}$$

for all $z,\xi,w \in \mathbb{T}$.

Take a real valued function $g_0 \in \mathrm{Lip}(I) \setminus C^1(I)$, and set $H_{g_0} = T_0(1_I \otimes g_0)$. Having in mind that g_0 is a real valued function, it follows from (4.7) and (4.25) that

$$(H_{g_0})_s(x) + (H_{g_0})_s'(m)z + \xi D(H_{g_0})_t(y) + D(H_{g_0})_t'(n)w = c_1g_0(\tau_2(s)) + c_1\xi\tau_2'(t)g_0'(\phi_3(\mathbf{z}_{z,\xi,w}))$$

for every $z,\xi,w \in \mathbb{T}$. Here, we note that $\phi_3(\mathbf{z}_{z,\xi,w})$ is independent from $z,\xi,w \in \mathbb{T}$ by Lemma 1.27. Since $z,\xi,w \in \mathbb{T}$ are arbitrarily chosen, we get $(H_{g_0})_s(x) = c_1g_0(\tau_2(s))$. This shows that $g_0(s) = g_0(\tau_2(\tau_2(s))) = \overline{c_1}\,(H_{g_0})^x(\tau_2(s))$. Because $\tau_2 \in \{id_I, 1_I - id_I\}$, we see that $g_0 \in C^1(I)$ by Remark 1.23. This contradicts the choice of $g_0 \in \mathrm{Lip}(I) \setminus C^1(I)$. Hence, we must have $\overline{c_1}\,T_0(1_I \otimes id_I) = H_{id} = 1_I \otimes \tau_2$. $\square$

LEMMA 1.36. *There exists $\tau_1 \in \{id_I, 1_I - id_I\}$ such that $\overline{c_1}\,T_0(id_I \otimes 1_I) = \tau_1 \otimes 1_I$.*

Proof. The same arguments in Lemmas from 1.28 to 1.34, applied to $\overline{c_1}\, T_0(id_I \otimes 1_I)$, there exists $\tau_1 \in \{id_I, 1_I - id_I\}$ such that $\overline{c_1}\, T_0(id_I \otimes 1_I) = \tau_1 \otimes 1_I$ or $\overline{c_1}\, T_0(id_I \otimes 1_I) = 1_I \otimes \tau_1$. Suppose that $\overline{c_1}\, T_0(id_I \otimes 1_I) = 1_I \otimes \tau_1$. By Lemma 1.35, $\overline{c_1}\, T_0(1_I \otimes id_I) = H_{id} = 1_I \otimes \tau_2$. Since T_0 is injective, we have $1_I \otimes \tau_1 \neq 1_I \otimes \tau_2$, that is, $\tau_1 \neq \tau_2$. Having in mind that $\tau_j \in \{id_I, 1_I - id_I\}$ for $j = 1, 2$, we conclude that $\tau_1 = 1_I - \tau_2$. Since $T_0(\mathbf{1}) = c_1 \mathbf{1}$ by Lemma 1.25, we derive from the real linearity of T_0 that

$$\begin{aligned}\overline{c_1}\, T_0(id_I \otimes 1_I) &= 1_I \otimes \tau_1 = 1_I \otimes (1_I - \tau_2) = \mathbf{1} - (1_I \otimes \tau_2) \\ &= \overline{c_1}\, T_0(\mathbf{1}) - \overline{c_1}\, T_0(1_I \otimes id_I) = \overline{c_1}\, T_0(\mathbf{1} - (1_I \otimes id_I)),\end{aligned}$$

and thus, $\overline{c_1}\, T_0(id_I \otimes 1_I) = \overline{c_1}\, T_0(\mathbf{1} - (1_I \otimes id_I))$. We see that $id_I \otimes 1_I = \mathbf{1} - (1_I \otimes id_I)$ because T_0 is injective. Evaluating the last identity at $(0, 0)$, we get

$$0 = (id_I \otimes 1_I)(0, 0) = (\mathbf{1} - (1_I \otimes id_I))(0, 0) = 1.$$

This is a contradiction. Therefore, we conclude that $\overline{c_1}\, T_0(id_I \otimes 1_I) = \tau_1 \otimes 1_I$. $\square$

From Lemmas 1.35 and 1.36, we can determine the forms of four maps ϕ_1, ϕ_2, ϕ_4, and ψ_0.

Lemma 1.37. *Let $\mathbf{x} = (s, x, m, z) \in X$, $\mathbf{y} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z$. Then the following identities hold:*

$$\phi_1(\mathbf{z}) = \tau_1(s), \quad \phi_2(\mathbf{z}) = \tau_2(x) \quad \text{and} \quad [\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})} = c_1 \tau_2'(m) z, \quad [\alpha\psi_0(\mathbf{z})]^{\varepsilon_2(\mathbf{z})} = \xi c_1 \tau_1'(t).$$

Proof. Set $H_{id} = \overline{c_1}\, T_0(1_I \otimes id_I)$. By Lemma 1.35, there exists $\tau_2 \in \{id_I, 1_I - id_I\}$ such that $H_{id} = 1_I \otimes \tau_2$. Applying (4.7) to $H_g = c_1 H_{id} = T_0(1_I \otimes id_I)$, we get

$$(c_1 H_{id})_s(x) + (c_1 H_{id})_s'(m) z + \xi D(c_1 H_{id})_t(y) + D(c_1 H_{id})_t'(n) w = c_1 \phi_2(\mathbf{z}) + [\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}. \tag{4.26}$$

We note that $H_{id} = 1_I \otimes \tau_2$, and then $(c_1 H_{id})_s = c_1 \tau_2$. We deduce from (2.5) that $(c_1 H_{id})_s' = c_1 \tau_2'$ and $D(c_1 H_{id})_t = D(c_1 H_{id})_t' = 0$. Entering these four identities into (4.26), we get

$$\tau_2(x) + \tau_2'(m) z = \phi_2(\mathbf{z}) + \overline{c_1}\, [\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}.$$

By the same argument as in Proof of Lemma 1.35, we have $\tau_2(x) = \phi_2(\mathbf{z})$ and $\tau_2'(m) z = \overline{c_1}\, [\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})}$.

Applying the same argument to $\overline{c_1}\, T_0(id_I \otimes 1_I)$ with Lemma 1.36, we see that $\tau_1(s) = \phi_1(\mathbf{z})$ and $[\alpha\psi_0(\mathbf{z})]^{\varepsilon_2(\mathbf{z})} = \xi c_1 \tau_1'(t)$. $\square$

Our next purpose is to determine the form of ψ_4. In order to prove it, we prepare two lemmas. First, we characterize $T_0(f \otimes 1_I)$ for $f \in C^1(I)$.

LEMMA 1.38. *Let $f \in C^1(I)$. For any $s, x \in I$, $T_0(f \otimes 1_I)_s(x) = c_1[f(\tau_1(s))]^{\varepsilon_0}$.*

PROOF. Let $f \in C^1(I)$ and we take $(s, x), (t, y) \in I^2$ and $m, n \in \mathcal{M}$ arbitrarily. Set $\mathbf{x}_z = (s, x, m, z) \in X$, $\mathbf{y}_{\xi,w} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z, \mathbf{y}_{\xi,w}) \in Z$ for each $z, \xi, w \in \mathbb{T}$. We put $G_f = T_0(f \otimes 1_I)$. We note that $\phi_1(\mathbf{z}) = \tau_1(s)$ and $[\alpha\psi_0(\mathbf{z})]^{\varepsilon_2(\mathbf{z})} = \xi c_1 \tau_1'(t)$ by Lemma 1.37. Applying (4.6) to $G_f = T_0(f \otimes 1_I)$, we derive from these two equalities that

$$(4.27) \quad (G_f)_s(x) + (G_f)_s'(m)z + \xi D(G_f)_t(y) + D(G_f)_t'(n)w \\ = c_1[f(\tau_1(s))]^{\varepsilon_0} + \xi c_1 \tau_1'(t)[f'(\psi_1(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_2(\mathbf{z}_{z,\xi,w})}$$

for any $z, \xi, w \in \mathbb{T}$. Because $\varepsilon_2 : Z \to \{\pm 1\}$ is a continuous function, the image of $\mathbb{T}^3$ under the map $(z, \xi, w) \mapsto \varepsilon_2(\mathbf{z}_{z,\xi,w})$ is connected. This implies that $\varepsilon_2(\mathbf{z}_{z,\xi,w}) = \varepsilon_2(\mathbf{z}_{1,1,1})$ for all $z, \xi, w \in \mathbb{T}$. Also, $\psi_1(\mathbf{z}_{z,\xi,w}) = \psi_1(\mathbf{z}_{1,1,1})$ by Lemma 1.27. Since $z, \xi, w \in \mathbb{T}$ are arbitrary, we conclude that $T_0(f \otimes 1_I)_s(x) = (G_f)_s(x) = c_1[f(\tau_1(s))]^{\varepsilon_0}$. $\square$

Next, we give a sufficient condition for $F \in C^1(I, \mathrm{Lip}(I))$ in order that $F = f \otimes 1_I$ for some $f \in C^1(I)$ in the next lemma.

LEMMA 1.39. *Let $F \in C^1(I, \mathrm{Lip}(I))$. Suppose that $D(F)_t'(n) = 0$ for all $t \in I$ and $n \in \mathcal{M}$ and that $F_{s_1}' = 0$ on $\mathcal{M}$ for some $s_1 \in I$. Then there exists $f \in C^1(I)$ such that $F = f \otimes 1_I$.*

PROOF. We define $f_0(t) = D(F)_t(0)$ for each $t \in I$, and then $f_0 \in C(I)$. By assumption, we see that $D(F)_t$ is a constant on I for every $t \in I$. It follows that

$$D(F)_t(y) = D(F)_t(0) = f_0(t) = (f_0 \otimes 1_I)_t(y)$$

for any $t, y \in I$, that is, $D(F) = f_0 \otimes 1_I$. We define $f_1 : I \to \mathbb{C}$ by $f_1(s) = \int_0^s f_0(u)du$ for each $s \in I$, then $f_1 \in C^1(I)$ with $f_1' = f_0$. Setting $G = F - (f_1 \otimes 1_I)$, we show that G satisfies the assumptions of Proposition 1.24.

We take $x \in I$ arbitrarily. Then

$$(G^x)' = (F^x)' - ((f_1 \otimes 1_I)^x)'.$$

Since $(F^x)' = D(F)^x$ by Remark 1.23, we have $(F^x)' = D(F)^x = (f_0 \otimes 1_I)^x = f_0$, and thus, $(F^x)' = f_0$. Having in mind that $((f_1 \otimes 1_I)^x)' = f_1' = f_0$, we deduce from the above equality that $(G^x)' = 0$. Since $x \in I$ is arbitrary, it follows that $(G^x)' = 0$ for any $x \in I$. Because

$F'_{s_1} = 0$ on $\mathcal{M}$ by assumption, we get $(F - f_1 \otimes 1_I)'_{s_1} = F'_{s_1} - f_1(s)1'_I = 0$, which shows that $(G_{s_1})' = 0$ on $\mathcal{M}$ for some $s_1 \in I$. We have proved that G satisfies the assumptions of Proposition 1.24. Hence, $F - (f_1 \otimes 1_I) = G = c_{f_1}(1_I \otimes 1_I)$ for some $c_{f_1} \in \mathbb{C}$. Putting $f = f_1 + c_{f_1}1_I$, we get $F = f_1 \otimes 1_I + c_{f_1}(1_I \otimes 1_I) = f \otimes 1_I$, and hence, $F = f \otimes 1_I$. $\square$

Let $F \in C^1(I, \mathrm{Lip}(I))$ and $\mathbf{x} = (s, x, m, z) \in X$, $\mathbf{y} = (\xi, (t, y, n, w)) \in Y$ with $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z$. Since $[\alpha]^{\varepsilon_0(\mathbf{z})} = c_1$, $\phi_1(\mathbf{z}) = \tau_1(s)$, $\phi_2(\mathbf{z}) = \tau_2(x)$, and $[\alpha\phi_4(\mathbf{z})]^{\varepsilon_1(\mathbf{z})} = c_1\tau_2'(m)z$ by Lemmas 1.25 and 1.37, we derive from (3.15) that

$$\Delta_1(F)(\mathbf{z}) = c_1[F_{\tau_1(s)}(\tau_2(x))]^{\varepsilon_0}, \quad \Delta_1'(F)(\mathbf{z}) = c_1\tau_2'(m)[F'_{\tau_1(s)}(\phi_3(\mathbf{z}))]^{\varepsilon_1(\mathbf{z})}z. \tag{4.28}$$

In the same way, we deduce from (3.16) that

$$\begin{aligned}\Delta_2(F)(\mathbf{z}) &= \xi c_1\tau_1'(t)[D(F)_{\psi_1(\mathbf{z})}(\psi_2(\mathbf{z}))]^{\varepsilon_2(\mathbf{z})}, \\ \Delta_2'(F)(\mathbf{z}) &= [\alpha D(F)'_{\psi_1(\mathbf{z})}(\psi_3(\mathbf{z})) \cdot \psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})},\end{aligned} \tag{4.29}$$

because $[\alpha(\mathbf{z})\psi_0(\mathbf{z})]^{\varepsilon_2(\mathbf{z})} = \xi c_1\tau_1'(t)$ by Lemma 1.37.

From the previous two lemmas, we can determine the form of $T_0(id_I \otimes id_I)$.

Lemma 1.40. *For any $s, x \in I$, we have $T_0(id_I \otimes id_I)_s(x) = c_1(\tau_1 \otimes \tau_2)_s(x)$.*

Proof. Fix arbitrary $s \in I$, and set $F = (id_I - \tau_1(s)1_I) \otimes id_I$ and $G = T_0(F)$. We first prove that $G_s = 0$ on I. We take $t, x, y \in I$ and $m, n \in \mathcal{M}$ arbitrarily. We put $\mathbf{x}_z = (s, x, m, z) \in X$, $\mathbf{y}_{\xi,w} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z, \mathbf{y}_{\xi,w}) \in Z$ for each $z, \xi, w \in \mathbb{T}$. We note that $F'_{\tau_1(s)} = 0$, $D(F) = 1_I \otimes id_I$, and $D(F)'_{\psi_1(\mathbf{z}_{z,\xi,w})} = 1_I$ by (2.5). We deduce from (4.28) and (4.29) that

$$\begin{aligned}\Delta_1(F)(\mathbf{z}_{z,\xi,w}) = \Delta_1'(F)(\mathbf{z}_{z,\xi,w}) = 0, \quad \Delta_2(F)(\mathbf{z}_{z,\xi,w}) = \xi c_1\tau_1'(t)\psi_2(\mathbf{z}_{z,\xi,w}), \\ \text{and} \quad \Delta_2'(F)(\mathbf{z}_{z,\xi,w}) = [\alpha\psi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})},\end{aligned}$$

where we have used $\psi_2(\mathbf{z}_{z,\xi,w}) \in I$. It follows from the above four equalities that

$$\begin{aligned}\Delta_1(F)(\mathbf{z}_{z,\xi,w}) + \Delta_1(F)'(\mathbf{z}_{z,\xi,w}) + \Delta_2(F)(\mathbf{z}_{z,\xi,w}) + \Delta_2(F)'(\mathbf{z}_{z,\xi,w}) \\ = \xi c_1\tau_1'(t)\psi_2(\mathbf{z}_{z,\xi,w}) + [\alpha\psi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})}.\end{aligned}$$

Applying (4.1) to $G = T_0(F)$, we infer from the above equality that

$$G_s(x) + G_s'(m)z + \xi D(G)_t(y) + D(G)_t'(n)w = \xi c_1\tau_1'(t)\psi_2(\mathbf{z}_{z,\xi,w}) + [\alpha\psi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})}.$$

Suppose that $G_s(x_0) \neq 0$ for some $x_0 \in I$. We deduce from the last equality that

$$|G_s(x_0) + G_s'(m)z + \xi\big(D(G)_t(y) - c_1\tau_1'(t)\psi_2(\mathbf{z}_{z,\xi,w})\big) + D(G)_t'(n)w| = 1$$

for any $z, \xi, w \in \mathbb{T}$. Applying Proposition 1.21 to the last equality, we conclude that $G_s'(m) = D(G)_t'(n) = 0$, because $G_s(x_0) \neq 0$. Since $m, n \in \mathcal{M}$ and $t \in I$ are arbitrarily chosen, we see that G satisfies the assumptions of Lemma 1.39. Hence, there exists $f_1 \in C^1(I)$ such that $G = f_1 \otimes 1_I$. We define $f_2(t) = [\overline{c_1} f_1(\tau_1(t))]^{\varepsilon_0}$ for each $t \in I$ and then $f_2 \in C^1(I)$, because $\tau_1 \in \{id_I, 1_I - id_I\}$. Having in mind that $\tau_1(\tau_1(t)) = t$ for $t \in I$, it follows from Lemma 1.38 and $G = f_1 \otimes 1_I$ that

$$T_0(f_2 \otimes 1_I)_t(y) = c_1[f_2(\tau_1(t))]^{\varepsilon_0} = f_1(t) = (f_1 \otimes 1_I)_t(y) = G_t(y)$$

for all $t, y \in I$, that is, $T_0(f_2 \otimes 1_I) = G$. Note that $G = T_0(F)$. Since T_0 is injective, we have $f_2 \otimes 1_I = F$. By the choice of F, we see that f_2 is a nonzero function. Thus we can choose $s_0 \in I$ satisfying $f_2(s_0) \neq 0$. Then we obtain

$$0 \neq f_2(s_0) = (f_2 \otimes 1_I)(s_0, 0) = F(s_0, 0) = 0,$$

which is impossible. Hence, we must have $G_s = 0$ on I.

By the real linearity of T_0, we get

$$T_0(F)_s = T_0((id_I - \tau_1(s)1_I) \otimes id_I)_s = T_0(id_I \otimes id_I)_s - \tau_1(s)T_0(1_I \otimes id_I)_s,$$

and hence,

$$T_0(F)_s = T_0(id_I \otimes id_I)_s - \tau_1(s)T_0(1_I \otimes id_I)_s.$$

Since $T_0(1_I \otimes id_I) = c_1(1_I \otimes \tau_2)$ for some $\tau_2 \in \{id_I, 1_I - id_I\}$ by Lemma 1.35, we derive from the above equality that $T_0(F)_s = T_0(id_I \otimes id_I)_s - c_1\tau_1(s)\tau_2$. Having in mind that $T_0(F)_s = G_s = 0$ on I, we obtain

$$T_0(id_I \otimes id_I)_s - c_1\tau_1(s)\tau_2 = 0.$$

Because $s \in I$ is chosen arbitrarily, we conclude that $T_0(id_I \otimes id_I)_s(x) = c_1(\tau_1 \otimes \tau_2)_s(x)$ for any $s, x \in I$. □

In the next lemma, we determine the form of ψ_4. Note that τ_1 in Lemma 1.36 satisfies that $\tau_1' = 1_I$ or -1_I, since $\tau_1 \in \{id_I, 1_I - id_I\}$.

LEMMA 1.41. *Let* $\mathbf{x} = (s, x, m, z) \in X$, $\mathbf{y} = (\xi, (t, y, n, w)) \in Y$, *and* $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in Z$. *Then the following identity holds:*

$$[\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})} = c_1\tau_1'(t)\tau_2'(n)w.$$

Proof. We put $F = id_I \otimes id_I$ and $G = T_0(F)$. Because $G = c_1(\tau_1 \otimes \tau_2)$ by Lemma 1.40, we infer from (2.5) that

$$G_s(x) = c_1\tau_1(s)\tau_2(x), \qquad G'_s(m) = c_1\tau_1(s)\tau'_2(m) \quad \text{and} \tag{4.30}$$

$$D(G)_t(y) = c_1\tau'_1(t)\tau_2(y), \qquad D(G)'_t(n) = c_1\tau'_1(t)\tau'_2(n). \tag{4.31}$$

We note that $\tau_j \in \{id_I, 1_I - id_I\}$ and $\tau'_j \in \{1_I, -1_I\}$ for $j = 1, 2$. Entering $F = id_I \otimes id_I$ into (4.28) and (4.29), we deduce from (2.5) that

$$\Delta_1(F)(\mathbf{z}) = c_1\tau_1(s)\tau_2(x), \qquad \Delta'_1(F)(\mathbf{z}) = c_1\tau_1(s)\tau'_2(m)z \quad \text{and} \tag{4.32}$$

$$\Delta_2(F)(\mathbf{z}) = \xi c_1\tau'_1(t)\psi_2(\mathbf{z}), \qquad \Delta'_2(F)(\mathbf{z}) = [\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})}, \tag{4.33}$$

where we have used $\psi_2(\mathbf{z}) \in I$. By (4.30) and (4.32), we notice that $G_s(x) = \Delta_1(F)(\mathbf{z})$ and $G'_s(m)z = \Delta'_1(F)(\mathbf{z})$. Substituting equalities from (4.30) through (4.33) into (4.1), we get

$$\xi c_1\tau'_1(t)\tau_2(y) + c_1\tau'_1(t)\tau'_2(n)w = \xi c_1\tau'_1(t)\psi_2(\mathbf{z}) + [\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})},$$

which implies that $\tau_2(y) + \overline{\xi}\tau'_2(n)w = \psi_2(\mathbf{z}) + \overline{\xi c_1}\tau'_1(t)[\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})}$. The argument in Proof of Lemma 1.35 yields $\tau_2(y) = \psi_2(\mathbf{z})$ and $\overline{\xi}\tau'_2(n)w = \overline{\xi c_1}\tau'_1(t)[\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})}$. This implies that $c_1\tau'_1(t)\tau'_2(n)w = [\alpha\psi_4(\mathbf{z})]^{\varepsilon_3(\mathbf{z})}$. □

We are now in a position to prove Main theorem.

Proof of Main Theorem. Let $F \in C^1(I, \mathrm{Lip}(I))$ and we put $G = T_0(F)$. We take $(s, x), (t, y) \in I^2$ and $m, n \in \mathcal{M}$ arbitrarily. For each $z, \xi, w \in Z$, we set $\mathbf{x}_z = (s, x, m, z) \in X$, $\mathbf{y}_{\xi,w} = (\xi, (t, y, n, w)) \in Y$, and $\mathbf{z}_{z,\xi,w} = (\mathbf{x}_z, \mathbf{y}_{\xi,w}) \in Z$. By Lemma 1.41, we have $[\alpha\psi_4(\mathbf{z}_{z,\xi,w})]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})} = c_1\tau'_1(t)\tau'_2(n)w$. It follows from (4.29) that

$$\begin{aligned}\Delta_2(F)(\mathbf{z}_{z,\xi,w}) &= \xi c_1\tau'_1(t)[D(F)_{\psi_1(\mathbf{z}_{z,\xi,w})}(\psi_2(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_2(\mathbf{z}_{z,\xi,w})} \quad \text{and} \\ \Delta'_2(F)(\mathbf{z}_{z,\xi,w}) &= c_1\tau'_1(t)\tau'_2(n)[D(F)'_{\psi_1(\mathbf{z}_{z,\xi,w})}(\psi_3(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})}w.\end{aligned} \tag{4.34}$$

Applying (4.1) to $G = T_0(F)$, we derive from (4.28) and (4.34) that

$$\begin{aligned}&G_s(x) + G'_s(m)z + \xi D(G)_t(y) + D(G)'_t(n)w \\ &= c_1[F_{\tau_1(s)}(\tau_2(x))]^{\varepsilon_0} + c_1\tau'_2(m)[F'_{\tau_1(s)}(\phi_3(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_1(\mathbf{z}_{z,\xi,w})}z \\ &\quad + \xi c_1\tau'_1(t)[D(F)_{\psi_1(\mathbf{z}_{z,\xi,w})}(\psi_2(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_2(\mathbf{z}_{z,\xi,w})} \\ &\quad + c_1\tau'_1(t)\tau'_2(n)[D(F)'_{\psi_1(\mathbf{z}_{z,\xi,w})}(\psi_3(\mathbf{z}_{z,\xi,w}))]^{\varepsilon_3(\mathbf{z}_{z,\xi,w})}w\end{aligned} \tag{4.35}$$

for any $z, \xi, w \in \mathbb{T}$. For each $j \in \{1,2,3\}$, $\varepsilon_j \colon Z \to \{\pm 1\}$ is a continuous function, and thus the image of $\mathbb{T}^3$ under the map $(z, \xi, w) \mapsto \varepsilon_j(\mathbf{z}_{z,\xi,w})$ is connected. This implies that the value $\varepsilon_j(\mathbf{z}_{z,\xi,w})$ is invariant with respect to $z, \xi, w \in \mathbb{T}$. In addition, $\phi_j(\mathbf{z}_{z,\xi,w})$ and $\psi_j(\mathbf{z}_{z,\xi,w})$ are invariant with $z, \xi, w \in \mathbb{T}$ by Lemmas 1.26 and 1.27 as well. Since $z, \xi, w \in \mathbb{T}$ are arbitrarily chosen, we conclude that $T_0(F)_s(x) = G_s(x) = c_1[F_{\tau_1(s)}(\tau_2(x))]^{\varepsilon_0}$, that is,

$$T_0(F)(s, x) = c_1[F(\tau_1(s), \tau_2(x))]^{\varepsilon_0}.$$

The proof is complete. $\square$

CHAPTER 2

Exploring new solutions to Tingley's problem for function algebras

Abstract

In this chapter, we present two new positive answers to Tingley's problem in certain subspaces of function algebras. In the first result, we prove that every surjective isometry between the unit spheres, $S(A)$ and $S(B)$, of two uniformly closed function algebras A and B on a locally compact Hausdorff spaces can be extended to a surjective real linear isometry from A onto B. In a second goal, we study surjective isometries between the unit spheres of two abelian JB^*-triples represented as spaces of continuous functions of the form

$$C_0^{\mathbb{T}}(X) = \{f \in C_0(X) \ : \ f(\lambda t) = \lambda f(t) \text{ for every } (\lambda, x) \in \mathbb{T} \times X\},$$

where X is a locally Hausdorff principal $\mathbb{T}$-bundle. We establish that every surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ admits an extension to a surjective real linear isometry between these two abelian JB*-triples.

1. Introduction

The problem of extending a surjective isometry between the unit spheres of two Banach spaces– named *Tingley's problem* after the contribution of D. Tingley in [**74**]– is nowadays a treding topic in functional analysis (see a representative sample in the references [**9, 12, 18, 19, 26, 28, 29, 30, 31, 56, 57, 60**] and the surveys [**59, 78**]). This isometric extension problem remains open for Banach spaces of dimension bigger than or equal to 3 though. In fact, it has not been until recently that a complete positive solution for 2– dimensional Banach spaces was obtained by T. Banakh in [**2**], a result culminating a tour-de-force by several researchers (cf. [**1, 3, 9**]).

In recent years, a growing interest on Tingley's problem for surjective isometries between the unit spheres of certain function algebras has attracted different specialists to approach this problem. The pioneering paper by R. Wang [**75**] inspired many subsequent results. O. Hatori, S. Oi and R.S. Togashi proved that each surjective isometry between the unit spheres of two uniform algebras can be always extended to a surjective real linear isometry between the uniform algebras (cf. [**38**]). We recall that a uniform algebra is a closed subalgebra of $C(K)$ which contains constants and separates the points of a compact Hausdorff space K, where $C(K)$ denotes the Banach algebra of all complex-valued continuous functions on K. This conclusion was improved by O. Hatori by showing that each uniform algebra $\mathcal{A}$ satisfies the complex Mazur-Ulam property, that is, every surjective isometry from its unit sphere onto the unit sphere of another Banach space E extends to a surjective real linear isometry from $\mathcal{A}$ onto E (see [**36**, Theorem 4.5])

This chapter is aimed to present our recent advances on Tingley's problem for some Banach spaces which are representable as certain function spaces. More concretely, we study Tingley's problem in the case of surjective isometry between the unit spheres of two uniformly closed function algebras. Note that uniformly closed function algebras constitute a strictly wider class than that given by uniform algebras. Indeed, we begin with a locally Hausdorff space X. Let $C_0(X)$ be a Banach algebra of all complex-valued continuous functions on X which vanishes at infinity. A *uniformly closed function algebra* A on X is a uniformly closed and strongly separating (i.e. there exist $f, g \in A$ such that $f(x) \neq 0$ and $g(y) \neq g(z)$ for each $x \in X$ and $y, z \in X$ with $y \neq z$) subalgebra of $C_0(X)$. We can obviously regard A as a subalgebra of $C(X \cup \{\infty\})$, where $X \cup \{\infty\}$ denotes the one-point compactification of X. However, it is worth observing that, under such an identification, A never contains the constant functions. Thus, it is not a uniform algebra.

The first main conclusion of this chapter proves that surjective isometry $\Delta : S(A) \to S(B)$ between the unit spheres of two uniformly closed function algebras A and B extends to a surjective real linear isometry $T : A \to B$ (see Theorem 2.1). Our arguments are based on an appropriate use of the Choquet boundary of each uniformly closed function algebra, the existence of Urysohn's lemma type properties for this Choquet boundary (as in [**32, 54, 65, 66**]) and a good description of the elements in the image of Δ at points in the Choquet boundary. The proof of the already mentioned result by Hatori, Oi and Togashi in [**38**] is inspired by some of Wang's original tools in [**75**]. In this chapter, we apply similar techniques, however, the arguments here provide a different point of view, and are not mere extensions to the case of uniformly closed function algebras as non-unital versions of uniform algebras.

The second main conclusion of this chapter is focused on the study of Tingley's problem for a surjective isometry between the unit spheres of two abelian JB*-triples. As it is well-known, and explained in the section 3, JB*-triples are precisely those complex Banach spaces whose open unit ball is a bounded symmetric domain ([**43**]). A JBW*-triple is a JB*-triple which is also a dual Banach space. It has recently shown ([**4, 41**]) that every surjective isometry from the unit sphere of a JBW*-triple onto the unit sphere of another Banach space extends to a surjective real linear isometry between the spaces. Few or nothing is known for general JB*-triples. The elements in the subclass of abelian JB*-triples can be identified, thanks to a Gelfand representation theory, with subspaces of continuous functions. Indeed, let X be a principal $\mathbb{T}$-bundle (i.e. a subset of a Hausdorff locally convex complex space such that $0 \notin X$, $X \cup \{0\}$ is compact, and $\mathbb{T}X \subset X$, where $\mathbb{T} = S(\mathbb{C})$. When X is regarded as a locally compact Hausdorff space, the closed subspace of $C_0(X)$ defined by

$$C_0^{\mathbb{T}}(X) := \{f \in C_0(X) \ : \ f(\lambda x) = \lambda f(x) \text{ for every } (\lambda, x) \in \mathbb{T} \times X\},$$

is closed for the triple product $\{f, g, h\} = f\overline{g}h$ $(f, g, h \in C_0^{\mathbb{T}}(X))$. In general, $C_0^{\mathbb{T}}(X)$ is not a subalgebra of $C_0(X)$. The Gelfand representation theory affirms that each abelian JB*-triple is isometrically isomorphic to some $C_0^{\mathbb{T}}(X)$ for a suitable principal $\mathbb{T}$-bundle X (see [**43**, Corollary 1.11]). These spaces are also related to Lindenstrauss spaces (cf. [**58**, Theorem12]).

The main conclusion section 3 establishes that each surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$, with X and Y being two principal $\mathbb{T}$-bundles, admits an extension to a surjective real linear isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ (see Theorem 2.29). This statement is complemented with Lemma 2.26 where it is shown that if $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ is a surjective real linear isometry, there exist a $\mathbb{T}$-invariant closed and open subset $D \subset X$ and a $\mathbb{T}$-equivariant homeomorphism $\tau : Y \to X$ satisfying

$$\begin{aligned} T(f)(y) &= f(\tau(x)) \quad (f \in C_0^{\mathbb{T}}(X), \quad y \in \tau^{-1}(D)) \quad \text{or} \\ T(f)(y) &= \overline{f(\tau(x))} \quad (f \in C_0^{\mathbb{T}}(X), \quad y \in \tau^{-1}(X \setminus D)). \end{aligned}$$

Tingley's problem for surjective isometries between the unit spheres of function spaces deserves its own attention, and a self-contained treatment.

2. Tingley's problem for uniformly closed function algebras

Let X be a locally compact Hausdorff space. Along this note we denote by $C_0(X)$ the set of all continuous complex-valued functions f on X, which vanish at infinity in the usual sense: for each $\varepsilon > 0$ the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is a compact subset of X. Then $C_0(X)$ is a commutative Banach algebra under pointwise operations and the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$ $(f \in C_0(X))$. A subset B of $C_0(X)$ is said to be *strongly separating*, if for each $x \in X$ and $y, z \in X$ with $y \neq z$, there exist $f, g \in B$ such that $f(x) \neq 0$ and $g(y) \neq g(z)$. A *uniformly* *closed function algebra* A on X is a uniformly closed and strongly separating subalgebra of $C_0(X)$.

For each function $f \in A$ the symbol $\mathrm{Ran}(f)$ will stand for the range of f. We set $\mathrm{Ran}_\pi(f) = \{z \in \mathrm{Ran}(f) : |z| = \|f\|\}$ $(f \in A)$. A *peaking function* g for A is a function of A with $\mathrm{Ran}_\pi(g) = \{1\}$; that is, if $g \in A$ satisfies $\|g\| = 1$ and $|g(x)| = 1$ for $x \in X$, then $g(x) = 1$. A compact subset $\mathcal{P} \subset X$ is called a *peak set* of A if there exists a peaking function $f \in A$ for which $\mathcal{P} = \{x \in X : f(x) = 1\}$. A subset which coincides with an intersection of a family of peak sets of A is called a *weak peak set* of A. A *peak point* (respectively, a *weak peak point*) of A is a set $x \in X$ satisfying that $\{x\}$ is a peak set (respectively, a *weak peak set*) of A. The *Choquet boundary* or the *strong boundary* for A, denoted by $\mathrm{Ch}(A)$, is the set of all weak peak points of A. It is shown in [**66**, Theorem 2.1] (see also [**65**]) that $\mathrm{Ch}(A)$ is precisely the set of all $x \in X$ such that the evaluation functional at the point x, δ_x, is an extreme point of the unit ball of the dual space of A (cf. [**32**, Definition 2.3.7]). It is well known that $\mathrm{Ch}(A)$ is indeed a boundary (norming set) for A; furthermore, $\mathrm{Ch}(A)$ satisfies the following properties (see, for example, [**54**, Propositions 2.2 and 2.3]):

(1) For each $f \in A$ there exists $x \in \mathrm{Ch}(A)$ such that $|f(x)| = \|f\|$;
(2) For each $\varepsilon > 0$, $x \in \mathrm{Ch}(A)$ and each open subset O in X with $x \in O$ there exists a peaking function $u \in A$ such that $u(x) = 1$ and $|u| < \varepsilon$ on $X \setminus O$.

The following is the main result of this section.

THEOREM 2.1. *Let $S(A)$ and $S(B)$ be unit spheres of two uniformly closed function algebras A and B, respectively. If $\Delta\colon S(A) \to S(B)$ is a surjective isometry, then there exists a surjective, real linear isometry $T\colon A \to B$ such that $T = \Delta$ on $S(A)$.*

REMARK 2.2. Let $T\colon A \to B$ be a surjective real linear isometry. In [**54**, Theorem 1.1], such an isometry T was characterized as a weighted composition operator, more concretely, there exist a continuous function $\alpha : \mathrm{Ch}(B) \to \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, a (possibly empty) clopen subset K of $\mathrm{Ch}(B)$, and a homeomorphism $\varphi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$T(f)(y) = \begin{cases} \alpha(y) f(\varphi(y)) & (y \in K) \\ \\ \alpha(y)\overline{f(\varphi(y))} & (y \in \mathrm{Ch}(B)\setminus K) \end{cases}$$

for all $f \in A$.

NOTATION. Under the previous assumptions, for each $f \in S(A)$, we write $|f|^{-1}(1)$ for the set $\{x \in X : |f(x)| = 1\}$, and we set

$$M_f = |f|^{-1}(1) \cap \mathrm{Ch}(A).$$

For $x \in \mathrm{Ch}(A)$, we denote by P_x the set of all peaking functions f for A with $f(x) = 1$. Define $\lambda P_x = \{\lambda f : f \in P_x\}$ for each $\lambda \in \mathbb{T}$. In the same way, we define Q_y the set of all peaking functions u for B with $u(y) = 1$. Here we note that

$$M_f = \{z \in \mathrm{Ch}(A) : |f(z)| = 1\} = \{z \in \mathrm{Ch}(A) : f(z) = \lambda\}$$

for all $f \in \lambda P_x$ and $\lambda \in \mathbb{T}$.

For each $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$, we define

$$\lambda V_x = \{f \in S(A) : f(x) = \lambda\}.$$

We see that $\lambda P_x \subset \lambda V_x$. In the same way, we define $\mu W_y = \{u \in S(B) : u(y) = \mu\}$ for each $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$.

LEMMA 2.3. *Let $f, g \in S(A)$ and $x_0 \in M_f$. If $f(x_0) \neq g(x_0)$, then there exists $h \in S(A)$ such that $\|f - h\| = 2 > \|g - h\|$.*

PROOF. Note first that $|f(x_0)| = 1$, since $x_0 \in M_f$. Set $2\delta = |f(x_0) - g(x_0)|$, and then $\delta > 0$. Define the open neighborhood O of x_0 by $O = \{x \in X : |g(x) - g(x_0)| < \delta\}$. Since $x_0 \in M_f \subset \mathrm{Ch}(A)$, there exists $u \in P_{x_0}$ such that $|u| < 2^{-1}$ on $X \setminus O$. We set $h = -f(x_0)u \in S(A)$. We have

$$2 = |2f(x_0)| = |f(x_0) - h(x_0)| \leq \|f - h\| \leq 2,$$

and thus $\|f - h\| = 2$.

Take an arbitrary $x \in X$. We shall prove that $|g(x) - h(x)| < 2$. If $x \in O$, then we observe that $|g(x) - h(x)| < 2$. Indeed, if $|g(x) - h(x)| = 2$, then $g(x) = -h(x)$ and $|h(x)| = 1$, since $g, h \in S(A)$. This implies that $|u(x)| = |h(x)| = 1$. Since u is a peaking function for A, we obtain $u(x) = 1$, and hence $g(x) = -h(x) = f(x_0)$. Since $x \in O$, we get $2\delta = |f(x_0) - g(x_0)| = |g(x) - g(x_0)| < \delta$, a contradiction. We have proved that $|g(x) - h(x)| < 2$ for all $x \in O$. Suppose now that $x \in X \setminus O$. Then $|u(x)| < 2^{-1}$. It follows that

$$|g(x) - h(x)| \leq |g(x)| + |f(x_0)u(x)| \leq 1 + \frac{1}{2} < 2.$$

Hence, $|g(x) - h(x)| < 2$, and consequently, $\|g - h\| < 2$. $\square$

In the rest of this section, we assume that A and B are uniformly closed function algebras on locally compact Hausdorff spaces X and Y, respectively, and that $\Delta\colon S(A) \to S(B)$ is a surjective isometry with respect to the supremum norms.

Lemma 2.4. *Let $f, g \in S(A)$. If $f = g$ on M_f, then $\Delta(f) = \Delta(g)$ on $M_{\Delta(f)}$.*

Proof. Arguing by contradiction, we suppose the existence of $y_0 \in M_{\Delta(f)}$ such that $\Delta(f)(y_0) \neq \Delta(g)(y_0)$. Applying Lemma 2.3 to $\Delta(f), \Delta(g) \in S(B)$ and $y_0 \in M_{\Delta(f)}$, we can choose $h \in S(A)$ so that $\|\Delta(f) - \Delta(h)\| = 2 > \|\Delta(g) - \Delta(h)\|$, where we have used that Δ is surjective. Since Δ is an isometry, we have $\|f - h\| = 2 > \|g - h\|$. Recall that $\mathrm{Ch}(A)$ is a boundary for A, and thus there exists $x_0 \in \mathrm{Ch}(A)$ with $|f(x_0) - h(x_0)| = 2$, and by the other condition $|g(x_0) - h(x_0)| < 2$. Since $f, h \in S(A)$, we get $|f(x_0)| = 1$, which implies $x_0 \in M_f$. Consequently, $f(x_0) \neq g(x_0)$ for $x_0 \in M_f$, which is impossible. □

Lemma 2.5. *Let $x \in \mathrm{Ch}(A)$, $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$. If $f_j \in \lambda P_x$ for each $j \in \mathbb{N}$ with $1 \leq j \leq n$, then $g = n^{-1} \sum_{j=1}^{n} f_j \in A$ satisfies $g \in \lambda P_x$ with $M_g \subset \cap_{j=1}^{n} M_{f_j}$.*

Proof. Since $f_j \in \lambda P_x$ for $j = 1, 2, \ldots, n$, then $f_j(x) = \lambda$ and $\|f_j\| = 1$ for every $j \in \{1, 2, \ldots, n\}$. We have

$$n = |n\lambda| = \left| \sum_{j=1}^{n} f_j(x) \right| \leq \sum_{j=1}^{n} |f_j(x)| \leq \sum_{j=1}^{n} \|f_j\| = n.$$

Hence, $g = n^{-1} \sum_{j=1}^{n} f_j \in A$ satisfies $\overline{\lambda} g(x) = 1 = \|g\|$.

We shall prove that $g \in \lambda P_x$. Suppose that $|\overline{\lambda} g(x')| = 1$ for $x' \in X$, and then $|\sum_{j=1}^{n} f_j(x')| = n$. Since $|f_j(x')| \leq 1$, it follows that $|\overline{\lambda} f_j(x')| = |f_j(x')| = 1$ for all $j \in \{1, 2, \ldots, n\}$, which implies that $\overline{\lambda} f_j(x') = 1$ because $\overline{\lambda} f_j \in P_x$, and thus $\overline{\lambda} g(x') = 1$. This shows that $\overline{\lambda} g \in P_x$ (consequently, $g \in \lambda P_x$) and $M_g \subset \cap_{j=1}^{n} M_{f_j}$. □

The characterization of compactness in terms of the finite intersection property is employed in our next result.

Lemma 2.6. *The intersection $\displaystyle\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)$ is non-empty for all λ in $\mathbb{T}$ and x in $\mathrm{Ch}(A)$.*

Proof. First, we note that $|u|^{-1}(1)$ is a compact subset of Y for each $u \in \Delta(\lambda P_x)$. So, by the characterization of compactness in terms of the finite intersection property, it is enough to show that $\cap_{j=1}^{n} |u_j|^{-1}(1) \neq \emptyset$ for each $n \in \mathbb{N}$ and $u_j \in \Delta(\lambda P_x)$ with $j = 1, 2, \ldots, n$.

Let $n \in \mathbb{N}$ and $u_j \in \Delta(\lambda P_x)$ for $j = 1, 2, \ldots, n$. Choose $f_j \in \lambda P_x$ so that $u_j = \Delta(f_j)$, and set $g = n^{-1} \sum_{j=1}^{n} f_j \in A$. We see that $g \in \lambda P_x$ with $M_g \subset \cap_{j=1}^{n} M_{f_j}$ by Lemma 2.5. We shall prove that $M_g = \cap_{j=1}^{n} M_{f_j}$. Here, we recall that

$$M_f = \{z \in \mathrm{Ch}(A) : |f(z)| = 1\} = \{z \in \mathrm{Ch}(A) : f(z) = \lambda\} \tag{2.1}$$

for all $f \in \lambda P_x$. Let $x_0 \in \cap_{j=1}^{n} M_{f_j}$. Since $f_j \in \lambda P_x$, we have $f_j(x_0) = \lambda$ for all $j \in \{1, 2, \ldots, n\}$ by (2.1). It follows that $g(x_0) = n^{-1} \sum_{j=1}^{n} f_j(x_0) = \lambda$. Since $g \in \lambda P_x$, equality (2.1) shows that $x_0 \in M_g$, and consequently, $\cap_{j=1}^{n} M_{f_j} \subset M_g$. Therefore, we conclude that $M_g = \cap_{j=1}^{n} M_{f_j}$, as claimed.

For each $z \in M_g = \cap_{j=1}^{n} M_{f_j}$, we have $g(z) = \lambda = f_j(z)$, that is, $g = f_j$ on M_g for each $j = 1, 2, \ldots, n$. If we apply Lemma 2.4, we deduce that $\Delta(g) = \Delta(f_j) = u_j$ on $M_{\Delta(g)}$. Then $|u_j(\zeta)| = |\Delta(g)(\zeta)| = 1$ for each $\zeta \in M_{\Delta(g)}$, and consequently $\cap_{j=1}^{n} |u_j|^{-1}(1) \neq \emptyset$, as claimed. □

We explore next the intersection of the non-empty set in the previous lemma with the Choquet boundary of B.

Lemma 2.7. *The intersection* $\mathrm{Ch}(B) \cap \left(\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1) \right)$ *is non-empty for each* $\lambda \in \mathbb{T}$ *and each* $x \in \mathrm{Ch}(A)$.

Proof. Let $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. There exists $y_0 \in \cap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)$ by Lemma 2.6. Take an arbitrary $u \in \Delta(\lambda P_x)$ (in particular, $|u(y_0)| = 1 = \|u\|$). Define the function $\tilde{u} \in B$ by

$$\tilde{u}(y) = \left(\overline{u(y_0)}^2 u^2(y) + \overline{u(y_0)} u(y) \right) /2, \ (y \in Y).$$

We observe that $\tilde{u} \in Q_{y_0}$. Namely, $1 = \tilde{u}(y_0) \le \|\tilde{u}\| \le 1$, and thus $\tilde{u} \in W_{y_0}$. Suppose now that $|\tilde{u}(y)| = 1$ for some $y \in Y$, and then $|\overline{u(y_0)} u^2(y) + u(y)| = 2$. It follows that

$$2 \le |\overline{u(y_0)} u^2(y)| + |u(y)| \le 2,$$

which shows that $|u(y)| = 1$. Hence it follows from $|\overline{u(y_0)} u^2(y) + u(y)| = 2$ that $|\overline{u(y_0)} u(y) + 1| = 2$, and consequently $\overline{u(y_0)} u(y) = 1$. This implies that $\tilde{u}(y) = 1$, and we have therefore proven that $\tilde{u} \in Q_{y_0}$. We see that $(\tilde{u})^{-1}(1)$ is a peak set for B with

$$(\tilde{u})^{-1}(1) = u^{-1}(u(y_0)) \subset |u|^{-1}(1).$$

By the arbitrariness of $u \in \Delta(\lambda P_x)$, we get $y_0 \in \cap_{u\in\Delta(\lambda P_x)}(\tilde{u})^{-1}(1)$. It is known that every non-empty weak peak set for B contains a weak peak point, that is, $\mathrm{Ch}(B) \cap \big(\cap_{u\in\Delta(\lambda P_x)}(\tilde{u})^{-1}(1)\big) \neq \emptyset$ (see, for example, [**54**, Proposition 2.1]). This shows that

$$\mathrm{Ch}(B) \cap \big(\cap_{u\in\Delta(\lambda P_x)}|u|^{-1}(1)\big) \neq \emptyset.$$

The proof is complete. $\square$

In the next result, we replace λP_x with λV_x.

LEMMA 2.8. *The intersection* $\mathrm{Ch}(B) \cap \left(\bigcap_{v\in\Delta(\lambda V_x)} |v|^{-1}(1)\right)$ *is non-empty for each* $\lambda \in \mathbb{T}$ *and each* $x \in \mathrm{Ch}(A)$.

PROOF. By Lemma 2.7, there exists $y \in \mathrm{Ch}(B) \cap \Big(\bigcap_{u\in\Delta(\lambda P_x)} |u|^{-1}(1)\Big)$. Take an arbitrary $v \in \Delta(\lambda V_x)$. We shall prove that $|v(y)| = 1$. Let $f \in \lambda V_x$ be such that $\Delta(f) = v$, and then $f(x) = \lambda$ and $\|f\| = 1$. Define the function $\tilde{f} \in S(A)$ by

$$\tilde{f}(z) = (\overline{\lambda}^2 f^2(z) + \overline{\lambda} f(z))/2, \ (z \in X).$$

We see that $\tilde{f} \in P_x$ with

$$M_{\tilde{f}} = \{z \in \mathrm{Ch}(A) : |\tilde{f}(z)| = 1\} = \{z \in \mathrm{Ch}(A) : f(z) = \lambda\}.$$

Recall that $M_{\tilde{f}} = \{z \in \mathrm{Ch}(A) : \tilde{f}(z) = 1\}$, since $\tilde{f} \in P_x$. For each $z \in M_{\tilde{f}}$, we have $\lambda\tilde{f}(z) = \lambda = f(z)$, and thus $\lambda\tilde{f} = f$ on $M_{\tilde{f}} = M_{\lambda\tilde{f}}$. Lemma 2.4 shows that

$$\Delta(\lambda\tilde{f}) = \Delta(f) \quad \text{on } M_{\Delta(\lambda\tilde{f})}.$$

Since $\lambda\tilde{f} \in \lambda P_x$, we obtain $|\Delta(\lambda\tilde{f})(y)| = 1$, that is, $y \in M_{\Delta(\lambda\tilde{f})}$. It follows that $v(y) = \Delta(f)(y) = \Delta(\lambda\tilde{f})(y)$, and consequenlty, $|v(y)| = |\Delta(\lambda\tilde{f})(y)| = 1$. Hence $y \in |v|^{-1}(1)$. We conclude from the arbitrariness of $v \in \Delta(\lambda V_x)$ that $y \in \mathrm{Ch}(B) \cap \big(\cap_{v\in\Delta(\lambda V_x)}|v|^{-1}(1)\big)$. $\square$

We determine next the behaviour of Δ on sets of the form λP_x.

LEMMA 2.9. *For each* $(\lambda, x) \in \mathbb{T} \times \mathrm{Ch}(A)$*, there exists a couple* (μ, y) *in* $\mathbb{T} \times \mathrm{Ch}(B)$ *such that* $\Delta(\lambda P_x) \subset \mu W_y$.

PROOF. Let us fix $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. By Lemma 2.8, there exists $y \in \mathrm{Ch}(B) \cap \big(\cap_{v\in\Delta(\lambda V_x)}|v|^{-1}(1)\big)$. For each $f \in \lambda P_x$, we have $|\Delta(f)(y)| = 1$ by the choice of y. Since $f \in \lambda P_x \subset S(A)$, we obtain $\|\Delta(f)\| = 1$. Hence, $\Delta(f) \in \mu W_y$ with $\mu = \Delta(f)(y) \in \mathbb{T}$.

Now, we prove that $\Delta(f)(y) = \Delta(g)(y)$ for all $f, g \in \lambda P_x$. Set $h = (f+g)/2 \in A$, and then $h \in \lambda P_x$ by Lemma 2.5. We observe that

$$M_h = h^{-1}(\lambda) \cap \mathrm{Ch}(A) = f^{-1}(\lambda) \cap g^{-1}(\lambda) \cap \mathrm{Ch}(A),$$

since $f, g, h \in \lambda P_x$, where $k^{-1}(\lambda) = \{z \in X : k(z) = \lambda\}$ for $k \in \lambda P_x$. Therefore, we have $f = h = g$ on M_h. We derive from Lemma 2.4 that $\Delta(f) = \Delta(h) = \Delta(g)$ on $M_{\Delta(h)}$. Since $\Delta(h) \in \Delta(\lambda V_x)$, we get $|\Delta(h)(y)| = 1$ by the choice of y. Thus, $y \in M_{\Delta(h)}$, and consequently $\Delta(f)(y) = \Delta(h)(y) = \Delta(g)(y)$.

The above arguments show that $\Delta(f) \in \mu W_y$ for all $f \in \lambda P_x$, where $\mu = \Delta(f)(y)$ is independent of the choice of $f \in \lambda P_x$. This shows that $\Delta(\lambda P_x) \subset \mu W_y$ for some $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$. □

Lemma 2.10. *For each $(\lambda, x) \in \mathbb{T} \times \mathrm{Ch}(A)$, there exists a couple (μ, y) in $\mathbb{T} \times \mathrm{Ch}(B)$ such that $\Delta(\lambda V_x) \subset \mu W_y$.*

Proof. Fix λ, x as in the statement. By Lemma 2.9, there exist $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$ such that $\Delta(\lambda P_x) \subset \mu W_y$. Let $v \in \Delta(\lambda V_x)$. We shall prove that $v \in \mu W_y$. Let $f \in \lambda V_x$ be such that $\Delta(f) = v$. Define the function $\tilde{f} \in A$ by

$$\tilde{f}(z) = (\overline{\lambda}^2 f^2(z) + \overline{\lambda} f(z))/2, \quad (z \in X).$$

We see that $\tilde{f} \in P_x$ with

$$M_{\tilde{f}} = \{z \in \mathrm{Ch}(A) : \tilde{f}(z) = 1\} = f^{-1}(\lambda) \cap \mathrm{Ch}(A).$$

For each $z \in M_{\tilde{f}}$, we have $\lambda\tilde{f}(z) = \lambda = f(z)$, and hence $\lambda\tilde{f} = f$ on $M_{\tilde{f}} = M_{\lambda\tilde{f}}$. Lemma 2.4 shows that $\Delta(\lambda\tilde{f}) = \Delta(f)$ on $M_{\Delta(\lambda\tilde{f})}$. Since $\tilde{f} \in P_x$, we have $\Delta(\lambda\tilde{f}) \in \Delta(\lambda P_x) \subset \mu W_y$. Thus $\Delta(\lambda\tilde{f}) \in \mu W_y$, that is, $\Delta(\lambda\tilde{f})(y) = \mu$. This implies that $|\Delta(\lambda\tilde{f})(y)| = 1$, which yields $y \in M_{\Delta(\lambda\tilde{f})}$. Therefore, $v(y) = \Delta(f)(y) = \Delta(\lambda\tilde{f})(y) = \mu$, and consequently $v \in \mu W_y$. This shows that $\Delta(\lambda V_x) \subset \mu W_y$. □

We shall discuss next the uniqueness of the couple (μ, y) in previous lemmas.

Lemma 2.11. *If $\lambda V_x \subset \lambda' V_{x'}$ holds for some $\lambda, \lambda' \in \mathbb{T}$ and $x, x' \in \mathrm{Ch}(A)$, then $\lambda = \lambda'$ and $x = x'$.*

Proof. Suppose, on the contrary, that $x \neq x'$. There exists $f \in P_x \subset V_x$ such that $|f(x')| < 1$ (cf. the properties in page 45). Then $\lambda f \in \lambda V_x \setminus (\lambda' V_{x'})$, since $|\lambda f(x')| < 1$. This contradicts $\lambda V_x \subset \lambda' V_{x'}$. Hence, we obtain $x = x'$, and thus $\lambda V_x \subset \lambda' V_x$ by the hypothesis.

For each $g \in V_x$, we have $\lambda g \in \lambda' V_x$, which shows that $\lambda = \lambda g(x) = \lambda'$. We thus conclude that $\lambda = \lambda'$. $\square$

LEMMA 2.12. *For each $(\lambda, x) \in \mathbb{T} \times \mathrm{Ch}(A)$, there exists a unique couple (μ, y) in $\mathbb{T} \times \mathrm{Ch}(B)$ such that $\Delta(\lambda V_x) = \mu W_y$.*

PROOF. Let us fix $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. By Lemma 2.10 there exist $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$ such that $\Delta(\lambda V_x) \subset \mu W_y$. Another application of Lemma 2.10, with $\mu \in \mathbb{T}$, $y \in \mathrm{Ch}(B)$ and Δ^{-1}, shows the existence of $\lambda' \in \mathbb{T}$ and $x' \in \mathrm{Ch}(A)$ such that $\Delta^{-1}(\mu W_y) \subset \lambda' V_{x'}$. Thus, we have $\Delta(\lambda V_x) \subset \mu W_y \subset \Delta(\lambda' V_{x'})$, and hence $\lambda V_x \subset \lambda' V_{x'}$. Therefore, we obtain $\lambda = \lambda'$ and $x = x'$ by Lemma 2.11, which shows that $\Delta(\lambda V_x) = \mu W_y$.

Suppose that $\Delta(\lambda V_x) = \mu' W_{y'}$ for some $\mu' \in \mathbb{T}$ and $y' \in \mathrm{Ch}(B)$. Then $\mu W_y = \Delta(\lambda V_x) = \mu' W_{y'}$, and hence $\mu W_y = \mu' W_{y'}$. Lemma 2.11 shows that $\mu = \mu'$ and $y = y'$, which proves the uniqueness of $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$. $\square$

We are now in a position to define the key functions describing the behaviour of Δ on sets of the form λV_x.

DEFINITION 2.13. By Lemma 2.12, there exist well-defined maps $\alpha\colon \mathbb{T} \times \mathrm{Ch}(A) \to \mathbb{T}$ and $\phi\colon \mathbb{T} \times \mathrm{Ch}(A) \to \mathrm{Ch}(B)$ with the following property:

$$\Delta(\lambda V_x) = \alpha(\lambda, x) W_{\phi(\lambda, x)} \qquad (\lambda \in \mathbb{T},\, x \in \mathrm{Ch}(A)).$$

Our next goal will consist in isolating the key properties of the just defined maps.

LEMMA 2.14. *For each $\mu, \mu' \in \mathbb{T}$ and $y, y' \in \mathrm{Ch}(B)$ with $y \neq y'$, there exist $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$.*

PROOF. Choose disjoint open sets $O, O' \subset Y$ so that $y \in O$ and $y' \in O'$. There exist $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $|\tilde{u}| < 1/3$ on $Y \setminus O$ and $|\tilde{v}| < 1/3$ on $Y \setminus O'$. For $z \in O$, we have $|\tilde{u}(z) - \tilde{v}(z)| \leq 1 + 1/3 < \sqrt{2}$, since $O \cap O' = \emptyset$. For $z \in Y \setminus O$, we obtain $|\tilde{u}(z) - \tilde{v}(z)| \leq 1/3 + 1 < \sqrt{2}$ by the choice of $\tilde{u}$. We thus conclude $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$, as is claimed. $\square$

LEMMA 2.15. *If $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$, then $\phi(\lambda, x) = \phi(-\lambda, x)$.*

PROOF. Let $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. We set $\mu = \alpha(\lambda, x)$, $\mu' = \alpha(-\lambda, x)$, $y = \phi(\lambda, x)$ and $y' = \phi(-\lambda, x)$. Then $\Delta(\lambda V_x) = \mu W_y$ and $\Delta((-\lambda) V_x) = \mu' W_{y'}$. Suppose, on the contrary, that $y \neq y'$. Lemma 2.14 assures the existence of $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$.

By the choice of $\tilde{u}$ and $\tilde{v}$, we see that $\Delta^{-1}(\tilde{u}) \in \Delta^{-1}(\mu Q_y) \subset \Delta^{-1}(\mu W_y) = \lambda V_x$ and $\Delta^{-1}(\tilde{v}) \in \Delta^{-1}(\mu' W_{y'}) \subset (-\lambda)V_x$. Then $\Delta^{-1}(\tilde{u})(x) = \lambda$ and $\Delta^{-1}(\tilde{v})(x) = -\lambda$, and therefore

$$\begin{aligned} 2 = |2\lambda| &= |\Delta^{-1}(\tilde{u})(x) - \Delta^{-1}(\tilde{v})(x)| \leq \|\Delta^{-1}(\tilde{u}) - \Delta^{-1}(\tilde{v})\| \\ &= \|\tilde{u} - \tilde{v}\| < \sqrt{2}, \end{aligned}$$

which is a contradiction. Consequently, we have $y = y'$, and hence $\phi(\lambda, x) = \phi(-\lambda, x)$. □

Lemma 2.16. *If $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$, then $\phi(\lambda, x) = \phi(1, x)$; hence, the point $\phi(\lambda, x)$ is independent of the choice of $\lambda \in \mathbb{T}$.*

Proof. Let $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. Set $\mu = \alpha(\lambda, x)$, $\mu' = \alpha(1, x)$, $y = \phi(\lambda, x)$ and $y' = \phi(1, x)$. Then $\Delta(\lambda V_x) = \mu W_y$ and $\Delta(V_x) = \mu' W_{y'}$. We shall prove that $y = y'$. Suppose that $y \neq y'$. Under this assumption, there exist $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$ (cf. Lemma 2.14). By the choice of $\tilde{u}$ and $\tilde{v}$, we obtain $\Delta^{-1}(\tilde{u}) \in \lambda V_x$ and $\Delta^{-1}(\tilde{v}) \in V_x$. Thus $\Delta^{-1}(\tilde{u})(x) = \lambda$ and $\Delta^{-1}(\tilde{v})(x) = 1$. If $\operatorname{Re}\lambda \leq 0$, then $|\lambda - 1| \geq \sqrt{2}$, which shows that

$$\begin{aligned} \sqrt{2} \leq |\lambda - 1| &= |\Delta^{-1}(u)(x) - \Delta^{-1}(v)(x)| \leq \|\Delta^{-1}(u) - \Delta^{-1}(v)\| \\ &= \|u - v\| < \sqrt{2}. \end{aligned}$$

We arrive at a contradiction, which yields $y = y'$ if $\operatorname{Re}\lambda \leq 0$. Now we consider the case when $\operatorname{Re}\lambda > 0$. Note that $\phi(-\lambda, x) = \phi(\lambda, x) = y$ by Lemma 2.15. Hence, $\Delta((-\lambda)V_x) = \nu W_y$ for some $\nu \in \mathbb{T}$. Since $\operatorname{Re}(-\lambda) < 0$, the above arguments can be applied to $\Delta((-\lambda)V_x) = \nu W_y$ and $\Delta(V_x) = \mu' W_{y'}$ to deduce that $y = y'$. Then we get $y = y'$ even if $\operatorname{Re}\lambda > 0$. □

Definition 2.17. By Lemma 2.16, we may and do write $\phi(\lambda, x) = \phi(x)$. We will also write $\alpha(\lambda, x) = \alpha_x(\lambda)$ for each $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$. Then we obtain

$$\Delta(\lambda V_x) = \alpha_x(\lambda) W_{\phi(x)} \qquad (\lambda \in \mathbb{T},\ x \in \mathrm{Ch}(A)). \tag{2.2}$$

The arguments above can be applied to the surjective isometry Δ^{-1} from $S(B)$ onto $S(A)$. Then there exist two maps $\beta\colon \mathbb{T} \times \mathrm{Ch}(B) \to \mathbb{T}$ and $\psi\colon \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$\Delta^{-1}(\mu W_y) = \beta_y(\mu) V_{\psi(y)} \qquad (\mu \in \mathbb{T},\ y \in \mathrm{Ch}(B)), \tag{2.3}$$

where $\beta_y(\mu) = \beta(\mu, y)$ for each $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$. We may regard α_x and β_y as maps from $\mathbb{T}$ into itself for each $x \in \mathrm{Ch}(A)$ and $y \in \mathrm{Ch}(B)$.

Lemma 2.18. *The maps $\alpha_x\colon \mathbb{T} \to \mathbb{T}$, for each $x \in \mathrm{Ch}(A)$, and $\phi\colon \mathrm{Ch}(A) \to \mathrm{Ch}(B)$ are both bijective with $\alpha_x^{-1} = \beta_{\phi(x)}$ and $\phi^{-1} = \psi$.*

Proof. Let $x \in \mathrm{Ch}(A)$. We will prove that α_x and ϕ are injective. Take $\lambda \in \mathbb{T}$ arbitrarily. If we apply (2.3) with $\mu = \alpha_x(\lambda)$ and $y = \phi(x)$, then we get

$$\Delta^{-1}(\alpha_x(\lambda) W_{\phi(x)}) = \beta_{\phi(x)}(\alpha_x(\lambda)) V_{\psi(\phi(x))}.$$

Combining the equality above with (2.2), we obtain

$$\lambda V_x = \Delta^{-1}(\alpha_x(\lambda) W_{\phi(x)}) = \beta_{\phi(x)}(\alpha_x(\lambda)) V_{\psi(\phi(x))}.$$

Lemma 2.11 shows that $\lambda = \beta_{\phi(x)}(\alpha_x(\lambda))$ and $x = \psi(\phi(x))$; since $\lambda \in \mathbb{T}$ is arbitrary, the first equality shows that α_x is injective. The second one shows that ϕ is injective, since $x \in \mathrm{Ch}(A)$ is arbitrary.

Now we prove that α_x and ϕ are both surjective. Let $\mu \in \mathbb{T}$ and $y \in \mathrm{Ch}(B)$. Applying (2.2) with $\lambda = \beta_y(\mu)$ and $x = \psi(y)$, we get $\Delta(\beta_y(\mu) V_{\psi(y)}) = \alpha_{\psi(y)}(\beta_y(\mu)) W_{\phi(\psi(y))}$. The last equality, together with (2.3), shows that

$$\mu W_y = \alpha_{\psi(y)}(\beta_y(\mu)) W_{\phi(\psi(y))}.$$

According to Lemma 2.11, we have

$$\mu = \alpha_{\psi(y)}(\beta_y(\mu)) \tag{2.4}$$

and $y = \phi(\psi(y))$. Since $y \in \mathrm{Ch}(B)$ is arbitrary, the second equality shows that ϕ is surjective. Then there exists $\phi^{-1} \colon \mathrm{Ch}(B) \to \mathrm{Ch}(A)$. We obtain $\phi(\phi^{-1}(y)) = y = \phi(\psi(y))$, which yields $\phi^{-1}(y) = \psi(y)$. We conclude, from the arbitrariness of $y \in \mathrm{Ch}(B)$, that $\phi^{-1} = \psi$. Since ψ is bijective with $\psi^{-1} = \phi$, for each $x \in \mathrm{Ch}(A)$ there exists $y \in \mathrm{Ch}(B)$ such that $x = \psi(y)$. By (2.4), $\mu = \alpha_{\psi(y)}(\beta_y(\mu)) = \alpha_x(\beta_{\phi(x)}(\mu))$ holds for all $\mu \in \mathbb{T}$. This implies that α_x is surjective for each $x \in \mathrm{Ch}(B)$. There exists α_x^{-1}, and then $\alpha_x(\alpha_x^{-1}(\mu)) = \mu = \alpha_x(\beta_{\phi(x)}(\mu))$ for all $\mu \in \mathbb{T}$. This shows that $\alpha_x^{-1} = \beta_{\phi(x)}$ for each $x \in \mathrm{Ch}(A)$. □

Lemma 2.19. *For each $x \in \mathrm{Ch}(A)$, the map $\alpha_x \colon \mathbb{T} \to \mathbb{T}$ is a surjective isometry.*

Proof. Let $x \in \mathrm{Ch}(A)$ and $\lambda_1, \lambda_2 \in \mathbb{T}$. Note that $\Delta(\lambda f)(\phi(x)) = \alpha_x(\lambda)$ for all $\lambda \in \mathbb{T}$ and $f \in V_x$ by (2.2). For each $f \in V_x$, we have

$$\begin{aligned} |\alpha_x(\lambda_1) - \alpha_x(\lambda_2)| &= |\Delta(\lambda_1 f)(\phi(x)) - \Delta(\lambda_2 f)(\phi(x))| \\ &\leq \|\Delta(\lambda_1 f) - \Delta(\lambda_2 f)\| = \|\lambda_1 f - \lambda_2 f\| \\ &= |\lambda_1 - \lambda_2|. \end{aligned}$$

Hence, $|\alpha_x(\lambda_1) - \alpha_x(\lambda_2)| \leq |\lambda_1 - \lambda_2|$. By applying the same argument to Δ^{-1}, we observe that β_y also is a contractive mapping. Having in mind that $\alpha_x^{-1} = \beta_{\phi(x)}$ (cf. Lemma 2.18), we obtain that α_x and $\beta_{\phi(x)}$ are surjective isometries on $\mathbb{T}$. $\square$

Fix $x \in \mathrm{Ch}(A)$. Since $\alpha_x : \mathbb{T} \to \mathbb{T}$ is a surjective isometry on the unit sphere of the complex plane, and Tingley's problem admits a positive solution in this case, α_x admits an extension to a surjective real linear isometry on $\mathbb{C}$, therefore one of the following statements hold:

$$\alpha_x(\lambda) = \alpha_x(1)\lambda \text{ for all } \lambda \in \mathbb{T}, \text{ or } \alpha_x(\lambda) = \alpha_x(1)\overline{\lambda} \text{ for all } \lambda \in \mathbb{T}. \tag{2.5}$$

One final technical result separates us from the main goal of this section.

Lemma 2.20. *Let $f \in S(A)$ and $x_0 \in \mathrm{Ch}(A)$ be such that $|f(x_0)| < 1$. We set $\lambda = f(x_0)/|f(x_0)|$ if $f(x_0) \neq 0$, and $\lambda = 1$ if $f(x_0) = 0$. For each r with $0 < r < 1$, there exists $g_r \in V_{x_0}$ such that*

$$rf + (1 - r|f(x_0)|)\lambda g_r \in \lambda V_{x_0}.$$

Proof. Note first that $1 - |f(x_0)| > 0$. We set

$$F_0 = \left\{ x \in X : |f(x) - f(x_0)| \geq \frac{1 - |f(x_0)|}{2} \right\}, \quad \text{and}$$
$$F_m = \left\{ x \in X : \frac{1 - |f(x_0)|}{2^{m+1}} \leq |f(x) - f(x_0)| \leq \frac{1 - |f(x_0)|}{2^m} \right\}$$

for each $m \in \mathbb{N}$. We see that F_n is a closed subset of X with $x_0 \notin F_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $x_0 \in \mathrm{Ch}(A)$, there exists $f_n \in P_{x_0}$ such that

$$|f_n| < \frac{1 - r}{1 - r|f(x_0)|} \quad \text{on } F_n \tag{2.6}$$

for each $n \in \mathbb{N} \cup \{0\}$. We set $g_r = f_0 \sum_{n=1}^{\infty} f_n/2^n$. Having in mind that the series converges in A from the choice of f_n, we observe that

$$1 = g_r(x_0) \leq \|g_r\| \leq \|f_0\| \sum_{n=1}^{\infty} \frac{\|f_n\|}{2^n} = 1,$$

and hence $g_r \in V_{x_0}$. Set $h_r = rf + (1 - r|f(x_0)|)\lambda g_r \in A$. We shall prove that $h_r \in \lambda V_{x_0}$. Since $g_r(x_0) = 1$ and $f(x_0) = |f(x_0)|\lambda$, we have $h_r(x_0) = \lambda$. Take $x \in X$ arbitrarily. To prove that $|h_r(x)| \leq 1$, we will consider three cases. If $x \in F_0$, then (2.6) shows that

$$|g_r(x)| \leq |f_0(x)| \sum_{n=1}^{\infty} \frac{|f_n(x)|}{2^n} < \frac{1 - r}{1 - r|f(x_0)|}.$$

We obtain

$$|h_r(x)| \leq r|f(x)| + (1 - r|f(x_0)|)|\lambda g_r(x)| < r + (1 - r) = 1.$$

Hence, $|h_r(x)| < 1$ if $x \in F_0$.

Suppose that $x \in F_m$ for some $m \in \mathbb{N}$. Then $|f(x) - f(x_0)| \leq (1 - |f(x_0)|)/2^m$ by the choice of F_m. We get

$$|f(x)| \leq |f(x_0)| + \frac{1 - |f(x_0)|}{2^m} = \left(1 - \frac{1}{2^m}\right)|f(x_0)| + \frac{1}{2^m}. \tag{2.7}$$

We derive from (2.6) that

$$\begin{aligned} |g_r(x)| &\leq |f_0(x)| \left(\frac{|f_m(x)|}{2^m} + \sum_{n \neq m} \frac{|f_n(x)|}{2^n} \right) \\ &< \frac{1}{2^m} \frac{1 - r}{1 - r|f(x_0)|} + 1 - \frac{1}{2^m}. \end{aligned}$$

It follows that

$$|(1 - r|f(x_0)|)\lambda g_r(x)| < \frac{1 - r}{2^m} + \left(1 - \frac{1}{2^m}\right)(1 - r|f(x_0)|). \tag{2.8}$$

We infer from (2.7) and (2.8) that

$$\begin{aligned} |h_r(x)| &\leq r|f(x)| + |(1 - r|f(x_0)|)\lambda g_r(x)| \\ &< r\left(1 - \frac{1}{2^m}\right)|f(x_0)| + \frac{r}{2^m} + \frac{1 - r}{2^m} + \left(1 - \frac{1}{2^m}\right)(1 - r|f(x_0)|) \\ &= 1, \end{aligned}$$

and hence, $|h_r(x)| < 1$ for $x \in \cup_{n=1}^{\infty} F_n$.

Now we consider the case in which $x \notin \cup_{n=0}^{\infty} F_n$. Then $x \in \cap_{n=0}^{\infty}(X \setminus F_n)$, which implies that $f(x) = f(x_0)$. We have

$$|h_r(x)| \leq r|f(x_0)| + 1 - r|f(x_0)| = 1,$$

and we thus conclude that $|h(x)| \leq 1$ for all $x \in X$, and consequently $h_r \in \lambda V_{x_0}$. $\square$

We have already gathered the tools to prove Theorem 2.1.

Proof of Theorem 2.1. Let $f \in S(A)$ and $y \in \mathrm{Ch}(B)$. To simplify the notation we set $x = \psi(y)$ and $\lambda = f(x)/|f(x)| \in \mathbb{T}$ if $f(x) \neq 0$, and $\lambda = 1$ if $f(x) = 0$, where $\psi = \phi^{-1}$ as in Lemma 2.18. We first prove that $|\Delta(f)(y)| = |f(x)|$. If $|f(x)| = 1$, then $f \in \lambda V_x$ and thus

$$|\Delta(f)(y)| = |\Delta(f)(\phi(x))| = |\alpha_x(\lambda)| = 1 = |f(x)|$$

by (2.2). We need to consider the case when $|f(x)| < 1$. By Lemma 2.20, there exists $g_r \in V_x$ such that $h_r = rf + (1 - r|f(x)|)\lambda g_r \in \lambda V_x$ for each r with $0 < r < 1$. We obtain

$$\begin{aligned}\|h_r - f\| &= \|(r-1)f + (1 - r|f(x)|)\lambda g_r\| \\ &\leq (1-r) + 1 - r|f(x)| = 2 - r - r|f(x)|.\end{aligned}$$

Since $h_r \in \lambda V_x$, we have $\Delta(h_r)(y) = \Delta(h_r)(\phi(x)) = \alpha_x(\lambda)$ by (2.2). Therefore, we get

$$\begin{aligned}1 - |\Delta(f)(y)| = |\alpha_x(\lambda)| - |\Delta(f)(y)| &\leq |\alpha_x(\lambda) - \Delta(f)(y)| \\ &= |\Delta(h_r)(y) - \Delta(f)(y)| \\ &\leq \|\Delta(h_r) - \Delta(f)\| = \|h_r - f\| \\ &\leq 2 - r - r|f(x)|.\end{aligned}$$

Since r with $0 < r < 1$ is arbitrary, we get

$$1 - |\Delta(f)(y)| \leq |\alpha_x(\lambda) - \Delta(f)(y)| \leq 1 - |f(x)|, \tag{2.9}$$

which shows that $|f(x)| \leq |\Delta(f)(y)| = |\Delta(f)(\phi(x))|$. By similar arguments, applied to Δ^{-1} instead of Δ, we have $|u(y)| \leq |\Delta^{-1}(u)(\psi(y))|$ for all $u \in S(B)$. In particular, $|\Delta(f)(y)| \leq |\Delta^{-1}(\Delta(f))(\psi(y))| = |f(x)|$, and consequently

$$|\Delta(f)(y)| = |f(\psi(y))|, \text{ for all } y \in \mathrm{Ch}(B),\ f \in A.$$

Next, we shall prove that

$$\Delta(f)(y) = \alpha_x(\lambda)|f(x)|. \tag{2.10}$$

Since $|\Delta(f)(y)| = |f(x)|$, we need to consider the case when $\Delta(f)(y) \neq 0$. It follows from (2.9) that

$$\begin{aligned}1 = |\alpha_x(\lambda)| &\leq |\alpha_x(\lambda) - \Delta(f)(y)| + |\Delta(f)(y)| \\ &\leq (1 - |f(x)|) + |f(x)| = 1,\end{aligned}$$

which shows that

$$|\alpha_x(\lambda)| = |\alpha_x(\lambda) - \Delta(f)(y)| + |\Delta(f)(y)|.$$

By the equality condition for the triangle inequality, there exists $t \geq 0$ such that $\alpha_x(\lambda) - \Delta(f)(y) = t\Delta(f)(y)$. Hence, we have $\Delta(f)(y) = \alpha_x(\lambda)/(1+t)$. On the other hand,

$$|f(x)| = |\Delta(f)(y)| = \left|\frac{\alpha_x(\lambda)}{1+t}\right| = \frac{1}{1+t},$$

which yields $\Delta(f)(y) = \alpha_x(\lambda)|f(x)|$.

Now, having in mind (2.5), we define two subsets K_+ and K_- of $\mathrm{Ch}(A)$ by

$$K_+ = \{x \in \mathrm{Ch}(A) : \alpha_x(\lambda) = \alpha_x(1)\lambda, \text{ for all } \lambda\}, \quad \text{and}$$
$$K_- = \{x \in \mathrm{Ch}(A) : \alpha_x(\lambda) = \alpha_x(1)\overline{\lambda} \text{ for all } \lambda\}.$$

We see that $\mathrm{Ch}(A)$ is the disjoint union of K_+ and K_- (cf. (2.5)). Recall that $\lambda = f(x)/|f(x)|$ if $f(x) \neq 0$, and $\lambda = 1$ if $f(x) = 0$. We derive from (2.10) that

$$\Delta(f)(y) = \alpha_x(\lambda)|f(x)| = \begin{cases} \alpha_x(1)f(x), \text{ if } x \in K_+ \\ \alpha_x(1)\overline{f(x)}, \text{ if } x \in K_-. \end{cases}$$

Setting $L_+ = \psi^{-1}(K_+)$ and $L_- = \psi^{-1}(K_-)$, we infer from the above equality with $x = \psi(y)$ that

$$\Delta(f)(y) = \begin{cases} \alpha_{\psi(y)}(1)f(\psi(y)), \text{ if } y \in L_+ \\ \alpha_{\psi(y)}(1)\overline{f(\psi(y))}, \text{ if } y \in L_-. \end{cases} \tag{2.11}$$

It follows from the bijectivity of ψ that $\mathrm{Ch}(B)$ is the disjoint union of L_+ and L_-. Consider finally, the positive homogenous extension $T\colon A \to B$ defined by

$$T(g) = \begin{cases} \|g\|\, \Delta\left(\frac{g}{\|g\|}\right), & \text{if } g \in A \setminus \{0\} \\ 0, & \text{if } g = 0. \end{cases}$$

The identity in (2.11) shows that

$$T(g)(y) = \begin{cases} \alpha_{\psi(y)}(1)g(\psi(y)), & \text{if } y \in L_+ \\ \alpha_{\psi(y)}(1)\overline{g(\psi(y))}, & \text{if } y \in L_- \end{cases} \qquad (g \in A). \tag{2.12}$$

For each $h \in B \setminus \{0\}$, we put $h_0 = h/\|h\| \in S(B)$. We derive from the surjectivity of Δ that $h_0 = \Delta(g_0)$ for some $g_0 \in S(A)$. Set $g = \|h\|g_0$, and then, it follows from $\|g\| = \|h\|$ that

$$T(g) = \|g\|\Delta(g/\|g\|) = \|h\|\Delta(g_0) = h.$$

Hence, T is surjective. Choose $g_1, g_2 \in A$. Since $\psi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ is bijective, we infer from the previous identity (2.12) that

$$\|T(g_1) - T(g_2)\| = \sup_{y \in \mathrm{Ch}(B)} |T(g_1)(y) - T(g_2)(y)| = \sup_{y \in \mathrm{Ch}(B)} |g_1(\psi(y)) - g_2(\psi(y))|$$
$$= \sup_{x \in \mathrm{Ch}(A)} |g_1(x) - g_2(x)| = \|g_1 - g_2\|,$$

where we have used that $\mathrm{Ch}(B)$ and $\mathrm{Ch}(A)$ are norming sets for B and A, respectively in the first and fourth equalities (see page 45). Therefore, we see that T is a surjective real linear isometry by the Mazur–Ulam theorem [**32**, p8, Theorem 1.3.5]. $\square$

The final argument in the proof of Theorem 2.1 can be also deduced from [**57**, Lemma 6] or [**25**, Lemma 2.1], the identity in (2.11) and the fact that Choquet boundaries are boundaries, and thus norming sets.

Although we do not make any use of the maximal convex subsets of the unit sphere of a uniformly closed function algebra, nor of the deep result asserting that a surjective isometry between the unit spheres of two Banach spaces maps maximal convex subsets to maximal convex subsets (see [**14**, Lemma 5.1] and [**70**, Lemma 3.5]), the conclusion in [**71**, Lemma 3.3] (see also [**38**, Lemma 3.1]) can be applied to deduce that every maximal convex subset $\mathcal{C}$ of the unit sphere of uniformly closed function algebra A on a locally compact Hausdorff space X is of the form

$$\mathcal{C} = \lambda V_x = \{f \in S(A) : f(x) = \lambda\},$$

for some $\lambda \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$ (this can be compared with [**38**, Lemma 3.2]).

3. Tingley's problem for commutative JB*-triples

Despite of having their own worth to be studied as important function spaces, there exist certain function spaces which are not given solution to their Tingley's problems. One example appears in the Gelfand representation for commutative JB*-triples. As a brief introduction, we shall mention that these complex spaces arose in holomorphic theory in the study and classification of bounded symmetric domains in arbitrary complex Banach spaces. These domains are the appropriate substitutes of simply connected domains to extend the Riemann mapping theorem to dimension greater than or equal to 2 (cf. [**43**] or the detailed presentation in [**11**, §5.6]).

For the sake of brevity, we shall omit a detailed presentation of the theory for general JB*-triples. However, it is worth recalling that the elements in the subclass of commutative JB*-triples can be represented as spaces of continuous functions by the Gelfand theory of JB*-triples for the purpose of this note (cf. [**43**], [**79**], [**8**, §3], [**10**, §4.2.1]). Indeed, let X be a subset of a Hausdorff locally convex complex space such that $0 \notin X$, $X \cup \{0\}$ is compact, and $\mathbb{T}X \subseteq X$, where $\mathbb{T} := \{\lambda \in \mathbb{T} : |\lambda| = 1\}$. Let us observe that under these hypotheses, $\lambda x = \mu x$ for $x \in X$, $\lambda, \mu \in \mathbb{T}$ implies $\lambda = \mu$. The space X is called a *principal* $\mathbb{T}$-*bundle* in [**43**].

A locally compact $\mathbb{T}_\sigma$-space is a locally compact Hausdorff space X together with a continuous mapping $\mathbb{T} \times X \to X$, $(\lambda, x) \mapsto \lambda x$, satisfying $\lambda(\mu x) = (\lambda\mu)x$ and $1x = x$ for all $\lambda, \mu \in \mathbb{T}$, $x \in X$. Each principal $\mathbb{T}$-bundle is a locally compact $\mathbb{T}_\sigma$-space. We can extend the product by elements in $\mathbb{T}$ to the one-point compactification $X \cup \{\omega\}$ of X by setting $\lambda\omega = \omega$ for all $\lambda \in \mathbb{T}$. We now consider the following subspace of continuous functions on a locally compact $\mathbb{T}_\sigma$-space X

$$C_0^{\mathbb{T}}(X) := \{a \in C_0(X) : a(\lambda t) = \lambda a(t) \text{ for every } (\lambda, t) \in \mathbb{T} \times X\}.$$

We shall regard $C_0^{\mathbb{T}}(X)$ as a norm closed subspace of $C_0(X)$ with the supremum norm. We observe that every $C_0(L)$ space is a $C_0^{\mathbb{T}}(X)$ space (cf. [**58**, Proposition 10]). However, there exist examples of principle $\mathbb{T}$-bundles X for which the space $C_0^{\mathbb{T}}(X)$ is not isometrically isomorphic to a $C_0(L)$ space (cf. [**43**, Corollary 1.13 and subsequent comments]). $C_0^{\mathbb{T}}(X)$ spaces, with X a locally compact $\mathbb{T}_\sigma$-space, are directly related to Lindenstrauss spaces (see [**58**, Theorem 12]).

Let us now fix a locally compact $\mathbb{T}_\sigma$-space X. We denote by $(C_0^{\mathbb{T}}(X)^*)_1$ the closed unit ball of the dual space of $C_0^{\mathbb{T}}(X)$. Although $C_0^{\mathbb{T}}(X)$ need not be a subalgebra of $C_0(X)$, it is closed for the triple product defined by $\{f, g, h\} = f\overline{g}h$ for $f, g, h \in C_0^{\mathbb{T}}(X)$. We shall write $f^{[1]} = f$, $f^{[3]} = \{f, f, f\}$ and $f^{[2n+1]} = \{f, f, f^{[2n-1]}\}$ for all natural n. For each $x \in X$, the mapping $\delta_x : C_0^{\mathbb{T}}(X) \to \mathbb{C}$ defined by $\delta_x(f) = f(x)$ for $f \in C_0^{\mathbb{T}}(X)$ is a bounded linear functional in $(C_0^{\mathbb{T}}(X)^*)_1$. According to the Arens–Kelley's theorem ([**32**, Theorem 2.3.5]), we see that δ_t for each $t \in L$ is an extreme point of the closed unit ball of the dual space of $C_0(L)$. However, this is not always true in the case of $C_0^{\mathbb{T}}(X)$. For example, if $x_0 \in X$ satisfies that $x_0 \in (\mathbb{T} \setminus \{1\})x_0$, that is, there exists $\lambda_0 \in \mathbb{T} \setminus \{1\}$ such that $\delta_{x_0} = \lambda_0 \delta_{x_0}$, then it is easy to check that $\delta_{x_0} = 0$ as a linear functional in $(C_0^{\mathbb{T}}(X)^*)_1$. By [**58**, Lemma 11], the extreme points of $(C_0^{\mathbb{T}}(X)^*)_1$ are precisely those δ_{x_0} which are non-zero, that is,

$$\operatorname{ext}(C_0^{\mathbb{T}}(X)^*)_1 = \{\delta_x \ : \ x \notin (\mathbb{T} \setminus \{1\})x\}. \tag{3.1}$$

We note that the set $\operatorname{ext}(C_0^{\mathbb{T}}(X)^*)_1$ is norming and a kind of Chouet boundary for $C_0^{\mathbb{T}}(X)$.

Those complex Banach spaces called JB*-triples are precisely the complex Banach spaces whose unit ball is a bounded symmetric domain, and were introduced by W. Kaup in [**43**] to classify these domains, and to establish a generalization of Riemann mapping theorem in dimension ≥ 2. A JB*-triple is a complex Banach space E admitting a continuous triple product $\{\cdot, \cdot, \cdot\} : E \times E \times E \to E$, which is symmetric and linear in the outer variables, conjugate-linear in the middle one, and satisfies the following axioms:

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, for all a,b,x,y in E, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(b) For all $a \in E$, $L(a,a)$ is a hermitian operator with non-negative spectrum;

(c) $\|\{a,a,a\}\| = \|a\|^3$, for all $a \in E$.

The class of JB*-triples includes all C*-algebras and all JB*-algebras (cf. [**43**, pages 522, 523 and 525]).

A JB*-triple E is called *abelian* or *commutative* if the set $\{L(a,b) : a,b \in E\}$ is a commutative subset of the space $\mathcal{B}(E)$ of all bounded linear operators on E (cf. [**43**, §1], [**10**, §4.1.47] or [**34**, §4] where commutative JB*-triples are called "associative"). Despite of the technical definition, every commutative JB*-triple can be isometrically represented, via a triple isomorphism, that is, a linear bijection preserving the triple product as a space of the form $C_0^{\mathbb{T}}(X)$ for a suitable principal $\mathbb{T}$-bundle X (see [**43**, Corollary 1.1], [**10**, Theorem 4.2.7], see also the interesting representation theorems in [**33**, §3] and [**34**, §4]).

Let X and Y be two principal $\mathbb{T}$-bundles. Each surjective complex linear isometry T from $C_0^{\mathbb{T}}(X)$ onto $C_0^{\mathbb{T}}(Y)$ is a triple isomorphism (i.e., it preserves the triple product seen above). Furthermore, that is the case, if and only if, there exists a $\mathbb{T}$-equivariant homeomorphism $\phi : Y \to X$ (i.e., $\phi(\lambda s) = \lambda\phi(s)$, for all $(\lambda, s) \in \mathbb{T} \times Y$) such that $T(f)(s) = f(\phi(s))$, for all $s \in Y$ and $f \in C_0^{\mathbb{T}}(X)$ (see [**43**, Proposition 1.12]). That is, surjective linear isometries and triple isomorphisms coincide, and they are precisely the composition operators with a $\mathbb{T}$-equivariant homeomorphism between the principle $\mathbb{T}$-bundles.

In some of the result of this section, we can apply tools and techniques in the theory of general JB*-triples. However, since the commutative objects of this category admit a concrete representation as function spaces, we strive for presenting basic arguments which do not require any knowledge on the general theory.

Our next goal will consist in determining the explicit form of all real linear isometries between $C_0^{\mathbb{T}}(X)$ spaces for principal $\mathbb{T}$-bundles (i.e. abelian JB*-triples), a description which materializes and concretizes the theoretical conclusions for real linear surjective isometries on C*-algebras and JB*-triples in [**16, 21**].

We recall a fundamental property of $C_0^{\mathbb{T}}(X)$. Let $\boldsymbol{\mu}$ denote the unit Haar measure on $\mathbb{T}$. For each $f \in C_0(X)$, we consider a function $\pi_{\mathbb{T}}(f) : X \to \mathbb{C}$ defined by

$$\pi_{\mathbb{T}}(f)(t) = \int_{\mathbb{T}} \lambda^{-1} f(\lambda t) d\boldsymbol{\mu}, \quad (t \in X).$$

It is known that $\pi_{\mathbb{T}}$ is a contractive projection of $C_0(X)$ onto $C_0^{\mathbb{T}}(X)$ (cf. [**58**]).

Remark 2.21. Suppose X is a locally compact $\mathbb{T}_\sigma$-space. Let W be a $\mathbb{T}$-invariant open neighbourhood of t_0 in X which is contained in a compact $\mathbb{T}$-invariant subset. We consider the following continuous function

$$\mathbb{T}\{t_0\} \cup (X \backslash W) \to \mathbb{C},$$

$$\lambda t_0 \mapsto \lambda, \text{ and } t \mapsto 0 \text{ for all } t \in X \backslash W.$$

Find, via Tiezte's theorem, a continuous function $\tilde{h} \in C_0(X)$ extending the previous mapping. Let $h = \pi_{\mathbb{T}}(\tilde{h}) \in C_0^{\mathbb{T}}(X)$. It is easy to check that $h(t_0) = 1$ and $h(t) = 0$ for all $t \in X \backslash W$. Clearly, $h \in S(C_0^{\mathbb{T}}(X))$ and $h(\lambda t_0) = \lambda$ for all $\lambda \in \mathbb{T}$. This construction, which was already considered in [**58**, Proof of Lemma 11], is a kind of Urysohn's lemma for $C_0^{\mathbb{T}}(X)$ spaces, and will be employed along this section.

In order to determine the form of all real linear isometries between $C_0^{\mathbb{T}}(X)$ and $C_0^{\mathbb{T}}(Y)$, we prepare some lemmas. First, we prove that if $x_1, x_2 \in X$ such that $\mathbb{T}\{x_1\} \cap \mathbb{T}\{x_2\}$ is empty, then we can choose some $\mathbb{T}$-invariant open neiborhood W of x_1 which is contained in a compact $\mathbb{T}$-invariant subset with $W \subset X \setminus \mathbb{T}\{x_2\}$.

Lemma 2.22. *Let $x_1 \in X$ and K a compact $\mathbb{T}$-invariant subset of X such that $\mathbb{T}\{x_1\} \cap K$ is empty. Then there exists a $\mathbb{T}$-invariant open neiborhood W of x_1 which is contained in a compact $\mathbb{T}$-invariant subset such that $W \subset X \setminus K$.*

Proof. Since X is a locally compact Hausdorff space and K is a compact subset, we can choose an open neighborhood V_0 of x_1 such that $V_0 \subset \mathrm{cl}(V_0) \subset X \setminus K$ and $\mathrm{cl}(V_0)$ is compact, where $\mathrm{cl}(V_0)$ denotes the closure of V_0. We note that $\mathbb{T}\{x_1\} \subset \mathbb{T}V_0$. For each $\lambda \in \mathbb{T}$, we define a map $\sigma_\lambda : X \to X$ by $\sigma_\lambda(x) = \lambda x$ for $x \in X$. Having in mind that $\sigma_\lambda : X \to X$ is continuous for each $\lambda \in \mathbb{T}$, we infer from $\sigma_{\overline{\lambda}}(\sigma_\lambda(x)) = x = \sigma_\lambda(\sigma_{\overline{\lambda}}(x))$ for any $x \in X$ that σ_λ is a homeomorphism on X. Hence, $\lambda V_0 = \sigma_\lambda(V_0)$ is an open neiborhood of λx_1. We put $W = \mathbb{T}V_0$, and then, we infer from $W = \mathbb{T}V_0 = \bigcup_{\lambda \in \mathbb{T}} \lambda V_0$ that W is an open neiborhood of x_1. Put $V_1 = \mathrm{cl}(V_0)$, and then, V_1 is compact by the choice of V_0. Because the mapping $\sigma : \mathbb{T} \times X \to X$, defined by $\sigma(\lambda, x) = \lambda x$ for $(\lambda, x) \in \mathbb{T} \times X$, is continuous and $\mathbb{T} \times V_1$ is a compact subset of $\mathbb{T} \times X$, we deduce from $\mathbb{T}V_1 = \sigma(\mathbb{T} \times V_1)$ that $\mathbb{T}V_1$ is a compact subset of X. Since K is a compact $\mathbb{T}$-invariant subset of X, we derive from $V_1 = \mathrm{cl}(V_0) \subset X \setminus K$ that $\mathbb{T}V_1 \subset X \setminus K$. We note that $V_0 \subset V_1$ and $W = \mathbb{T}V_0$. Therefore, we conclude that W is an

$\mathbb{T}$-invariant neiborhood of x in X with $W \subset X \setminus K$ which is contained in a compact $\mathbb{T}$-invariant subset $\mathbb{T}V_1$ of X. $\square$

LEMMA 2.23. *Let X be a principal $\mathbb{T}$-bundle. We define $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ by*

$$\tau_X(x) = \delta_x \qquad (x \in X).$$

Then the mapping τ_X is a homeomorphism from X onto $\mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ with the relative weak*-topology.

PROOF. By the assumption that X is a principal $\mathbb{T}$-bundle, we note that X is a locally compact $\mathbb{T}_\sigma$-space. Because $\lambda x = \mu x$ for $x \in X$ $\lambda, \mu \in \mathbb{T}$ implies $\lambda = \mu$, we see that $x \notin \mathbb{T}\setminus\{1\}x$ for all $x \in X$. It follows from (3.1) that

$$\mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1 = \{\delta_x \ : \ x \in X\}. \tag{3.2}$$

Hence, the map $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X))^*$ is well-defined and surjective.

We prove that τ_X is injective. Choose $x, y \in X$ with $x \neq y$ and fix them. If $x = \lambda_0 y$ for some $\lambda_0 \in \mathbb{T} \setminus \{1\}$, then we infer from Remark 2.21 that there exists $h \in S(C_0^{\mathbb{T}}(X))$ such that $h(y) = 1$. Having in mind that $x = \lambda_0 y$ and $h(\lambda_0 y) = \lambda_0 h(y)$, we get

$$\delta_x(h) = h(x) = h(\lambda_0 y) = \lambda_0 \neq 1 = h(y) = \delta_y(h),$$

and thus, $\tau_X(x) = \delta_x \neq \delta_y = \tau_X(y)$. Next, we assume that $x \notin \mathbb{T}\{y\}$. Then $\mathbb{T}\{x\} \cap \mathbb{T}\{y\}$ is empty. Since $\mathbb{T}\{y\}$ is a compact $\mathbb{T}$-invariant subset of X such that $\mathbb{T}\{x\} \cap \mathbb{T}\{y\}$ is empty, it follows from Lemma 2.22 that we can choose an $\mathbb{T}$-invariant open neiborhood W of x which is contained in a compact $\mathbb{T}$-invariant subset such that $W \subset X \setminus \mathbb{T}\{y\}$. Applying Remark 2.21 to $\mathbb{T}\{x\}$ with W, we can choose $h \in S(C_0^{\mathbb{T}}(X))$ such that $h(x) = 1$ and $h(y) = 0$. This implies that

$$\delta_x(h) = h(x) = 1 \neq 0 = h(y) = \delta_y(h),$$

and hence, $\tau_X(x) = \delta_x \neq \delta_y = \tau_X(y)$. Therefore, we observe that $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ injective.

Finally, we prove that $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ and $\tau_X^{-1} : \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1 \to X$ are continuous. Take any element $x \in X$ and net $\{x_\gamma\}_{\gamma\in\Gamma}$ which converges to x. For all $f \in C_0^{\mathbb{T}}(X)$, we have

$$\lim_{\gamma\in\Gamma} \tau_X(x_\gamma)(f) = \lim_{\gamma\in\Gamma} \delta_{x_\gamma}(f) = \lim_{\gamma\in\Gamma} f(x_\gamma) = f(x) = \delta_x(f) = \tau_X(x)(f).$$

We derive from the definition of the weak*-topology that $\lim_{\gamma\in\Gamma} \tau_X(x_\gamma) = \tau_X(x)$. Hence, $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ is continuous. On the other hand, we take $\delta_x \in \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$

arbitrarily. Let $\{\delta_{x_\gamma}\}_{\gamma\in\Gamma}$ be a net which converges to $\delta_x \in C_0^{\mathbb{T}}(X)^*$ with respect to the relative weak*-topology. By the definition of the weak*-topology, we have

$$\lim_{\gamma\in\Gamma} f(x_\gamma) = \lim_{\gamma\in\Gamma} \delta_{x_\gamma}(f) = \delta_x(f) = f(x) \tag{3.3}$$

for every $f \in C_0^{\mathbb{T}}(X)$. Having in mind that $C_0^{\mathbb{T}}(X)$ strongly separates the points of X by Remark 2.21, we deduce from [**42**, Proposition 2.2.14] that the topology of X is equivalent to the weak topology induced by $\{f : f \in C_0^{\mathbb{T}}(X)\}$, and thus, it follows fron (3.3) that $x_\gamma \to x$ with respect to the original topology in X. Therefore, we conclude that τ_X^{-1} is continuous. The proof is complete. $\square$

Next, we explore the correspondence between $C_0^{\mathbb{T}}(Y)^*$ and $C_0^{\mathbb{T}}(X)^*$ induced by a surjective real linear isometry T between $C_0^{\mathbb{T}}(X)$ and $C_0^{\mathbb{T}}(Y)$, where X and Y are principal $\mathbb{T}$-bundles. Because $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ is not necessarily complex linear, the adjoint operator $T^* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ is not well-defined. In place of T^*, we define $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ by

$$T_*(\eta)(f) = \mathrm{Re}\big(\eta(T(f))\big) - i\mathrm{Re}\big(\eta(T(if))\big) \quad (\eta \in C_0^{\mathbb{T}}(Y)^*,\ f \in C_0^{\mathbb{T}}(X)). \tag{3.4}$$

It is well known that $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ is a surjective real linear isometry (see [**67**, Proposition 5.17] and [**55**]). We see that T_* preserves the extreme points of the closed unit ball of the dual spaces, that is, $T_*(\mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1) = \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$.

Let $\tau_X : X \to \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$ and $\tau_Y : Y \to \mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1$ be as in Lemma 2.23. So as to characterize T_* on $\mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1$, we define a map $\tau : Y \to X$ by $\tau = \tau_X^{-1} \circ T_*|_{\mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1} \circ \tau_Y$. We note that $\tau : Y \to X$ is a homeomorphism by Lemma 2.23 and $T_* : C_0^{\mathbb{T}}(Y) \to C_0^{\mathbb{T}}(X)$ is a surjective isometry with $T_*(\mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1) = \mathrm{ext}(C_0^{\mathbb{T}}(X)^*)_1$. It follows from the definition of τ that $T_*|_{\mathrm{ext}(C_0^{\mathbb{T}}(Y)^*)_1} \circ \tau_Y = \tau_X \circ \tau$, and thus,

$$T_*(\delta_y) = \delta_{\tau(y)} \qquad (y \in Y). \tag{3.5}$$

In the following two lemmas, we investigate the property of the homeomorphism $\tau : Y \to X$ induced by a surjective real linear isometry $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ as in the last paragraph.

LEMMA 2.24. *Let $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ be a surjective real linear isometry and $\tau : Y \to X$ a homeomorphism induced by the map $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$. For each $y \in Y$, we have*

$$\tau(\mathbb{T}\{y\}) \subset \mathbb{T}\{\tau(y)\}.$$

Proof. Let $y \in Y$ and fix it. First, we prove that $\tau(\lambda y) \in \mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\}$ for all $\lambda \in \mathbb{T}$. Suppose that there exists $\lambda_0 = a_0 + ib_0 \in \mathbb{T} \setminus \{1, i\}$ such that

$$\tau(\lambda_0 y) \notin \mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\}. \tag{3.6}$$

Having in mind that $\delta_{\lambda_0 y}(g) = g(\lambda_0 y) = \lambda_0 g(y) = \lambda_0 \delta_y(g)$ for all $g \in C_0^{\mathbb{T}}(Y)$, we get $\delta_{\lambda_0 y} = \lambda_0 \delta_y$. Because T_* is a surjective real linear isometry, we obtain

$$T_*(\delta_{\lambda_0 y}) = T_*(\lambda_0 \delta_y) = T_*(a_0 \delta_y) + T_*(ib_0 \delta_y) = a_0 T_*(\delta_y) + b_0 T_*(\delta_{iy}),$$

and thus $T_*(\delta_{\lambda_0 y}) = a_0 T_*(\delta_y) + b_0 T_*(\delta_{iy})$. We note that $T_*(\delta_{\lambda_0 y}) = \delta_{\tau(\lambda_0 y)}$, $T_*(\delta_y) = \delta_{\tau(y)}$, and $T_*(\delta_{iy}) = \delta_{\tau(iy)}$ by (3.5). Entering these three equalities into $T_*(\delta_{\lambda_0 y}) = a_0 T_*(\delta_y) + b_0 T_*(\delta_{iy})$, we have

$$\delta_{\tau(\lambda_0 y)} = a_0 \delta_{\tau(y)} + b_0 \delta_{\tau(iy)}. \tag{3.7}$$

It follows from (3.6) that $\mathbb{T}\{\tau(\lambda_0 y)\} \cap (\mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\})$ is empty. Because $\mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\}$ is a compact $\mathbb{T}$-invariant subset, we deduce from Remark 2.21 with Lemma 2.22 that there exists $g_1 \in C_0^{\mathbb{T}}(Y)$ such that $g_1(\tau(\lambda_0 y)) = 1$ and $g(\tau(y)) = 0 = g(\tau(iy)$. Evaluating the equality (3.7) at g_1, we get $1 = 0$. This is a contradiction. Hence, we must have

$$\tau(\lambda y) \in \mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\} \qquad (\lambda \in \mathbb{T}). \tag{3.8}$$

Since the map $\lambda \mapsto \tau(\lambda y)$ is a continuous map between $\mathbb{T}$ into X and $\mathbb{T}$ is a connected set, the subset $\tau(\mathbb{T}\{y\})$ of X is also connected. We deduce from the continuity of the scalar multiplication that $\mathbb{T}\tau(\mathbb{T}\{y\})$ is a connected subset of X. It follows from (3.8) that $\mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\} = \mathbb{T}\tau(\mathbb{T}\{y\})$, and thus, $\mathbb{T}\{\tau(y)\} \cup \mathbb{T}\{\tau(iy)\}$ is connected. This shows that $\mathbb{T}\{\tau(y)\} \cap \mathbb{T}\{\tau(iy)\}$ is not empty. Hence $\tau(iy) = \lambda_i \tau(y)$ for some $\lambda_i \in \mathbb{T}$, which implies that $\mathbb{T}\{\tau(iy)\} = \mathbb{T}\{\tau(y)\}$. We infer from (3.8) with the last equality that $\tau(\mathbb{T}\{y\}) \subset \mathbb{T}\{\tau(y)\}$. $\square$

Lemma 2.25. *Let $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ be a surjective real linear isometry and $\tau : Y \to X$ a homeomorphism induced by $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$. There exists a function $\varepsilon : Y \to \{\pm 1\}$ such that*

$$\tau(\lambda y) = \lambda^{\varepsilon(y)} \tau(y) \qquad (\lambda \in \mathbb{T},\ y \in Y).$$

Proof. Let $y \in Y$ and fix it. Note that $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ is real linear. We infer from $\delta_{\lambda y} = \lambda \delta_y$ for $\lambda \in \mathbb{T}$ that

$$T_*(\delta_{\lambda y}) = T_*((a + ib)\delta_y) = aT_*(\delta_y) + bT_*(i\delta_y) = aT_*(\delta_y) + bT_*(\delta_{iy}),$$

and thus, $T_*(\delta_{\lambda y}) = aT_*(\delta_y) + bT_*(\delta_{iy})$ for each $\lambda = a + ib \in \mathbb{T}$. Having in mind that $T_*(\delta_{\lambda y}) = \delta_{\tau(\lambda y)}$ by (3.5), we deduce from the last equality that

$$\delta_{\tau(\lambda y)} = a\delta_{\tau(y)} + b\delta_{\tau(iy)} \qquad (\lambda = a + ib \in \mathbb{T}). \tag{3.9}$$

It follows from Lemma 2.24 that there exists $\mu_\lambda \in \mathbb{T}$ such that $\tau(\lambda y) = \mu_\lambda \tau(y)$ for each $\lambda \in \mathbb{T}$. Having in mind that $\delta_{\tau(\lambda y)} = \delta_{\mu_\lambda \tau(y)} = \mu_\lambda \delta_{\tau(y)}$ for $\lambda \in \mathbb{T}$, we can rewrite the equality (3.9) as

$$\mu_\lambda \delta_{\tau(y)} = a\delta_{\tau(y)} + b\mu_i \delta_{\tau(y)} \qquad (\lambda = a + ib \in \mathbb{T}). \tag{3.10}$$

Entering $\lambda_0 = (1 - i)/\sqrt{2} \in \mathbb{T}$ into (3.10), we obtain

$$\mu_{\lambda_0} \delta_{\tau(y)} = \frac{1}{\sqrt{2}} \delta_{\tau(y)} - \frac{1}{\sqrt{2}} \mu_i \delta_{\tau(y)}.$$

We take $f_0 \in C_0^{\mathbb{T}}(X)$ with $f_0(\tau(y)) = 1$. Evaluating the last equality at f_0, we obtain

$$\sqrt{2} = |\sqrt{2}\mu_{\lambda_0} f_0(\tau(y))| = |f_0(\tau(y)) - \mu_i f_0(\tau(y))| = |1 - \mu_i|.$$

We derive from $\mu_i \in \mathbb{T}$ that $\mu_i = i$ or $-i$. Hence, we derive from $\tau(iy) = \mu_i \tau(y)$ that $\tau(iy) = i\tau(y)$ or $\tau(iy) = -i\tau(y)$. There exists $\varepsilon(y) \in \{\pm 1\}$ such that $\tau(iy) = \varepsilon(y) i \tau(y)$ for each $y \in Y$. Because $\delta_{\mu_\lambda \tau(y)} = \mu_\lambda \delta_{\tau(y)}$ for $\lambda \in \mathbb{T}$, we have

$$\mu_i \delta_{\tau(y)} = \delta_{\mu_i \tau(y)} = \delta_{\tau(iy)} = \delta_{\varepsilon(y) i \tau(y)} = \varepsilon(y) i \delta_{\tau(y)},$$

and thus, $\mu_i \delta_{\tau(y)} = \varepsilon(y) i \delta_{\tau(y)}$. We deduce from (3.10) that

$$\mu_\lambda \delta_{\tau(y)} = (a + \varepsilon(y) ib) \delta_{\tau(y)} = \lambda^{\varepsilon(y)} \delta_{\tau(y)}$$

for all $\lambda = a + ib \in \mathbb{T}$. We choose $f_1 \in C_0^{\mathbb{T}}(X)$ such that $f_1(\tau(y)) = 1$. Evaluating the last equality at f_1, we obtain $\mu_\lambda = \lambda^{\varepsilon(y)}$. By the choice of $\mu_\lambda \in \mathbb{T}$, $\tau(\lambda y) = \mu_\lambda \tau(y) = \lambda^{\varepsilon(y)} \tau(y)$ for all $\lambda \in \mathbb{T}$. Since $y \in Y$ is arbitrarily chosen, the proof is complete. $\square$

We are now in position to determine the form of surjective real linear isometries between two $C_0^{\mathbb{T}}(X)$ spaces. We denote by $[z]^1 = 1$ and $[z]^{-1} = \overline{z}$ for $z \in \mathbb{C}$.

LEMMA 2.26. *If $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ be a surjective real linear isometry, then there exist a homeomorphism $\tau : Y \to X$ and a $\mathbb{T}$-invariant clopen subset D of Y satysfing*

$$T(f)(y) = f(\tau(y)), \quad \tau(\lambda y) = \lambda \tau(y) \qquad (f \in C_0^{\mathbb{T}}(X), \lambda \in \mathbb{T}, y \in D) \quad \textit{and}$$

$$T(f)(y) = \overline{f(\tau(y))}, \quad \tau(\lambda y) = \overline{\lambda} \tau(y) \qquad (f \in C_0^{\mathbb{T}}(X), \lambda \in \mathbb{T}, y \in Y \setminus D).$$

Proof. Let $\tau : Y \to X$ be a homeomorphism induced by $T_* : C_0^{\mathbb{T}}(Y)^* \to C_0^{\mathbb{T}}(X)^*$ and $\varepsilon : Y \to \{\pm 1\}$ a function as in Lemma 2.25. Take $y \in Y$, $f \in C_0^{\mathbb{T}}(X)$ arbitrarily and fix them. Entering $\eta = \delta_y, \delta_{iy}$ into (3.4) respectively, we get the following two equalities;

$$T_*(\delta_y)(f) = \mathrm{Re}\big(\delta_y(T(f))\big) - i\mathrm{Re}\big(\delta_y(T(if))\big),$$
$$T_*(\delta_{iy})(f) = \mathrm{Re}\big(\delta_{iy}(T(f))\big) - i\mathrm{Re}\big(\delta_{iy}(T(if))\big).$$

Taking the real part in the above two equalities, we obtain

$$\mathrm{Re}\big(T_*(\delta_y)(f)\big) = \mathrm{Re}\big(\delta_y(T(f))\big) \quad \text{and} \quad \mathrm{Re}\big(T_*(\delta_{iy})(f)\big) = \mathrm{Re}\big(\delta_{iy}(T(f))\big).$$

By the equality (3.5), we can rewrite the last two equalities as

$$\mathrm{Re}\big(f(\tau(y))\big) = \mathrm{Re}\big(T(f)(y)\big) \quad \text{and} \quad \mathrm{Re}\big(f(\tau(iy))\big) = \mathrm{Re}\big(iT(f)(y)\big). \tag{3.11}$$

Having in mind that $\tau(iy) = \varepsilon(y)i\tau(y)$ by Lemma 2.25, we infer from $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ for $z \in \mathbb{C}$ with the second equality in (3.11) that

$$-\mathrm{Im}(T(f)(y)) = \mathrm{Re}(iT(f)(y)) = \mathrm{Re}(f(\tau(iy))) = \mathrm{Re}(\varepsilon(y)if(\tau(y))) = -\varepsilon(y)\mathrm{Im}(f(\tau(y))).$$

Consequently, $\mathrm{Im}(T(f)(y)) = \varepsilon(y)\mathrm{Im}(f(\tau(y)))$. Combining the last equality with the first equality in (3.11), we obtain

$$\begin{aligned} T(f)(y) &= \mathrm{Re}\big(T(f)(y)\big) + i\mathrm{Im}\big(T(f)(y)\big) \\ &= \mathrm{Re}\big(f(\tau(y))\big) + \varepsilon(y)i\mathrm{Im}\big(f(\tau(y))\big) = [f(\tau(y))]^{\varepsilon(y)}, \end{aligned}$$

and thus, $T(f)(y) = [f(\tau(y))]^{\varepsilon(y)}$. Because $y \in Y$ and $f \in C_0^{\mathbb{T}}(X)$ are arbitrary, we have

$$T(f)(y) = [f(\tau(y))]^{\varepsilon(y)} \qquad (f \in C_0^{\mathbb{T}}(X), y \in Y). \tag{3.12}$$

We set $D = \{y \in Y \ : \ \varepsilon(y) = 1\}$. Having in mind that $\varepsilon(Y) = \{1, -1\}$, we see that $Y \setminus D = \{y \in Y \ : \ \varepsilon(y) = -1\}$. We shall prove that $\varepsilon : Y \to \{\pm 1\}$ is continuous and the subset D is a $\mathbb{T}$-invariant clopen subset of Y in the rest part.

Now, we prove that $\varepsilon : Y \to \{\pm 1\}$ is continuous on Y. Let $y_0 \in Y$ and fix it. We take any net $\{y_\gamma\}_{\gamma \in \Gamma}$ in Y which converges to y_0. Choose $g \in C_0^{\mathbb{T}}(Y)$ with $g(y_0) = 1$. Since $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ is surjective, there exists $f \in C_0^{\mathbb{T}}(X)$ such that $T(f) = g$. We deduce from (3.12) that

$$T(if)(y) = [if(\tau(y))]^{\varepsilon(y)} = \varepsilon(y)i[f(\tau(y))]^{\varepsilon(y)} = \varepsilon(y)iT(f)(y) = \varepsilon(y)ig(y),$$

and hence, $T(if)(y) = \varepsilon(y)ig(y)$ for all $y \in Y$. Having in mind that $\lim_{\gamma\in\Gamma} y_\gamma = y_0$ with $g(y_0) = 1$, we may assume that $g(y_\gamma) \neq 0$ for all $\gamma \in \Gamma$. It follows from the last equality that

$$\lim_{\gamma\in\Gamma} \varepsilon(y_\gamma) = \lim_{\gamma\in\Gamma} \bar{i}\frac{T(if)(y_\gamma)}{g(y_\gamma)} = \bar{i}\frac{T(if)(y)}{g(y)} = \varepsilon(y).$$

Therefore, we conclude that $\varepsilon : Y \to \{\pm 1\}$ is continuous. Because $\varepsilon : Y \to \{\pm 1\}$ is a continuous function on Y, we see that D and $Y \setminus D$ are closed subset in Y, and hence, the subset D is a clopen subset in Y.

Finally, we prove that the subset D is $\mathbb{T}$-invariant. Let $y_0 \in D$ and fix it. Choose $\lambda \in \mathbb{T}$ arbitrarily. It follows from Lemma 2.25 with $\varepsilon(y_0) = 1$ that $\tau(\lambda y_0) = \lambda_0\tau(y_0)$ and $\tau((i\lambda)y_0) = i\lambda\tau(y_0)$. Having in mind that $i^{\varepsilon(\lambda y_0)}\tau(\lambda y_0) = \tau(i(\lambda y_0))$ by Lemma 2.25, these two equalities imply that

$$\varepsilon(\lambda y_0)i\tau(\lambda y_0) = i^{\varepsilon(\lambda y_0)}\tau(\lambda y_0) = \tau((i\lambda)y_0) = (i\lambda)\tau(y_0) = i(\lambda\tau(y_0)) = i\tau(\lambda y_0),$$

and thus, $\varepsilon(\lambda y_0)i\tau(\lambda y_0) = i\tau(\lambda y_0)$. This implies that $\varepsilon(\lambda y_0)i\delta_{\tau(\lambda y_0)} = i\delta_{\tau(\lambda y_0)}$, where we have used $\mu\delta_{\tau(y_0)} = \delta_{\mu\tau(y_0)}$ for $\mu \in \mathbb{T}$. We take $g \in C_0^{\mathbb{T}}(X)$ such that $g(\tau(\lambda y_0)) = \bar{i}$, and then, we derive from $\varepsilon(\lambda y_0)i\delta_{\tau(\lambda y_0)} = i\delta_{\tau(\lambda y_0)}$ that $\varepsilon(\lambda y_0) = \varepsilon(\lambda y_0)i\delta_{\tau(\lambda y_0)}(g) = i\delta_{\tau(\lambda y_0)}(g) = 1$. Since $\lambda \in \mathbb{T}$ is arbitrarily chosen, we conclude that $\varepsilon(\lambda y_0) = 1$ for all $\lambda \in \mathbb{T}$. This shows that $\mathbb{T}\{y_0\} \subset D$ for every $y_0 \in D$. Therefore, we conclude that the clopen subset D is $\mathbb{T}$-invariant. The proof is complete. $\square$

Let us continue with a rudimentary *continuous triple functional calculus* in our setting. Let $\mathcal{B}_{\mathbb{C}}$ denote the closed unit ball of $\mathbb{C}$, regarded as principal $\mathbb{T}$-bundle. For each $f \in C_0^{\mathbb{T}}(X)$ with $\|f\| \leq 1$ and each h in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}) = \{h \in C(\mathcal{B}_{\mathbb{C}}) : h(0) = 0,\ h(\lambda\zeta) = \lambda h(\zeta),\ \lambda \in \mathbb{T}, \zeta \in \mathcal{B}_{\mathbb{C}}\}$, the composition $h \circ f$ lies in $C_0^{\mathbb{T}}(X)$, and it will be denoted by $h_t(f) = h \circ f$. This coincides with the so-called continuous triple functional calculus in the wider setting of JB*-triples. For each $n \in \mathbb{N} \cup \{0\}$, we define $h_n : \mathcal{B}_{\mathbb{C}} \to \mathbb{C}$ by $h_n(\zeta) = |\zeta|^{2n}\zeta$ for $\zeta \in \mathcal{B}_{\mathbb{C}}$. We observe that $(h_n)_t(f) = f^{[2n+1]}$ for each f in the closed unit ball of $C_0^{\mathbb{T}}(X)$.

Let $(C_0^{\mathbb{T}}(X))_1$ be the closed unit ball of $C_0^{\mathbb{T}}(X)$. A face V of $(C_0^{\mathbb{T}}(X))_1$ is a convex subset of $(C_0^{\mathbb{T}}(X))_1$ such that if $f_1, f_2 \in (C_0^{\mathbb{T}}(X))_1$ with $(f_1 + f_2)/2 \in F$, then $f_1, f_2 \in F$. The next result is a type of concretized version of [**24**, Lemma 3.3].

LEMMA 2.27. *Let V be a norm-closed face of $(C_0^{\mathbb{T}}(X))_1$, where X is a principle $\mathbb{T}$-bundle, and let h be a function in the closed unit ball of $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ such that h is the identity on $\mathbb{T}$. Then, for all elements f in V, the element $h_t(f)$ lies in V.*

Proof. Since each $h \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ satisfies $h(\lambda\zeta) = \lambda h(\zeta)$ for $\lambda \in \mathbb{T}$ and $\zeta \in \mathcal{B}_{\mathbb{C}}$, the values of h on the interval $[0,1]$ determine the whole function h. We can now repeat, almost literally the argument in [**24**, Lemma 3.3]. Choose a positive $\epsilon < 1/2$ and fix it. Let h_ϵ and g_ϵ denote the functions in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose restrictions to $[0,1]$ are given by

$$h_\epsilon(t) = \begin{cases} 0, & 0 \le t \le \epsilon/2, \\ \frac{2}{\epsilon}h(\epsilon)t - h(\epsilon), & \epsilon/2 \le t \le \epsilon, \\ h(t), & \epsilon \le t < 1-\epsilon, \\ \big(\frac{2}{\epsilon}(1-t) - 1\big)h(1-\epsilon) + \frac{2}{\epsilon}(t-1) + 2, & 1-\epsilon \le t \le 1-\epsilon/2, \\ 1, & 1-\epsilon/2 \le t \le 1 \end{cases}$$

and

$$g_\epsilon(t) = (1 - \frac{\epsilon}{2})^{-1}(t - \frac{\epsilon}{2}h_\epsilon),\ t \in [0,1],$$

respectively. Having in mind that $|h_\epsilon(t)| \le 1$ and $h(\lambda t) = \lambda h(t)$ for $t \in (0,1]$ and $\lambda \in \mathbb{T}$, we observe that $h_\epsilon \in (C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}))_1$.

Next, we show that $g_\epsilon \in (C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}))_1$. Since the values of g_ϵ on the interval $[0,1]$ determine the values of g_ϵ on $\mathcal{B}_{\mathbb{C}}$, it is enough to consider only in the case that $t \in [0,1]$. Let $t \in [0,1]$ and fix it. If $t \in [0,\epsilon/2] \cup [1-\epsilon/2, 1]$, then $h_\epsilon(t) = 0$ or 1. By the definition of g_ϵ, we infer from $\epsilon < 1/2$ that

$$|t - h_\epsilon(t)| \le \max\{\epsilon, (1 - \frac{\epsilon}{2})\} \le 1 - \frac{\epsilon}{2}.$$

We assume that $t \in [\epsilon/2, 1-\epsilon]$, and then, it follows from $h_\epsilon \in (C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}))_1$ and $t \le 1-\epsilon$ that

$$|t - \frac{\epsilon}{2}h_\epsilon(t)| \le t + \frac{\epsilon}{2} \le 1 - \frac{\epsilon}{2}.$$

Finally, we consider the case that $t \in [1-\epsilon, 1-\epsilon/2]$. Then it follows from $-1 \le -t - \epsilon/2 \le \epsilon/2 - 1$ that

$$|t - \frac{\epsilon}{2}h_\epsilon(t)| = |(1 - t - \frac{\epsilon}{2})h(1-\epsilon) + 1 - \epsilon| \le |1 - t - \frac{\epsilon}{2}| + (1-\epsilon) \le 1 - \frac{\epsilon}{2}.$$

Therefore, we conclude that $|g_\epsilon(t)| \le 1$ for all $t \in [0,1]$, and thus, $g_\epsilon \in (C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}))_1$.

By the definition of g_ϵ, we have $(1-\epsilon/2)g_\epsilon + (\epsilon/2)h_\epsilon = id_{\mathcal{B}_{\mathbb{C}}}$, where $id_{\mathcal{B}_{\mathbb{C}}} : \mathcal{B}_{\mathbb{C}} \to \mathcal{B}_{\mathbb{C}}$ is the identity map on $\mathcal{B}_{\mathbb{C}}$. Having in mind that $(g_\epsilon)_t(f) = g_\epsilon \circ f$ and $(h_\epsilon)_t(f) = h_\epsilon \circ f$ are in $(C_0^{\mathbb{T}}(X))_1$, we obtain

$$(1 - \frac{\epsilon}{2})(g_\epsilon)_t(f) + \frac{\epsilon}{2}(h_\epsilon)_t(f) = f.$$

Since V is a face of $(C_0^{\mathbb{T}}(X))_1$, we see that $(h_\epsilon)_t(f) \in V$ for each $0 < \epsilon < 1/2$.

According to the definition of h_ϵ, $\|h - h_\epsilon\|_\infty$ tends to zero when ϵ tends to zero. In fact, we choose $d > 0$ arbitrarily. Because h is uniformly continuous on $\mathcal{B}_{\mathbb{C}}$, there exists $\epsilon_1 > 0$ such that if $0 < \epsilon < \min\{\epsilon_1, 1/2\}$, then $|h(s) - h(t)| < d$ for every $s, t \in [0,1]$ with $|s-t| \le \epsilon$. Let $t \in [0,1]$ and fix it. If $t \in [0, \epsilon/2]$, then $|h(t) - h_\epsilon(t)| = |h(t) - 0| = |h(t) - h(0)| < d$. We assume that $t \in [\epsilon/2, \epsilon]$. We note that $1 \le (2/\epsilon)t \le 2$ from $t \in [\epsilon/2, \epsilon]$. It follows from the choice of ϵ that

$$|h(t) - h_\epsilon(t)| = |h(t) - \big(\frac{2}{\epsilon}t - 1\big)h(\epsilon)| \le |h(t) - h(\epsilon)| + \big(2 - \frac{2}{\epsilon}\big)|h(\epsilon) - h(0)| < 2d,$$

and thus, $|h(t) - h_\epsilon(t)| < 2d$. By a similar calculation, we obtain $|h(t) - h_\epsilon(t)| < 2d$ when $t \in [1-\epsilon, 1-\epsilon/2]$. Finally, we consider the case that $t \in [1-\epsilon/2, 1]$. It follows from $h_\epsilon(t) = 1 = h(1)$ that $|h(t) - h_\epsilon(t)| = |h(t) - h(1)| < d$. Because $|h(t) - h_\epsilon(t)| = 0$ for $t \in [\epsilon, 1-\epsilon]$, we deduce from the above argument that $\|h - h_\epsilon\|_\infty < 2d$ for all $0 < \epsilon < \min\{\epsilon_1, 1/2\}$. Hence, we conclude that $\|h - h_\epsilon\|_\infty \to 0$ as $\epsilon \to 0$. Since V is a norm-closed face of $(C_0^{\mathbb{T}}(X))_1$, it follows from $(h_\epsilon)_t(f) \in V$ for any $\epsilon \in (0, 1/2)$ that $h_t(f) \in V$. $\square$

$C_0^{\mathbb{T}}(X)$ spaces lack of peaking functions, since for each $f \in S(C_0^{\mathbb{T}}(X))$, we have $\mathbb{T} \subseteq f(X)$. We can combine the description in (3.1) with the facial theory of JB*-triples in [**24**] to determine the maximal proper faces of the closed unit ball of $C_0^{\mathbb{T}}(X)$, however we prioritize a self-contained argument for function spaces more accessible for a wider audience.

Pick $x_0 \in X$ with $x_0 \notin (\mathbb{T}\backslash\{1\})x_0$ and $\mu \in \mathbb{T}$. We shall define the set

$$V_{\mu,x_0}^X := \{f \in S(C_0^{\mathbb{T}}(X)) : f(x_0) = \mu\}.$$

According to Remark 2.21, we observe that the set V_{μ,x_0} is non-empty.

Let E be a Banach space. As observed by R. Tanaka in [**71**, Lemma 3.3] and [**72**, Lemma 3.2], Eidelheit's separation theorem or the geometric Hahn-Banach theorem can be employed to deduce that a convex subset $C \subseteq S(E)$ is a maximal convex subset if and only if it is a maximal norm closed proper face of the closed unit ball, $(E)_1$, of E.

We give next a concrete description of the norm closed faces of $(C_0^{\mathbb{T}}(X))_1$. The conclusion can be also derived from the study of norm closed faces of the closed unit ball of a general JB*-triple [**24**] and a good knowledge on the minimal tripotents in the second dual and its relation with the extreme point of the closed unit ball of the first dual. For the sake of simplicity, we include here an alternative argument with techniques of function algebras.

Lemma 2.28. *Let X be a principal $\mathbb{T}$-bundle. Then every maximal convex subset (equivalently, each maximal proper norm closed face) of the closed unit ball of $C_0^{\mathbb{T}}(X)$ is of the form*

$$V_{1,x_0}^X := \{f \in S(C_0^{\mathbb{T}}(X)) : f(x_0) = 1\},$$

for some $x_0 \in X$.

PROOF. Let V be a maximal convex subset of $S(C_0^{\mathbb{T}}(X))$. By [**38**, Lemma 3.5], there exists $\eta \in \operatorname{ext}(C_0^{\mathbb{T}}(X)^*)_1$ such that

$$V = \eta^{-1}(1) \cap S(C_0^{\mathbb{T}}(X)). \tag{3.13}$$

Since X is a principal $\mathbb{T}$-bundle, It follows from (3.2) that $\eta = \delta_{x_0}$ for some $x_0 \in X$. Combining (3.13) with $\eta = \delta_{x_0}$, we obtain $V = \{f \in S(C_0^{\mathbb{T}}(X)) : f(x_0) = 1\} = V_{1,x_0}$.

Conversely, we prove that V_{1,x_0} is a maximal convex subset of $S(C_0^{\mathbb{T}}(X)^*)$ for all x_0. Let $x_0 \in X$ and fix it. By Zorn's lemma, there exists a maximal norm closed proper face V of $(C_0^{\mathbb{T}}(X))_1$ such that $V_{1,x_0} \subset V$. We suppose that there exists $f_1 \in V \setminus V_{1,x_0}$. By Remark 2.21, we can choose $f_2 \in V_{1,x_0}$. It follows from $f_1 \in V \setminus V_{1,x_0}$ and the convexity of V that the function $f = (f_1 + f_2)/2 \in V$ satisfies $|f(x_0)| < \delta < 1$ for an appropriate δ. The set $U = \{x \in X \ : \ |f(x)| < \delta\}$ is a $\mathbb{T}$-invariant open neiborhood of x_0, because $f \in C_0^{\mathbb{T}}(X)$. As we commented after Remark 2.21, we can find $h \in C_0^{\mathbb{T}}(X)$ with $\|h\|_\infty = 1$, $h(x_0) = 1$, and $h|_{X\setminus U} \equiv 0$.

Let $\widetilde{k}$ be a function defined by

$$\widetilde{k}(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ \frac{1}{1-\delta}(t-\delta), & \delta < t \le 1. \end{cases}$$

We define a function $k : \mathcal{B}_{\mathbb{C}} \to \mathbb{C}$ by $\widetilde{k}(0) = 0$ and $k(\lambda t) = \lambda \widetilde{k}(t)$ for $t \in (0,1]$ and $\lambda \in \mathbb{T}$. Lemma 2.27 assures that $k_t(f) = k \circ f \in V$. We put $h_1 = (h + k_t(f))/2$. Having in mind that $k(f(x)) = 0$ for all $x \in U$, we see that $k_t(f)|_U \equiv 0$. Since $h|_{X\setminus U} \equiv 0$, it can be easily seen that $k_t(f)h = 0$. This implies that $\|h_1\|_\infty = \|(h + k_t(f))/2\|_\infty = \max\{\|h/2\|_\infty, \|k_t(f)/2\|_\infty\} = 1/2$. We note that $h \in V_{1,x_0}$ by the choice of h. Since V is a face which contains V_{1,x_0}, $h_1 = (h + k_t(f))/2 \in V$. It follows from $(1/2)0 + (1/2)(2h_1) = h_1$ with $0, 2h_1 \in (C_0^{\mathbb{T}}(X))_1$ that $0 \in V$. Because $(f + (-f))/2 = 0 \in V$ for all $f \in (C_0^{\mathbb{T}}(X))_1$, we conclude that $V = (C_0^{\mathbb{T}}(X))_1$. This contradicts that V is proper. Hence, we must have V_{1,x_0} is a maximal norm closed proper face of $(C_0^{\mathbb{T}}(X))_1$. By [**71**, Lemma 3.3], V_{1,x_0} is a maximal convex subset of $S(C_0^{\mathbb{T}}(X))$. $\square$

Labelling the maximal convex subsets, V^X_{μ,x_0}, of the unit sphere of $C_0^{\mathbb{T}}(X)$ in terms of pairs (μ, x_0) with $\mu \in \mathbb{T}$ and $x_0 \in X$ does not produce an unambiguous association because $V^X_{\mu,x_0} = V^X_{\lambda\mu,\lambda x_0}$ for all $\lambda \in \mathbb{T}$. To avoid repetitions, let us consider the following property: a non-empty subset $\mathcal{S}$ of a principal $\mathbb{T}$-bundle X satisfies the *non-overlapping property* if for each $t \in \mathcal{S}$ we have $\mathcal{S} \cap \mathbb{T}t = \{t\}$. Thanks to Zorn's lemma, we can always find a maximal non-overlapping subset X_0 of X. Let us observe that in this case, $\mathbb{T}X_0 = X$, actually, for each $t \in X$ there exist unique $t_0 \in X_0$ and $\mu \in \mathbb{T}$ such that $t = \mu t_0$. Consequently, the set $\{\delta_{t_0} : t_0 \in X_0\}$ is norming. Furthermore, the set

$$\{V^X_{\mu,x_0} : \mu \in \mathbb{T}, x_0 \in X_0\}$$

covers all possible proper maximal convex subsets of $S(C_0^{\mathbb{T}}(X))$. Actually, we take arbitrary maximal convex subset V of $S(C_0^{\mathbb{T}}(X))$. it follows from Lemma 2.28 that $V = V^X_{1,x}$ for some $x \in X$. Since $X = \mathbb{T}X_0$, there exist unique $\mu \in \mathbb{T}$ and $x_0 \in X_0$ such that $x = \mu x_0$. This shows that $V = V^X_{1,\mu x_0} = V^X_{\overline{\mu},x_0}$. According to the above argument, we obtain the following;

$$\text{there exist unique } \mu \in \mathbb{T} \text{ and } x_0 \in X_0 \text{ such that } V = V^X_{\mu,x_0} \tag{3.14}$$

for each proper maximal convex subset V of $S(C_0^{\mathbb{T}}(X))$.

An alternative proof for Lemma 2.28 can be deduced from [**71**, Lemma 3.3] (see also [**38**, Lemma 3.1].

The main result of this section is a solution to Tingley's problem in the case of abelian JB*-triples.

THEOREM 2.29. *Let X and Y be two principal $\mathbb{T}$-bundles. Then each surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ extends to a surjective real linear isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$.*

Furthermore, there exist a $\mathbb{T}$-invariant clopen subset $D \subseteq Y$ and a homeomorphism $\phi : Y \to X$ satisfying

$$\begin{aligned} &\Delta(f)(x) = f(\phi(x)), \quad \phi(\lambda x) = \lambda\phi(x) \qquad (f \in S(C_0^{\mathbb{T}}(X)), \lambda \in \mathbb{T}, x \in D), \quad \textit{and} \\ &\Delta(f)(x) = \overline{f(\phi(x))}, \quad \phi(\lambda x) = \overline{\lambda}\phi(x) \qquad (f \in S(C_0^{\mathbb{T}}(X)), \lambda \in \mathbb{T}, x \in X \setminus D). \end{aligned}$$

Consequently, there exists a surjective isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ such that $T|_{C_0^{\mathbb{T}}(D)}$ is complex linear, $T|_{C_0^{\mathbb{T}}(X\setminus D)}$ is conjugate-linear and $T(f) = \Delta(f)$ for all $f \in S(C_0^{\mathbb{T}}(X))$.

The proof will be given after a series of technical lemmas. Let us begin by recalling a key result in the techniques developed to study the problem of extension of isometries which is essentially due to L. Cheng and Y. Dong [**14**, Lemma 5.1] and R. Tanaka [**71**] (see also [**70**, Lemma 3.5], [**73**, Lemmas 2.1 and 2.2]).

PROPOSITION 2.30. ([**14**, Lemma 5.1], [**71**, Lemma 3.3], [**70**, Lemma 3.5]) *Let $\Delta : S(E) \to S(F)$ be a surjective isometry between the unit spheres of two Banach spaces, and let $\mathcal{M}$ be a convex subset of $S(E)$. Then $\mathcal{M}$ is a maximal proper face of $\mathcal{B}_E$ (equivalently, a maximal convex* *subset of $S(E)$) if and only if $\Delta(\mathcal{M})$ is a maximal proper (closed) face of $\mathcal{B}_F$ (equivalently, a maximal convex subset of $S(F)$).*

The next corollary is a consequence of Proposition 2.30 and Lemma 2.28.

COROLLARY 2.31. *Let X and Y be two principal $\mathbb{T}$-bundles and $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ a surjective isometry. For each $x \in X$, there exist elements $y \in Y$ such that*

$$\Delta(F_{1,x}^X) = F_{1,y}^Y.$$

We have already given some arguments showing that the element $(1, y) \in \mathbb{T} \times Y$ in the conclusion of the previous corollary need not be unique. To avoid the problem we consider the next lemma.

LEMMA 2.32. *Let X and Y be two principal $\mathbb{T}$-bundles and X_0 a maximal non-overlapping subset of X. We assume that $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ a surjective isometry. For each x_0 in X_0, there exists a unique $y_0 = \tau(x_0) \in Y$ satisfying*

$$\Delta(V_{1,x_0}^X) = V_{1,\tau(x_0)}^Y.$$

The mapping $\tau : X_0 \to Y$ is well-defined and injective.

PROOF. By Corollary 2.31, there exists $y_0 \in Y$ such that

$$\Delta(V_{1,x_0}^X) = V_{1,y_0}^Y. \tag{3.15}$$

We prove that the element $y_0 \in Y$ satisfying the identity in (3.15) is unique. Indeed, if $V_{1,y_0}^Y = V_{1,y_1}^Y$ for $y_1 \notin \mathbb{T}y_0$, then we can find a function $g \in S(C_0^{\mathbb{T}}(Y))$ with $g(y_1) = 1$ and $g(y_0) = 0$ by Remark 2.21 with Lemma 2.22, which is impossible. Hence, $y_1 = \mu y_0$ for some $\mu \in \mathbb{T}$. We choose some $g \in V_{1,y_0}^Y$. It follows from $V_{1,y_0}^Y = V_{1,y_1}^Y$ with $y_0 = \overline{\mu} y_1$ that $1 = g(y_0) = g(\overline{\mu} y_1) = \overline{\mu} g(y_1) = \overline{\mu}$, and hence, $y_0 = y_1$. We set $\phi(x_0) = y_0$ with the element y_0 given by (3.15). The rest is clear from the previous arguments. □

Henceforth we fix a surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$, where X and Y are two principal $\mathbb{T}$-bundles, a maximal non-overlapping subset $X_0 \subseteq X$ and the injective mapping $\tau : X_0 \to Y$ given by Lemma 2.32.

The next step in our strategy isolates a crucial property of τ.

LEMMA 2.33. *Let x_0 be an element in X_0, and let f be an element in $S(C_0^{\mathbb{T}}(X))$ satisfying $f(x_0) = 0$. Then $\Delta(f)(\tau(x_0)) = 0$.*

PROOF. Let $id_{\mathcal{B}_{\mathbb{C}}} : \mathcal{B}_{\mathbb{C}} \to \mathbb{C}$ be the identity function defined by $id_{\mathcal{B}_{\mathbb{C}}}(z) = z$ for each $z \in \mathcal{B}_{\mathbb{C}}$. For each $0 < \epsilon < 1/2$, let φ_ϵ denotes the functions in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose restrictions to $[0,1]$ are given by

$$\varphi_\epsilon(t) = \begin{cases} 0, & 0 \leq t \leq \epsilon/2, \\ 2t - \epsilon, & \epsilon/2 \leq t \leq \epsilon, \\ t, & \epsilon \leq t < 1 - \epsilon, \\ 2t - (1-\epsilon), & 1 - \epsilon \leq t \leq 1 - \epsilon/2, \\ 1, & 1 - \epsilon/2 \leq t \leq 1. \end{cases}$$

According to the definition of φ_ϵ, it is easy to check that $|id_{\mathcal{B}_{\mathbb{C}}}(t) - \varphi_\epsilon(t)| = |t - \varphi_\epsilon(t)| < \epsilon$ for all $t \in [0,1]$. This implies that $\|id_{\mathcal{B}_{\mathbb{C}}} - \varphi_\epsilon\|_\infty$ tends to 0 as $\epsilon \to 0$.

Since $f \in S(C_0^{\mathbb{T}}(X))$, there exists $x_1 \in X_0$ such that $f \in V_{1,x_1}$. It is easy to check that V_{1,x_1} is a norm closed face of $(C_0^{\mathbb{T}}(X))_1$ by Lemma 2.28 with Proposition 2.30. Having in mind that $\varphi_\epsilon|_{\mathbb{T}} = id_{\mathbb{T}}$ for any $\epsilon \in (0,1/2)$, it follows from Lemma 2.27 that $(\varphi_\epsilon)_t(f) = \varphi_\epsilon \circ f \in V_{1,x_1}$, since V_{1,x_1} is a norm closed face of $(C_0^{\mathbb{T}}(X))_1$. Taking a sequence $(a_n)_{n=1}^\infty$ in (0,1/2) which converges to 0, we set $f_n = (\varphi_{a_n})_t(f) = \varphi_{a_n} \circ f \in V_{1,x_1}$ for each $n \in \mathbb{N}$. Because $\lim_{n\to\infty} \|id_{\mathcal{B}_{\mathbb{C}}} - \varphi_{a_n}\|_\infty = 0$ from the above argument, we see that

$$\lim_{n\to\infty} \|f - f_n\|_\infty = \lim_{n\to\infty} \|id_{\mathcal{B}_{\mathbb{C}}} \circ f - \varphi_{a_n} \circ f\|_\infty = 0. \tag{3.16}$$

We choose $n \in \mathbb{N}$ arbitrarily and fix it. Put $U_n = \{x \in X \ : \ |f_n(x)| < a_n/2\}$. Since $f_n(x_0) = \varphi_{a_n}(f(x_0)) = 0$ and $f_n \in S(C_0^{\mathbb{T}}(X))$, the subset U_n is a $\mathbb{T}$-invariant open neiborhood of x_0. We deduce from the definition of φ_{a_n} that $f_n(x) = 0$ for $x \in U_n$, and thus, $f_n|_{U_n} \equiv 0$. By Remark 2.21, we can choose a function $h_n \in S(C_0^{\mathbb{T}}(X))$ satisfying $h_n(x_0) = 1$ and $h_n|_{X\setminus U_n} \equiv 0$. It is clear that $h_n \in V_{1,x_0}^X$ and $\|f_n \pm h_n\|_\infty = 1$, since $f|_{U_n} \equiv 0$ and $h|_{X\setminus U_n} \equiv 0$. By [**56**, Proposition 2.3(*a*)], we have $\Delta(-V_{1,x_0}^X) = -\Delta(V_{1,x_0}^X)$. Therefore, there exists $k_n \in V_{1,x_0}^X$ such that $-\Delta(k_n) = \Delta(-h_n)$. It follows from Lemma 2.32 that $\Delta(V_{1,x_0}^X) = V_{1,\tau(x_0)}^Y$. This implies that $\Delta(-h_n) = -\Delta(k_n) \in -V_{1,\tau(x_0)}^Y$, and hence, $\Delta(-h_n)(\tau(x_0)) = -\Delta(k_n)(\tau(x_0)) = -1$. Because $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ is an isometry, we deduce from $\|f_n \pm h_n\|_\infty = 1$ that

$$\begin{aligned} |\Delta(f_n)(\tau(x_0)) + 1| = |\Delta(f_n)(\tau(x_0)) - \Delta(-h_n)(\tau(x_0))| &\leq \|\Delta(f) - \Delta(-h_n)\|_\infty \\ &= \|f_n - (-h_n)\|_\infty = 1, \\ |\Delta(f_n)(\tau(x_0)) - 1| = |\Delta(f_n)(\tau(x_0)) - \Delta(h_n)(\tau(x_0))| &\leq \|\Delta(f_n) - \Delta(h_n)\|_\infty \\ &= \|f_n - h_n\|_\infty = 1. \end{aligned}$$

Consequently, we obtain $|\Delta(f_n)(\tau(x_0)) + 1| \le 1$ and $|\Delta(f_n)(\tau(x_0)) - 1| \le 1$. These two inequalities show that $\Delta(f_n)(\tau(x_0)) = 0$. Since $n \in \mathbb{N}$ is arbitrarily chosen and $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ is an isometry, we deduce from (3.16) that $\Delta(f)(\tau(x_0)) = \lim_{n\to\infty} \Delta(f_n)(\tau(x_0)) = 0$. The proof is complete. $\square$

LEMMA 2.34. *The set $Y_0 = \{\tau(x_0) : x_0 \in X_0\}$ is a maximal non-overlapping subset of Y, and hence, the set $\{\delta_y : y \in Y_0\}$ is norming in $C_0^{\mathbb{T}}(Y)$. Furthermore, the mapping $\tau : X_0 \to Y_0$ is a bijection satisfying*

$$\Delta(V^X_{1,x_0}) = V^Y_{1,\tau(x_0)} \text{ for all } x_0 \in X_0,$$

and τ^{-1} is precisely the mapping given by Lemma 2.32 for Δ^{-1} and Y_0.

PROOF. We shall first show that Y_0 is non-overlapping. Let $x_1 \in X_0$ and fix it. We choose $x_2 \in X_0$ with $x_1 \neq x_2$. Since $\tau : X_0 \to Y$ is injective, we have $\tau(t_1) \neq \tau(t_2)$ in Y_0. Having in mind that X_0 is non-overlapping, we can find $f \in V^X_{1,x_1}$ with $f(x_2) = 0$ (cf. Remark 2.21). It follows from Lemma 2.32 that $\Delta(f) \in V^Y_{1,\tau(x_1)}$, and thus, $\Delta(f)(\tau(x_1)) = 1$. On the other hand, Lemma 2.33 implies that $\Delta(f)(\tau(x_2)) = 0$, which implies that $\tau(x_2) \notin \mathbb{T}\{\tau(x_1)\}$. Because $x_2 \in X_0 \setminus \{x_1\}$ is arbitrarily chosen, we conclude that $Y_0 \cap \mathbb{T}\{\tau(x_1)\} = \{\tau(x_1)\}$. This shows that Y_0 is non-overlapping.

Thanks to Zorn's lemma, there exists a maximal non-overlapping subset $\tilde{Y}_0$ of Y such that $Y_0 \subset \tilde{Y}_0$. Applying Lemma 2.32 to Δ^{-1} and $\tilde{Y}_0$, we deduce the existence of an injective mapping $\sigma : \tilde{Y}_0 \to X$ satisfying

$$\Delta^{-1}(V^Y_{1,y_0}) = V^X_{1,\sigma(y_0)} \tag{3.17}$$

for all $y_0 \in \tilde{Y}_0$. By the first part of our argument, applied to Δ^{-1} and $\tilde{Y}_0$, we know that $\sigma(\tilde{Y}_0)$ must be non-overlapping subset of X and contains X_0. The maximality of X_0 implies that $\sigma(\tilde{Y}_0) = X_0$, and thus, $\sigma : \tilde{Y}_0 \to X_0$ is a bijective map.

Next, we prove that $\sigma(\tau(x_0)) = x_0$ for $x_0 \in X_0$. Choose $x_0 \in X_0$ arbitrarily and fix it. We infer from Lemma 2.32 that $\Delta(V^X_{1,x_0}) = V^Y_{1,\tau(x_0)}$. Combining (3.17) with the last equality, we obtain

$$V_{1,\sigma(\tau(x_0))} = \Delta^{-1}(V^Y_{1,\tau(x_0)}) = V^X_{1,x_0}.$$

According to Remark 2.21, we must have $\sigma(\tau(x_0)) \in \mathbb{T}\{x_0\}$. We derive from $\sigma(\tau(x_0)) \in \mathbb{T}\{x_0\}$ with $\sigma(\tau(x_0)), x_0 \in \sigma(\tilde{Y}_0)$ that $\sigma(\tau(x_0)) = x_0$. Since $x_0 \in X_0$ is arbitrarily chosen, we conclude that $\sigma(\tau(x_0)) = x_0$ for all $x_0 \in X_0$.

Finally, we show that Y_0 is a maximal non-overlapping subset and $\sigma = \tau^{-1}$. Suppose that there exists $y_1 \in \tilde{Y}_0 \backslash Y_0$. It follows from (3.17) that

$$\Delta^{-1}(V^Y_{1,y_1}) = V^X_{1,\sigma(y_1)}. \tag{3.18}$$

We put $y_2 = \tau(\sigma(y_1)) \in Y_0 \subset \tilde{Y}_0$. Having in mind that $\tilde{Y}_0$ is non-overlapping, $y_1 \in \tilde{Y}_0 \setminus Y_0$ and $y_2 \in Y_0$, we see that $y_2 \notin \mathbb{T}\{y_1\}$. By Remark 2.21, we can find $g \in V^Y_{1,y_1}$ vanishing at y_2. Lemma 2.33, applied to Δ^{-1}, g and y_2, implies that $\Delta^{-1}(g)(\sigma(y_2)) = 0$. Since $\sigma(\tau(x_0)) = x_0$ for all $x_0 \in X_0$, we obtain $\sigma(y_2) = \sigma(\tau(\sigma(y_1))) = \sigma(y_1)$, and hence, $\Delta^{-1}(g)(\sigma(y_1)) = 0$. On the other hand, we infer from (3.18) with $g \in V^Y_{1,y_1}$ that $\Delta^{-1}(g) \in V^X_{1,\sigma(y_1)}$, and hence $1 = \Delta^{-1}(g)(\sigma(y_1)) = 0$, leading to a contradiction. We must have $Y_0 = \tilde{Y}_0$, which shows that Y_0 is a maximal non-overlapping subset of Y. Applying the argument in the last paragraph to τ and σ, we obtain $\tau(\sigma(y_0)) = y_0$ for $y_0 \in Y_0$. Therefore, we derive from $\sigma(\tau(x_0)) = x_0$ for $x_0 \in X_0$ that $\sigma = \tau^{-1}$. $\square$

Having in mind that X_0 and Y_0 are maximal non-overlapping subsets of X and Y, each maximal convex subset of $S(C_0^{\mathbb{T}}(X))$ and $S(C_0^{\mathbb{T}}(Y))$ can be labelled by $\mathbb{T} \times X_0$ and $\mathbb{T} \times Y_0$, respectively in the next lemma.

LEMMA 2.35. *Let X_0 and Y_0 be as in Lemma 2.34. There exist two maps $\phi : \mathbb{T} \times X_0 \to Y_0$ and $\alpha_\Delta : \mathbb{T} \times X_0 \to \mathbb{T}$ such that*

$$\Delta(V^X_{\lambda,x}) = V^Y_{\alpha_\Delta(\lambda,x),\phi(\lambda,x)} \qquad ((\lambda, x) \in \mathbb{T} \times X_0).$$

PROOF. Let $(\lambda, x) \in \mathbb{T} \times X_0$ and fix it. Having in mind that $V^X_{\lambda,x} = V^X_{1,\bar{\lambda}x}$, we see that $V^X_{\lambda,x}$ is a proper maximal convex subset of $S(C_0^{\mathbb{T}}(X))$ by Lemma 2.28. It follows from Lemma 2.30 that $\Delta(V^X_{\lambda,x})$ is also a proper maximal convex subset of $S(C_0^{\mathbb{T}}(Y))$. Since Y_0 is a maximal non-overlapping subset of Y_0, there exists $(\mu, y) \in \mathbb{T} \times Y_0$ uniquely such that $\Delta(V^X_{\lambda,x}) = V^Y_{\mu,y}$ by (3.14). Set $\mu = \alpha_\Delta(\lambda, x)$ and $y = \phi(\lambda, x)$. Since $(\lambda, x) \in \mathbb{T} \times X_0$ is arbitrary chosen, the mappings $\alpha_\Delta : \mathbb{T} \times X_0 \to \mathbb{T}$ and $\phi : \mathbb{T} \times X_0 \to Y_0$ are well defined and satisfy

$$\Delta(V^X_{\lambda,x}) = V^Y_{\alpha_\Delta(\lambda,x),\phi(\lambda,x)}$$

for any $(\lambda, x) \in \mathbb{T} \times X_0$. $\square$

LEMMA 2.36. *The mappings α_Δ and ϕ satisfy*

$$\alpha_\Delta(-\lambda, x) = -\alpha_\Delta(\lambda, x), \text{ and } \phi(-\lambda, x) = \phi(\lambda, x),$$

for all $\lambda \in \mathbb{T}$ and $x \in X_0$.

Proof. A new application of [**56**, Proposition 2.3(a)] with Lemma 2.35 gives

$$\begin{aligned} V^Y_{\alpha_\Delta(-\lambda,x),\phi(-\lambda,x)} &= \Delta(V^X_{-\lambda,x}) = \Delta(-V^X_{\lambda,x}) = -\Delta(V^X_{\lambda,x}) \\ &= -V^Y_{\alpha_\Delta(\lambda,x),\phi(\lambda,x)} = V^Y_{-\alpha_\Delta(\lambda,x),\phi(\lambda,x)}, \end{aligned}$$

and thus,

$$V^Y_{\alpha_\Delta(-\lambda,x),\phi(-\lambda,x)} = V^Y_{-\alpha_\Delta(\lambda,x),\phi(\lambda,x)}. \tag{3.19}$$

Suppose that $\phi(-\lambda,x) \neq \phi(\lambda,x)$ in Y_0. Since Y_0 is a non-overlapping subset of Y, $\mathbb{T}\{\phi(-\lambda,x)\} \cap \mathbb{T}\{\phi(\lambda,x)\}$ is empty. By Remark 2.21 with Lemma 2.22, there exists a function g_0 in $V^Y_{\alpha_\Delta(-\lambda,x),\phi(-\lambda,x)}$ vanishing at $\phi(\lambda,x)$, contradicting the equality (3.19). Therefore $\phi(-\lambda,x) = \phi(\lambda,x)$ and $\alpha_\Delta(-\lambda,x) = -\alpha_\Delta(\lambda,x)$. □

Proposition 2.37. *The identity*

$$\phi(\lambda,x) = \phi(1,x) = \tau(x)$$

hold for all $\lambda \in \mathbb{T}$ and $x \in X_0$.

Proof. Let $x \in X_0$ and fix it. We note that $\tau(x) \in Y_0$ and Y_0 is a maximal overlapping subset of Y by Lemma 2.34. Combining Lemmas 2.32 and 2.35, we obtain

$$V^Y_{\alpha_\Delta(1,x),\phi(1,x)} = \Delta(V^X_{1,x}) = V^Y_{1,\tau(x)}.$$

We note that $\tau(x) \in Y_0$ and Y_0 is a maximal overlapping subset of Y by Lemma 2.34. Applying the same argument in the proof of Lemma 2.36 to the above equality, we have $\phi(1,x) = \tau(x)$.

Next, we prove that $\phi(\lambda,x) = \phi(1,x)$ for all $\lambda \in \mathbb{T}$. Suppose that $\phi(\lambda_0,x) \neq \phi(1,x)$ for some $\lambda_0 \in \mathbb{T}$ with $\mathrm{Re}(\lambda_0) \leq 0$. Let us observe that $\phi(\lambda_0,x), \phi(x) \in Y_0$ and the subset Y_0 is a maximal non-overlapping set of Y. Thus, the subset $\mathbb{T}\{\phi(\lambda_0,x)\} \cap \mathbb{T}\{\phi(1,x)\}$ is empty. By Lemma 2.22, we can find two open disjoint neighborhoods of these two points, and hence it follows from Remark 2.21 that there exist two functions $g_1, g_2 \in S(C_0^{\mathbb{T}}(Y))$ such that $g_1 \in F^Y_{\alpha_\Delta(\lambda,x),\phi(\lambda,x)}$ and $g_2 \in F^Y_{\alpha_\Delta(1,x),\phi(1,x)}$ with $\|g_1 \pm g_2\| = 1$. It follows from Lemma 2.35 that $f_1 = \Delta^{-1}(g_1) \in V^X_{\lambda_0,x}$ and $f_2 = \Delta^{-1}(g_2) \in V^X_{1,x}$. Since $\Delta : S(C_0^{\mathbb{T}}(X)) \to C_0^{\mathbb{T}}(Y)$ is an isometry, we deduce from $\mathrm{Re}(\lambda_0) \leq 0$ that

$$\sqrt{2} < |\lambda_0 - 1| = |f_1(x) - f_2(x)| \leq \|f_1 - f_2\| = \|\Delta(f_1) - \Delta(f_2)\| = \|g_1 - g_2\| = 1,$$

which is impossible. Hence, we must have $\phi(\lambda_0,x) = \phi(1,x_0)$ for all $\lambda_0 \in \mathbb{T}$ with $\mathrm{Re}(\lambda_0) \leq 0$.

We consider that case that $\lambda \in \mathbb{T}$ with $\mathrm{Re}(\lambda) > 0$. We note that $\phi(\lambda_0, x) = \phi(-\lambda_0, x)$ by Lemma 2.36. It follows from the argument in the last paragraph with $\mathrm{Re}(-\lambda_0) < 0$ that $\phi(\lambda_0, x) = \phi(-\lambda_0, x) = \phi(1, x)$. Therefore, we conclude that $\phi(\lambda, x) = \phi(1, x)$ for all $\lambda \in \mathbb{T}$. $\square$

For simplicity of notation, we shall write $\phi(\lambda, x) = \phi(x)$ for all $\lambda \in \mathbb{T}$ and $x \in X_0$. This is well defined by Proposition 2.37. We can rewrite the identity in Lemma 2.35 as

$$\Delta(V^X_{\lambda,x}) = V^Y_{\alpha_\Delta(\lambda,x),\phi(x)} \qquad ((\lambda, x) \in \mathbb{T} \times X_0). \tag{3.20}$$

Applying the same argument to $\Delta^{-1} : S(C^{\mathbb{T}}_0(Y)) \to S(C^{\mathbb{T}}_0(X))$ with $\tau^{-1} : Y_0 \to X_0$, there exists two maps $\alpha_{\Delta^{-1}} : \mathbb{T} \times Y_0 \to \mathbb{T}$ and $\psi : \mathbb{T} \times Y_0 \to X_0$ satisfying

$$\psi(\mu, y) = \psi(1, y) = \tau^{-1}(y) \qquad ((\mu, y) \in \mathbb{T} \times Y_0) \quad \text{and}$$

$$\Delta^{-1}(V^Y_{\mu,y}) = V^X_{\alpha_{\Delta^{-1}}(\mu,y),\psi(y)} \qquad ((\mu, y) \in \mathbb{T} \times Y_0), \tag{3.21}$$

where we have rewritten $\psi(\mu, y) = \psi(y)$ for all $(\mu, y) \in \mathbb{T} \times Y_0$.

In the following two lemmas, we shall prove that $\alpha_\Delta(\cdot, x) : \mathbb{T} \to \mathbb{T}$ and $\alpha_{\Delta^{-1}}(\cdot, \phi(x)) : \mathbb{T} \to \mathbb{T}$ are surjective isometries on $\mathbb{T}$ for each $x \in X_0$ and $\psi : Y_0 \to X_0$ is the inverse of $\phi : X_0 \to Y_0$.

Lemma 2.38. *For each $x \in X_0$, the mappings $\alpha_\Delta(\cdot, x), \alpha_{\Delta^{-1}}(\cdot, \phi(x)) : \mathbb{T} \to \mathbb{T}$ are bijective and $\alpha_{\Delta^{-1}}(\cdot, \phi(x))$ is the inverse of $\alpha_\Delta(\cdot, x)$. Moreover, the mapping $\phi : X_0 \to Y_0$ is a bijective map whose inverse is $\psi : Y_0 \to X_0$.*

Proof. Fix an arbitrary $x \in X_0$ and $\lambda \in \mathbb{T}$. Combining (3.20) and (3.21), we obtain

$$V^X_{\lambda,x} = \Delta^{-1}(\Delta(V^X_{\lambda,x})) = \Delta^{-1}(V^Y_{\alpha_\Delta(\lambda,x),\phi(x)}) = V^X_{\alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda,x),\phi(x)),\psi(\phi(x))},$$

and thus, $V^X_{\lambda,x} = V^X_{\alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda,x),\phi(x)),\psi(\phi(x))}$. Since $\lambda \in \mathbb{T}$ and $x \in X_0$ are arbitrarily chosen, it follows from (3.14) that

$$\alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda, x), \phi(x)) = \lambda \quad \text{and} \quad \psi(\phi(x)) = x \qquad ((\lambda, x) \in \mathbb{T} \times X_0). \tag{3.22}$$

Interchanging the roles of Δ and Δ^{-1} in the last paragraph, we infer from (3.20) and (3.21) that $V^Y_{\mu,y} = V^Y_{\alpha_\Delta(\alpha_{\Delta^{-1}}(\mu,y),\psi(y)),\phi(\psi(y))}$ for any $\mu \in \mathbb{T}$ and $y \in Y_0$. It follows from (3.14) that

$$\alpha_\Delta(\alpha_{\Delta^{-1}}(\mu, y), \psi(y)) = \mu \quad \text{and} \quad \phi(\psi(y)) = y \qquad ((\mu, y) \in \mathbb{T} \times Y_0). \tag{3.23}$$

The second equalities in (3.22) and (3.23) assure that $\phi : X_0 \to Y_0$ is a bijective map whose inverse is $\psi : Y_0 \to X_0$.

For each $x \in X_0$, we define two maps $\alpha_x : \mathbb{T} \to \mathbb{T}$ and $\alpha_{\phi(x)} : \mathbb{T} \to \mathbb{T}$ by

$$\alpha_x(\lambda) = \alpha_\Delta(\lambda, x) \quad \text{and} \quad \alpha_{\phi(x)}(\mu) = \alpha_{\Delta^{-1}}(\mu, \phi(x)) \quad (\lambda, \mu \in \mathbb{T}).$$

By the first equality of (3.22), we derive from the definitions of α_x and $\alpha_{\phi(x)}$ that

$$\alpha_{\phi(x)}(\alpha_x(\lambda)) = \alpha_{\Delta^{-1}}(\alpha_x(\lambda), \phi(x)) = \alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda, x), \phi(x)) = \lambda \quad (\lambda \in \mathbb{T}).$$

Having in mind that $\psi(\phi(x)) = x$, we infer from the first equality of (3.23) that

$$\alpha_x(\alpha_{\phi(x)}(\mu)) = \alpha_\Delta(\alpha_{\Delta^{-1}}(\mu, \phi(x)), \psi(\phi(x))) = \mu \quad (\mu \in \mathbb{T}).$$

The last two equalities show that $\alpha_\Delta(\cdot, x) = \alpha_x : \mathbb{T} \to \mathbb{T}$ is a bijective map and $\alpha_\Delta(\cdot, x)^{-1} = \alpha_x^{-1} = \alpha_{\phi(x)} = \alpha_{\Delta^{-1}}(\cdot, \phi(x))$. $\square$

We can argue as in Lemma 2.19 to deduce that $\alpha(\cdot, x) : \mathbb{T} \to \mathbb{T}$ is an isometric mapping for each $x \in X_0$.

LEMMA 2.39. *For each $x \in X_0$, the mappings $\alpha_\Delta(\cdot, x), \alpha_{\Delta^{-1}}(\cdot, \phi(x)) : \mathbb{T} \to \mathbb{T}$ are surjective isometries.*

PROOF. Let $x \in X_0$ and fix it. We choose λ_1, λ_2 in $\mathbb{T}$ arbitrarily. Take an element $f \in F_{1,x}$, and then we note that $\lambda_1 f \in V^X_{\lambda_1,x}$ and $\lambda_2 f \in V^X_{\lambda_2,x}$. Since $\Delta(F^X_{\lambda_j,x}) = F^Y_{\alpha_\Delta(\lambda_j,x),\phi(x)}$ for $j = 1, 2$ by (3.20), it can be easily seen that

$$\begin{aligned} |\alpha_\Delta(\lambda_1, x) - \alpha_\Delta(\lambda_2, x)| &= |\Delta(\lambda_1 f)(\phi(x)) - \Delta(\lambda_2 f)(\phi(x))| \\ &\leq \|\Delta(\lambda_1 f) - \Delta(\lambda_2 f)\| = \|(\lambda_1 - \lambda_2) f\| = |\lambda_1 - \lambda_2|. \end{aligned}$$

This proves that the mapping $\alpha_\Delta(\cdot, x) : \mathbb{T} \to \mathbb{T}$ is contractive. Replacing Δ with Δ^{-1}, we observe that $\alpha_{\Delta^{-1}}(\cdot, \phi(x)) : \mathbb{T} \to \mathbb{T}$ is contractive too. Having in mind that $\alpha_{\Delta^{-1}}(\cdot, \phi(x))$ is the inverse of $\alpha_\Delta(\cdot, x)$ by Lemma 2.38, we get

$$\begin{aligned} |\lambda_1 - \lambda_2| &= |\alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda_1, x), \phi(x)) - \alpha_{\Delta^{-1}}(\alpha_\Delta(\lambda_2, x), \phi(x))| \\ &\leq |\alpha_\Delta(\lambda_1, x) - \alpha_\Delta(\lambda_2, x)| \leq |\lambda_1 - \lambda_2|, \end{aligned}$$

and hence, $|\alpha_\Delta(\lambda_1, x) - \alpha_\Delta(\lambda_2, x)| = |\lambda_1 - \lambda_2|$ for all $\lambda_1, \lambda_2 \in \mathbb{T}$. Therefore, $\alpha_\Delta(\cdot, x) : \mathbb{T} \to \mathbb{T}$ is an isometry on $\mathbb{T}$. Interchanging the roles of $\alpha_\Delta(\cdot, x)$ and $\alpha_{\Delta^{-1}}(\cdot, \phi(x))$, we see that $\alpha_{\Delta^{-1}}(\cdot, \phi(x)) : \mathbb{T} \to \mathbb{T}$ is also an isometry. $\square$

It follows from the previous lemma that $\alpha_\Delta(\cdot, x) : \mathbb{T} \to \mathbb{T}$ is a surjective isometry. By the solution to Tingley's problem for $\mathbb{T} = S(\mathbb{C})$ (see, for example [**38**]), we conclude that

$$\alpha_\Delta(\lambda, x) = \alpha_\Delta(1, x)\lambda \;\; (\lambda \in \mathbb{T}), \quad \text{or} \quad \alpha_\Delta(\lambda, x) = \alpha_\Delta(1, x)\overline{\lambda} \;\; (\forall \lambda \in \mathbb{T}). \tag{3.24}$$

for each $x \in X_0$. The just stated property determines a partition of $X_0 = X_0^+ \cup X_0^-$ with respect to the following subsets

$$X_0^+ = \{x \in X_0 : \alpha_\Delta(\lambda, x) = \alpha_\Delta(1, x)\lambda \quad (\lambda \in \mathbb{T})\}, \tag{3.25}$$

$$X_0^- = \{t_0 \in X_0 : \alpha_\Delta(\lambda, x) = \alpha_\Delta(1, x)\overline{\lambda} \quad (\lambda \in \mathbb{T})\}. \tag{3.26}$$

The continuous triple functional calculus explained before Lemma 2.27 is now applied in our next technical result.

Lemma 2.40. *Let $x_0 \in X_0$ and $f \in S(C_0^{\mathbb{T}}(X))$ with $|f(x)| < 1$. Set $\lambda = \frac{f(x)}{|f(x))|}$ if $f(x) \neq 0$ and $\lambda = 1$ if $f(x) = 0$. We take $\varepsilon > 0$ arbitrarily and fix it. Then there exist $g_\varepsilon \in V_{1,x}^X$ and $f_\varepsilon \in S(C_0^{\mathbb{T}}(X))$ satisfying $f_\varepsilon(x) = f(x)$, $\|f - f_\varepsilon\| < \varepsilon$, and*

$$rf_\varepsilon + (1 - r|f(x_0)|)\lambda g_\varepsilon \in V_{\lambda,x}^X$$

for all $0 < r < 1$.

Proof. We choose any $\epsilon > 0$ arbitrarily and fix it. The case for $f(x_0) = 0$ is easier. In this case, it follows from the argument in the proof of Lemma 2.33 that there exists a sequence $f_\epsilon \in S(C_0^{\mathbb{T}}(X))$ such that $f_\epsilon = 0$ on some $\mathbb{T}$-invariant open neighborhood U_ϵ of x_0 and $\|f - f_\epsilon\|_\infty < \epsilon$. Applying Remark 2.21 to f_ϵ with U_ϵ, there exists $g_\epsilon \in S(C_0^{\mathbb{T}}(X))$ such that $g_\epsilon|_{X \setminus U_\epsilon} = 0$ and $g_\epsilon(x_0) = 1$. Because $f_\epsilon|_{U_n} = 0$ and $g_\epsilon|_{X \setminus U_n} = 0$, we observe that $\|rf_\epsilon + g_\epsilon\|_\infty = \max\{r\|f_\epsilon\|_\infty, \|g_\epsilon\|_\infty\} = 1$ and $f(x_0) = 0 = f_\epsilon(x_0)$. It follows from $f_\epsilon(x_0) = 0$ and $g_\epsilon(x_0) = 1$ that $(rf_\epsilon + (1 - r|f(x_0)|)\lambda g_\epsilon)(x_0) = \lambda$, and thus $rf_\epsilon + (1 - r|f(x_0)|)\lambda g_\epsilon \in V_{\lambda,x_0}^X$.

Suppose next that $0 < |f(x_0)| < 1$, and choose a positive ϵ such that $|f(x_0)| + \epsilon < 1$ and $0 < |f(x_0)| - \epsilon$. We set $V_\epsilon = \{x \in X : |f(x) - f(x_0)| < \epsilon/2\}$ and $W_\epsilon = \mathbb{T}V_\epsilon$. Having in mind that $\mathbb{T}V_\epsilon = \bigcup_{\lambda \in \mathbb{T}} \lambda V_\epsilon$ and λV_ϵ is an open subset in X for $\lambda \in \mathbb{T}$, we observe that $W_\epsilon = \mathbb{T}V_\epsilon$ is a $\mathbb{T}$-invariant open neighborhood of x_0. Put $K_\epsilon = \{x \in X : |f(x_0)| - \epsilon/2 \leq |f(x)| \leq |f(x_0)| + \epsilon/2\}$, and then, W_ϵ is contained in K_ϵ. In fact, choose $y \in W_\epsilon$ arbitrarily. There exist $\lambda_0 \in \mathbb{T}$ and $y_0 \in V_\epsilon$ such that $y = \lambda_0 y_0$. It follows from $y_0 \in V_\epsilon$ that $|f(y) - f(\lambda_0 x_0)| = |\lambda_0(f(y_0) - f(x_0))| < \epsilon/2$, and thus, $|f(y)| < |f(\lambda_0 x_0)| + \epsilon/2 = |f(x_0)| + \epsilon/2$. This implies that $y \in K_\epsilon$, and hence, we see that $W_\epsilon \subset K_\epsilon$ and

$$|f(y)| < |f(x_0)| + \epsilon/2 \qquad (y \in W_\epsilon). \tag{3.27}$$

Let us find, via Remark 2.21, a function $g_\varepsilon \in F_{1,x_0}^X$ such that $g_\varepsilon|_{X \setminus W_\varepsilon} \equiv 0$.

We define two maps $\eta_1 : [|f(x_0)|, |f(x_0)|+\epsilon/2] \to [0,1]$ and $\eta_2 : [|f(x_0)|+\epsilon/2, |f(x_0)|+\epsilon] \to [0,1]$ by

$$\eta_1(s) = \frac{2}{\epsilon}(s - |f(x_0)|) \quad (|f(x_0)| \leq s \leq |f(x_0)| + \frac{\epsilon}{2}),$$
$$\eta_2(s) = \frac{2}{\epsilon}\big(s - (|f(x_0)| + \frac{\epsilon}{2})\big) \quad (|f(x_0)| + \frac{\epsilon}{2} \leq s \leq |f(x_0)| + \epsilon).$$

It is easy to check that η_i is a bijective continuous function for $i = 1, 2$. Let us consider the following $h_\varepsilon \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose values on $[0,1]$ are the following:

$$h_\epsilon(s) = \begin{cases} s, & 0 \leq s \leq |f(x_0)|, \\ \big(1-\eta_1(s)\big)|f(x_0)| + \eta_1(s)(|f(x_0)| - \frac{\epsilon}{2}) & |f(x_0)| \leq s \leq |f(x_0)| + \varepsilon/2, \\ |f(x_0)| - \varepsilon/2, & s = |f(x_0)| + \varepsilon/2, \\ \big(1-\eta_2(s)\big)\big(|f(x_0)| - \frac{\epsilon}{2}\big) + \eta_2(s)(|f(x_0)| + \epsilon) & |f(x_0)| + \varepsilon/2 \leq s \leq |f(x_0)| + \varepsilon, \\ s, & |f(x_0)| + \varepsilon \leq s \leq 1. \end{cases}$$

Let $id_{\mathcal{B}_{\mathbb{C}}} : \mathcal{B}_{\mathbb{C}} \to \mathcal{B}_{\mathbb{C}}$ denote the identity mapping. We note that $id_{\mathcal{B}_{\mathbb{C}}} \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$. By the definition of h_ϵ, we see that $|id_{\mathcal{B}_{\mathbb{C}}}(z) - h_\epsilon(z)| \leq \epsilon$ for all $z \in \mathcal{B}_{\mathbb{C}}$. Set $f_\varepsilon = (h_\varepsilon)_t(f)$. Since $\|id_{\mathcal{B}_{\mathbb{C}}} - h_\epsilon\|_\infty \leq \epsilon$, it follows that

$$\|f - f_\varepsilon\| = \sup_{x \in X} |id_{\mathcal{B}_{\mathbb{C}}}(f(x)) - (h_\epsilon)(f(x))| \leq \varepsilon.$$

Clearly $f_\epsilon(x_0) = f(x_0)$. For any $x \in X \setminus W_\varepsilon$, we have

$$|(rf_\epsilon + (1 - r|f(t_0)|)\lambda g_\epsilon)(x)| = |rf_\epsilon(x)| \leq r \leq 1.$$

Choose $x \in W_\epsilon$ arbitrarily. It follows from (3.27) that $|f(x)| < |f(x_0)| + \epsilon/2$. Hence, we observe that

$$|f_\epsilon(x)| = |h_\epsilon(f(x))| = |h_\epsilon(e^{i\theta_x}|f(x)|)| = |h_\epsilon(|f(x)|)| \leq |f(x_0)|$$

by the choice of h_ϵ. Therefore,

$$|(rf_\varepsilon + (1 - r|f(x_0)|)\lambda g_\varepsilon)(s)| \leq r|f(x_0)| + 1 - r|f(x_0)| = 1.$$

Finally the identity

$$\begin{aligned}(rf_\epsilon + (1 - r|f(x_0)|)\lambda g_\epsilon)(x_0) &= rf_\epsilon(x_0) + (1 - r|f(x_0)|)\frac{f_\epsilon(x_0)}{|f_\epsilon(x_0)|} \\ &= \frac{f_\epsilon(x_0)}{|f_\epsilon(x_0)|} = \frac{f(x_0)}{|f(x_0)|} = \lambda,\end{aligned}$$

proves that $rf_\epsilon + (1 - r|f(x_0)|)\lambda g_\epsilon \in F^X_{\lambda, x_0}$, as desired. $\square$

In the next proposition, we shall determine the point evaluations of elements in the image of Δ at the points of the form $\phi(t_0)$.

LEMMA 2.41. *For each $x_0 \in X_0$ and each $f \in S(C_0^{\mathbb{T}}(X))$, we have*

$$\Delta(f)(\phi(x_0)) = \alpha_\Delta\left(\frac{f(x_0)}{|f(x_0)|}, x_0\right)|f(x_0)| = \begin{cases} \alpha_\Delta(1, x_0)\, f(x_0), & \text{if } x_0 \in X_0^+, \\ \\ \alpha_\Delta(1, x_0)\, \overline{f(x_0)}, & \text{if } x_0 \in X_0^-. \end{cases}$$

PROOF. Let us fix $x_0 \in X_0$ and $f \in S(C_0^{\mathbb{T}}(X))$. The case $f(x_0) = 0$ follows from Lemma 2.33. If $|f(x_0)| = 1$, then we have $f \in V_{f(x_0),x_0}^X$, and thus,

$$\Delta(f)(\phi(x_0)) = \alpha_\Delta(f(x_0), x_0) = \begin{cases} \alpha_\Delta(1, x_0)\, f(x_0), & \text{if } x_0 \in X_0^+, \\ \\ \alpha_\Delta(1, x_0)\, \overline{f(x_0)}, & \text{if } x_0 \in X_0^- \end{cases}$$

by (3.20) with (3.24). We can therefore assume that $0 < |f(x_0)| < 1$. Set $\lambda = \frac{f(x_0)}{|f(x_0)|}$.

We shall first show that

$$|\Delta(f)(\phi(x_0))| = |f(x_0)|. \tag{3.28}$$

For each $\varepsilon > 0$, there exist $g_\epsilon \in F_{1,x_0}^X$ and $f_\epsilon \in S(C_0^{\mathbb{T}}(X))$ satisfying

$$h_{r,\epsilon} = rf_\epsilon + (1 - r|f(x_0)|)\lambda g_\epsilon \in V_{\lambda,x_0}^X \quad (0 < r < 1),$$

$f_\epsilon(x_0) = f(x_0)$ and $\|f - f_\epsilon\| \leq \epsilon$ by Lemma 2.40. In particular,

$$\Delta(h_{r,\epsilon})(\phi(x_0)) = \alpha_\Delta(\lambda, x_0), \tag{3.29}$$

and by definition,

$$\|h_{r,\epsilon} - f_\epsilon\| \leq (1 - r) + 1 - r|f(x_0)| = 2 - r - r|f(x_0)|.$$

On the other hand, it follows from (3.29) with the last equality that

$$\begin{aligned} 1 - |\Delta(f_\epsilon)(\phi(t_0))| &= |\alpha_\Delta(\lambda, x_0)| - |\Delta(f_\epsilon)(\phi(x_0))| \\ &\leq |\alpha_\Delta(\lambda, x_0) - \Delta(f_\epsilon)(\phi(x_0))| \\ &= |\Delta(h_{r,\epsilon})(\phi(x_0)) - \Delta(f_\epsilon)(\phi(x_0))| \\ &\leq \|\Delta(h_{r,\epsilon}) - \Delta(f_\epsilon)\| = \|h_{r,\epsilon} - f_\epsilon\| \\ &\leq 2 - r - r|f(x_0)|, \end{aligned} \tag{3.30}$$

which implies that $r + r|f(x_0)| - 1 \le |\Delta(f_\epsilon)(\phi(x_0))|$ for all $0 < r < 1$. Letting $r \to 1$, we get $|f(x_0)| \le |\Delta(f_\epsilon)(\phi(x_0))|$. Now, it follows from $\|\Delta(f) - \Delta(f_\epsilon)\|_\infty = \|f - f_\epsilon\|_\infty \le \epsilon$ that

$$|f(x_0)| \le |\Delta(f_\epsilon(\phi(x_0)))| \le |\Delta(f)(\phi(x_0))| + \epsilon.$$

Since $\epsilon > 0$ is arbitrarily chosen, we conclude that $|f(x_0)| \le |\Delta(f)(\phi(x_0))|$.

We note that $\psi = \phi^{-1}$ by Lemma 2.38. Applying the same argument to Δ^{-1}, ψ, $\Delta(f)$ and $\phi(x_0)$ in the roles of Δ, ϕ, f and x_0, we get

$$|\Delta(f)(\phi(x_0))| \le |\Delta^{-1}\Delta(f)(\psi(\phi(x_0))| = |f(x_0)|,$$

which concludes the proof of (3.28).

If we take limits $r \to 1$ and $\varepsilon \to 0$ in the inequalities given by the second and last lines of (3.30), we arrive to

$$|\alpha_\Delta(\lambda, x_0) - \Delta(f)(\phi(x_0))| \le 1 - |f(x_0)|. \tag{3.31}$$

Consequently, we deduce from (3.28) that

$$\begin{aligned} 1 = |\alpha_\Delta(\lambda, t_0)| &\le |\alpha_\Delta(\lambda, x_0) - \Delta(f)(\phi(x_0))| + |\Delta(f)(\phi(x_0))| \\ &\le 1 - |f(x_0)| + |f(x_0)| = 1. \end{aligned}$$

It then follows that the equality holds in a triangular inequality, so there exists a positive number $t > 0$ such that $t\alpha_\Delta(\lambda, x_0) = \Delta(f)(\phi(x_0))$. In particular $t = |\Delta(f)(\phi(x_0))| = |f(x_0)|$. Having in mind that $\lambda = f(x_0)/|f(x_0)|$, we have proved that

$$\Delta(f)(\phi(x_0)) = \alpha_\Delta\left(\frac{f(x_0)}{|f(x_0)|}, x_0\right)\ |f(x_0)|.$$

The rest is clear from the equalities (3.24), (3.25) and (3.26). □

Proof of Theorem 2.29. We denote by X_0 and Y_0 the maximal non-overlapping subsets employed in the previous arguments. Let $\phi : X_0 \to Y_0$ be the bijection presented in Lemmas 2.35 and 2.38. As we have already commented after Lemma 2.28, the sets $\{\delta_{x_0} : x_0 \in X_0\}$ and $\{\delta_{\phi(x_0)} : x_0 \in X_0\}$ are norming in $C_0^{\mathbb{T}}(X)$ and $C_0^{\mathbb{T}}(Y)$, respectively.

We define a mapping $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ by

$$T(g) = \begin{cases} \|f\|\,\Delta\left(\dfrac{f}{\|f\|}\right), & \text{if } f \in C_0^{\mathbb{T}}(X) \setminus \{0\}, \\[2ex] 0, & \text{if } f = 0. \end{cases}$$

We can follow a similar argument to that in the proof of Theorem 2.1, and employ the identity in Lemma 2.41 to prove that the mapping $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ is a surjective real linear isometry which is an extension of Δ. The final conclusions are straightforward consequences of Lemma 2.26. $\square$

CHAPTER 3

Tingley's problem for a Banach space of Lipschitz functions on the closed unit interval

Abstract

We prove that every surjective isometry on the unit sphere of $\mathrm{Lip}(I)$ of all Lipschitz continuous functions on the closed unit interval I is extended to a surjective real linear isometry on $\mathrm{Lip}(I)$ with the norm $\|f\|_\sigma = |f(0)| + \|f'\|_{L^\infty}$.

1. Introduction and main results

Let E and F be Banach spaces whose unit spheres are $S(E)$ and $S(F)$, respectively. In 1987, Tingley [**74**] asks whether each surjective isometry $\Delta\colon S(E) \to S(F)$ is extended to a surjective, real linear isometry from E onto F. Since then, many mathematicians have given affirmative answers to the Tingley's problem for particular Banach spaces. There is a huge list of the research of the problem, here we show only some of them. Tingley's problem is treated for function spaces in [**20, 38, 47, 48, 76, 77**], and for operator spaces in [**26, 27, 28, 29, 30, 31, 61, 62, 63, 71, 73, 72**]. Besides the Tingley's problem, the Mazur–Ulam property for Banach spaces has been studying actively; a Banach space E has the Mazur–Ulam property if F is any Banach space, every surjective isometry from $S(E)$ onto $S(F)$ admits a unique extension to a surjective real linear isometry from E onto F. See, for example, [**2, 19, 36, 57, 68, 69**].

Let $\mathrm{Lip}(I)$ be the complex linear space of all Lipschitz continuous complex valued functions on the closed unit interval $I = [0,1]$. For each Banach space E, we denote by $S(E)$ the unit sphere of E. We define $\|f\|_\sigma$ for $f \in \mathrm{Lip}(I)$ by

$$\|f\|_\sigma = |f(0)| + \|f'\|_{L^\infty},$$

where $\|\cdot\|_{L^\infty}$ denotes the essential supremum norm on I. It is well known that each $f \in \mathrm{Lip}(I)$ has essentially bounded derivative f' almost everywhere. Hence, f' belongs to $L^\infty(I)$, the commutative Banach algebra of all essentially bounded measurable functions on I with the essential supremum norm $\|\cdot\|_{L^\infty}$. Consequently, $\|\cdot\|_\sigma$ is a well defined norm on $\mathrm{Lip}(I)$. The purpose of this paper is to prove that every surjective isometry on $S(\mathrm{Lip}(I))$ admits a surjective real linear extension to $\mathrm{Lip}(I)$, which gives a solution to Tingley's problem for $\mathrm{Lip}(I)$. The followings are the main results of this paper.

THEOREM 3.1. *Let* $\Delta\colon S(\mathrm{Lip}(I)) \to S(\mathrm{Lip}(I))$ *be a surjective isometry with* $\|\cdot\|_\sigma$. *Then* Δ *is extended to a surjective, real linear isometry on* $\mathrm{Lip}(I)$.

COROLLARY 3.2. *For each surjective isometry* $T_1\colon \mathrm{Lip}(I) \to \mathrm{Lip}(I)$ *with* $\|\cdot\|_\sigma$, *there exist a constant* α *of modulus* 1, $h_0 \in S_{L^\infty(I)}$ *and a real algebra automorphism* Ψ *on* $L^\infty(I)$ *such that*

$$T_1(f)(t) = \Delta_1(0)(t) + \alpha f(0) + \int_0^t h_0\Psi(f')\,dm \qquad (t \in I,\ f \in \mathrm{Lip}(I)), \quad \text{or}$$

$$T_1(f)(t) = \Delta_1(0)(t) + \alpha\overline{f(0)} + \int_0^t h_0\Psi(f')\,dm \qquad (t \in I,\ f \in \mathrm{Lip}(I)),$$

where m *denotes the Lebesgue measure on* I.

REMARK 3.3. We should note that Theorem 3.1 is deduced from [**77**, Theorem 3.5]. In fact, $\mathrm{Lip}(I)$ equipped with $\|\cdot\|_\sigma$ is identified with the ℓ^1-sum of $\mathbb{R}^2$ and $C(X, \mathbb{R}^2)$ for some compact Hausdorff space X. Here, $C(X, \mathbb{R}^2)$ is the Banach space of all continuous $\mathbb{R}^2$ valued maps on X with the supremum norm. In this paper, we will give a different proof from that of [**77**] of Tingley's problem for $\mathrm{Lip}(I)$.

Koshimizu [**45**, Theorem 1.2] gave the characterization of surjective complex linear isometries on $\mathrm{Lip}(I)$ with $\|\cdot\|_\sigma$. We will characterize surjective isometries on $\mathrm{Lip}(I)$ in Corollary 3.2.

2. Preliminaries and auxiliary lemmas

We denote by $\mathbb{T}$ the unit circle in the complex number field $\mathbb{C}$. Let $\mathcal{M}$ be the maximal ideal space of $L^\infty(I)$: Then $\mathcal{M}$ is a compact Hausdorff space so that the Gelfand transform, defined by $\widehat{h}(\eta) = \eta(h)$ for $h \in L^\infty(I)$ and $\eta \in \mathcal{M}$, is a continuous function from $\mathcal{M}$ to $\mathbb{C}$. Let $C(X)$ be the commutative Banach algebra of all continuous complex valued functions on a compact Hausdorff space X with the supremum norm $\|\cdot\|_\infty$ on X. The Gelfand–Naimark theorem states that the Gelfand transformation $\Gamma\colon L^\infty(I) \to C(\mathcal{M})$, defined by $\Gamma(h) = \widehat{h}$ for $h \in L^\infty(I)$, is an isometric isomorphism. Thus, $\|h\|_{L^\infty} = \sup_{\eta\in\mathcal{M}} |\widehat{h}(\eta)| = \|\widehat{h}\|_\infty$ for $h \in L^\infty(I)$. We define

$$\widetilde{f}(\eta, z) = f(0) + \widehat{f'}(\eta)z \tag{2.1}$$

for $f \in \mathrm{Lip}(I)$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Then the function $\widetilde{f}$ is continuous on $\mathcal{M} \times \mathbb{T}$ with the product topology. We set

$$B = \{\widetilde{f} \in C(\mathcal{M} \times \mathbb{T}) : f \in \mathrm{Lip}(I)\}.$$

Then B is a normed linear subspace of $C(\mathcal{M} \times \mathbb{T})$ equipped with the supremum norm $\|\cdot\|_\infty$ on $\mathcal{M} \times \mathbb{T}$.

We define a mapping $U\colon (\mathrm{Lip}(I), \|\cdot\|_\sigma) \to (B, \|\cdot\|_\infty)$ by $U(f) = \widetilde{f}$ for $f \in \mathrm{Lip}(I)$. We see that U is a surjective complex linear map from $\mathrm{Lip}(I)$ onto B. In addition, $\|U(f)\|_\infty = \|f\|_\sigma$ holds for all $f \in \mathrm{Lip}(I)$: In fact, for each $f \in \mathrm{Lip}(I)$, there exist $z_0, z_1 \in \mathbb{T}$ and $\eta_0 \in \mathcal{M}$ such that $f(0) = |f(0)|z_0$ and $\widehat{f'}(\eta_0) = \|\widehat{f'}\|_\infty z_1$. Then

$$\begin{aligned} |U(f)(\eta_0, z_0\overline{z_1})| &= |f(0) + \widehat{f'}(\eta_0)z_0\overline{z_1}| = |(|f(0)| + \|\widehat{f'}\|_\infty)z_0| \\ &= |f(0)| + \|\widehat{f'}\|_\infty = |f(0)| + \|f'\|_{L^\infty} = \|f\|_\sigma. \end{aligned}$$

We thus obtain $\|f\|_\sigma \leq \|U(f)\|_\infty$. For each $(\eta, z) \in \mathcal{M} \times \mathbb{T}$, we have

$$|U(f)(\eta, z)| = |f(0) + \widehat{f'}(\eta)z| \leq |f(0)| + |\widehat{f'}(\eta)| \leq |f(0)| + \|\widehat{f'}\|_\infty = \|f\|_\sigma,$$

which yields $\|U(f)\|_\infty \leq \|f\|_\sigma$. Consequently,

$$\|\widetilde{f}\|_\infty = \|U(f)\|_\infty = \|f\|_\sigma \qquad (f \in \mathrm{Lip}(I)).$$

Therefore, the map U is a surjective complex linear isometry from $(\mathrm{Lip}(I), \|\cdot\|_\sigma)$ onto $(B, \|\cdot\|_\infty)$. In particular, $U(S(\mathrm{Lip}(I))) \subset S(B)$. Since U^{-1} has the same property as U, we obtain $U^{-1}(S(B)) \subset S(\mathrm{Lip}(I))$, and hence, $U(S(\mathrm{Lip}(I))) = S(B)$.

For each $f \in \mathrm{Lip}(I)$, we observe that f is absolutely continuous on I. Thus, the following identity holds:

$$f(t) - f(0) = \int_0^t f'\,dm \qquad (t \in I), \tag{2.2}$$

where m denotes the Lebesgue measure on I (see, for example, [**67**, Theorem 7.20]). Having in mind $\{\widehat{h} : h \in L^\infty(I)\} = C(\mathcal{M})$, for each $u \in C(\mathcal{M})$ there exists a unique $h \in L^\infty(I)$ such that $u = \widehat{h}$. We define $\mathcal{I}(u)$ by

$$\mathcal{I}(u)(t) = \int_0^t h\,dm \qquad (t \in I).$$

We observe that $\mathcal{I}(u)$ is a Lipschitz function on I with

$$\mathcal{I}(u)(0) = 0 \quad \text{and} \quad \mathcal{I}(u)' = h \quad \text{a.e.}$$

In particular, we obtain

$$\widehat{\mathcal{I}(u)'} = u. \tag{2.3}$$

Here, we note that $\mathcal{I}(u) \in S(\mathrm{Lip}(I))$ for $u \in S(C(\mathcal{M}))$: In fact,

$$\|\mathcal{I}(u)\|_\sigma = |\mathcal{I}(u)(0)| + \|\mathcal{I}(u)'\|_{L^\infty} = \|\widehat{\mathcal{I}(u)'}\|_\infty = \|u\|_\infty = 1,$$

which yields $\mathcal{I}(u) \in S(\mathrm{Lip}(I))$. Hence, $\mathcal{I}(S(C(\mathcal{M}))) \subset S(\mathrm{Lip}(I))$.

Let $\Delta\colon (S(\mathrm{Lip}(I)), \|\cdot\|_\sigma) \to (S(\mathrm{Lip}(I)), \|\cdot\|_\sigma)$ be a surjective isometry. We define $T = U\Delta U^{-1}$; we see that T is a well defined surjective isometry from $(S(B), \|\cdot\|_\infty)$ onto itself, since U is a surjective complex linear isometry from $(\mathrm{Lip}(I), \|\cdot\|_\sigma)$ onto $(B, \|\cdot\|_\infty)$ with $U(S(\mathrm{Lip}(I))) = S(B)$.

$$\begin{array}{ccc} S(\mathrm{Lip}(I)) & \xrightarrow{\Delta} & S(\mathrm{Lip}(I)) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ S(B) & \xrightarrow[T]{} & S(B) \end{array}$$

The identity $TU = U\Delta$ implies that

$$T(\widetilde{f}) = \widetilde{\Delta(f)} \qquad (f \in S(\mathrm{Lip}(I))). \tag{2.4}$$

For each $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, we define

$$\lambda V_x = \{\widetilde{f} \in S(B) : \widetilde{f}(x) = \lambda\},$$

which plays an important role in our arguments. In the rest of this paper, we denote $\mathbf{1}_I$ and $\mathbf{1}_{\mathcal{M}}$ by the constant functions taking the value only 1 defined on I and $\mathcal{M}$, respectively.

Lemma 3.4. *If $\lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$ for some $(\lambda_1, x_1), (\lambda_2, x_2) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, then $(\lambda_1, x_1) = (\lambda_2, x_2)$.*

Proof. We first note that $\widetilde{\mathbf{1}_I}$ is a constant function on $\mathcal{M} \times \mathbb{T}$ by (2.1). Then $\lambda_1 \widetilde{\mathbf{1}_I} \in \lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$, which yields $\lambda_1 = \lambda_1 \widetilde{\mathbf{1}_I}(x_1) = \lambda_1 \widetilde{\mathbf{1}_I}(x_2) = \lambda_2$. This implies $\lambda_1 = \lambda_2$.

Setting $x_j = (\eta_j, z_j)$ for $j = 1, 2$, we first prove $\eta_1 = \eta_2$. Suppose, on the contrary, that $\eta_1 \neq \eta_2$. There exists $u \in S(C(\mathcal{M}))$ such that $u(\eta_1) = 1$ and $u(\eta_2) = 0$. We set $f = \mathcal{I}(\lambda_1 \overline{z_1} u) \in S(\mathrm{Lip}(I))$, and then $\widetilde{f}(\eta_1, z_1) = \lambda_1$ and $\widetilde{f}(\eta_2, z_2) = 0$ by (2.3). This shows that $\widetilde{f} \in \lambda_1 V_{x_1} \setminus \lambda_2 V_{x_2}$, which contradicts the assumption that $\lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$. Consequently, we have $\eta_1 = \eta_2$.

Finally, we shall prove $z_1 = z_2$. By (2.3), we see that $g = \mathcal{I}(\lambda_1 \overline{z_1}\,\mathbf{1}_{\mathcal{M}})$ satisfies $\widetilde{g} \in S(B)$ and $\widetilde{g}(\eta_1, z_1) = \lambda_1$. We thus obtain $\widetilde{g} \in \lambda_1 V_{x_1} \subset \lambda_2 V_{x_2}$, and hence $\lambda_2 = \widetilde{g}(\eta_2, z_2) = \lambda_1 \overline{z_1} z_2$ by the choice of g. This implies $z_1 = z_2$, since $\lambda_1 = \lambda_2$. We have proven that $(\lambda_1, x_1) = (\lambda_2, x_2)$. $\square$

We denote by $\mathcal{F}(B)$ the set of all maximal convex subsets of $S(B)$. Let $\mathrm{ext}(B_1^*)$ be the set of all extreme points of the closed unit ball B_1^* of the dual space of B. It is proved in [**38**, Lemma 3.1] that for each $F \in \mathcal{F}(B)$ there exists $\xi \in \mathrm{ext}(B_1^*)$ such that $F = \xi^{-1}(1) \cap S(B)$, where $\xi^{-1}(1) = \{\widetilde{F} \in B : \xi(\widetilde{F}) = 1\}$. Let $\mathrm{Ch}(B)$ be the Choquet boundary for B, that is, $\mathrm{Ch}(B)$ is the set of all $x \in \mathcal{M} \times \mathbb{T}$ such that the point evaluation $\delta_x \colon B \to \mathbb{C}$ at x is in $\mathrm{ext}(B_1^*)$. By the Arens–Kelley theorem (cf. [**32**, Corollary 2.3.6]), we see that $\mathrm{ext}(B_1^*) = \{\lambda\delta_x \in B_1^* : \lambda \in \mathbb{T},\, x \in \mathrm{Ch}(B)\}$.

Lemma 3.5. *For each $x_0 = (\eta_0, z_0) \in \mathcal{M} \times \mathbb{T}$, the Dirac measure concentrated at x_0 is unique representing measure for δ_{x_0}.*

Proof. Fix an arbitrary open set O in $\mathcal{M}$ with $\eta_0 \in O$. By Urysohn's lemma, we can find $u \in S(C(\mathcal{M}))$ such that $u(\eta_0) = 1$ and $u = 0$ on $\mathcal{M} \setminus O$. Take any representing measure σ for δ_{x_0}, that is, σ is a regular Borel measure on $\mathcal{M} \times \mathbb{T}$ satisfying $\delta_{x_0}(\widetilde{g}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{g}\, d\sigma$ for all $\widetilde{g} \in B$ and $\|\sigma\| = 1$, where $\|\sigma\|$ is the total variation of σ. Having in mind that the operator norm $\|\delta_{x_0}\|$ of δ_{x_0} satisfies $\|\delta_{x_0}\| = 1 = \delta_{x_0}(\widetilde{\mathbf{1}_I})$, we observe that σ is a positive measure (see, for example, [**7**, p.81]). Setting $f = \mathcal{I}(u) \in S(\mathrm{Lip}(I))$, we obtain $\widetilde{f}(\eta, z) = u(\eta)z$ for $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ by (2.1) and (2.3). Since $u = 0$ on $\mathcal{M} \setminus O$, we get

$$\begin{aligned} 1 = |z_0| = |\delta_{x_0}(\widetilde{f})| &= \left| \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f}\, d\sigma \right| \leq \left| \int_{O\times\mathbb{T}} \widetilde{f}\, d\sigma \right| + \left| \int_{(\mathcal{M}\times\mathbb{T})\setminus(O\times\mathbb{T})} \widetilde{f}\, d\sigma \right| \\ &\leq \int_{O\times\mathbb{T}} |\widetilde{f}|\, d\sigma \leq \|\widetilde{f}\|_\infty \sigma(O\times\mathbb{T}) = \sigma(O\times\mathbb{T}) \leq \|\sigma\| = 1. \end{aligned}$$

Consequently, $\sigma(O \times \mathbb{T}) = 1$ for all open sets O in $\mathcal{M}$ with $\eta_0 \in O$, and therefore, we observe that $\sigma(\{\eta_0\} \times \mathbb{T}) = 1$ by the regularity of σ. We thus obtain

$$z_0 = \delta_{x_0}(\widetilde{f}) = \int_{\{\eta_0\}\times\mathbb{T}} \widetilde{f}\, d\sigma = \int_{\{\eta_0\}\times\mathbb{T}} u(\eta) z\, \delta\sigma = \int_{\{\eta_0\}\times\mathbb{T}} z\, \delta\sigma.$$

We derive from $\sigma(\{\eta_0\} \times \mathbb{T}) = 1$ that $\int_{\{\eta_0\}\times\mathbb{T}} (z_0 - z)\, d\sigma = 0$. Setting $Z = \{\eta_0\} \times (\mathbb{T} \setminus \{z_0\})$, we obtain $\int_Z (1 - \overline{z_0} z)\, d\sigma = -\overline{z_0} \int_Z (z - z_0)\, d\sigma = 0$, which yields $\int_Z \mathrm{Re}(1 - \overline{z_0} z)\, d\sigma = 0$. As $\mathrm{Re}(1 - \overline{z_0} z) > 0$ on Z, we conclude $\sigma(Z) = 0$, and thus $\sigma(\{\eta_0\} \times \{z_0\}) = 1$. This proves that any representing measure for δ_{x_0} is the Dirac measure concentrated at x_0. $\square$

Lemma 3.6. *For each $x_0 = (\eta_0, z_0) \in \mathcal{M}\times\mathbb{T}$, we have $x_0 \in \mathrm{Ch}(B)$, that is,* $\mathrm{Ch}(B) = \mathcal{M}\times\mathbb{T}$.

PROOF. We shall prove that δ_{x_0} belongs to $\mathrm{ext}(B_1^*)$. Suppose that $\delta_{x_0} = (\xi_1 + \xi_2)/2$ for $\xi_1, \xi_2 \in B_1^*$. For $j = 1, 2$, there exists a representing measure σ_j for ξ_j by the Hahn–Banach theorem and the Riesz representation theorem (see, for example, [**67**, Theorems 5.16 and 2.14]). Since $\xi_1(\widetilde{\mathbf{1}}_I) + \xi_2(\widetilde{\mathbf{1}}_I) = 2\delta_{x_0}(\widetilde{\mathbf{1}}_I) = 2$ with $|\xi_j(\widetilde{\mathbf{1}}_I)| \leq 1$, we have $\xi_j(\widetilde{\mathbf{1}}_I) = 1 = \|\xi_j\|$ for $j = 1, 2$. Applying the same argument in [**7**, p.81] to σ_j, we see that σ_j is a positive measure. We put $\sigma = (\sigma_1 + \sigma_2)/2$, and then σ is a positive measure.

First, we prove that σ is a representing measure for δ_{x_0}. Because σ_j is a representing measure for ξ_j, we get

$$\int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d\sigma = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d(\frac{\sigma_1 + \sigma_2}{2}) = \frac{\xi_1(\widetilde{f}) + \xi_2(\widetilde{f})}{2} = \delta_{x_0}(\widetilde{f}) \quad (\widetilde{f} \in B).$$

Entering $\widetilde{f} = \widetilde{\mathbf{1}}_I$ into the above equality, we have $\sigma(\mathcal{M} \times \mathbb{T}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{\mathbf{1}}_I d\sigma = 1$, which shows that $\|\sigma\| = 1 = \|\delta_{x_0}\|$. Therefore, σ is a representing measure for δ_{x_0}. By Lemma 3.5, $\sigma = (\sigma_1 + \sigma_2)/2$ is the Dirac measure, τ_{x_0}, concentrated at x_0.

We note that σ_j is a positive measure with $j = 1, 2$. For each Borel set D with $x_0 \notin D$, we obtain $(\sigma_1(D) + \sigma_2(D))/2 = \sigma(D) = 0$, and thus, $\sigma_j(D) = 0$. Having in mind that $\|\sigma_j\| = \|\xi_j\| = 1$, we conclude that $\sigma_j = \tau_{x_0}$ for $j = 1, 2$. Hence, $\xi_j(\widetilde{f}) = \int_{\mathcal{M}\times\mathbb{T}} \widetilde{f} d\sigma_j = \widetilde{f}(x_0) = \delta_{x_0}(\widetilde{f})$ for any $\widetilde{f} \in B$, which implies that $\xi_1 = \delta_{x_0} = \xi_2$. This proves $\delta_{x_0} \in \mathrm{ext}(B_1^*)$, which yields $x_0 \in \mathrm{Ch}(B)$. $\square$

We now characterize the set of all maximal convex subsets $\mathcal{F}(B)$ of $S(B)$. The following result is proved by Hatori, Oi and Shindo Togashi in [**38**] for uniform algebras. The proof below of the next proposition is quite similar to that of [**38**, Lemma 3.2].

PROPOSITION 3.7. *Let F be a subset of $S(B)$. Then $F \in \mathcal{F}(B)$ if and only if there exist $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$ such that $F = \lambda V_x$.*

PROOF. Suppose that F is a maximal convex subset of $S(B)$. By [**38**, Lemma 3.1], $F = \xi^{-1}(1) \cap S(B)$ for some $\xi \in \mathrm{ext}(B_1^*) = \{\lambda\delta_x \in B_1^* : \lambda \in \mathbb{T}, x \in \mathcal{M} \times \mathbb{T}\}$, where we have used Lemma 3.6. There exist $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$ such that $\xi = \lambda\delta_x$. Now we can write

$$F = (\lambda\delta_x)^{-1}(1) \cap S(B) = \{\widetilde{f} \in S(B) : \lambda\widetilde{f}(x) = 1\} = \overline{\lambda} V_x.$$

We thus obtain $F = \overline{\lambda} V_x$ with $\overline{\lambda} \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$.

Conversely, suppose that $F = \lambda V_x$ for some $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. It is routine to check that F is a convex subset of $S(B)$. Using Zorn's lemma, we can prove that there exists a maximal convex subset K of $S(B)$ with $F \subset K$. By the above paragraph, we see that

$K = \mu V_y$ for some $\mu \in \mathbb{T}$ and $y \in \mathcal{M} \times \mathbb{T}$. Then $\lambda V_x = F \subset K = \mu V_y$. Lemma 3.4 shows that $(\lambda, x) = (\mu, y)$, which implies that $F = K$. Consequently, F is a maximal convex subset of $S(B)$. $\square$

Tanaka [**70**, Lemma 3.5] proved that every surjective isometry between the unit spheres of two Banach spaces preserves maximal convex subsets of the spheres (see also [**14**, Lemma 5.1]). By these results, we can prove the following lemma.

LEMMA 3.8. *There exist maps* $\alpha\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathbb{T}$ *and* $\phi\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathcal{M} \times \mathbb{T}$ *such that*

$$T(\lambda V_x) = \alpha(\lambda, x) V_{\phi(\lambda, x)} \tag{2.5}$$

for all $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$.

PROOF. For each $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, λV_x is a maximal convex subset of $S(B)$ by Proposition 3.7. By [**70**, Lemma 3.5], surjective isometry $T\colon S(B) \to S(B)$ preserves maximal convex subsets of $S(B)$, that is, there exists $(\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$ such that $T(\lambda V_x) = \mu V_y$. If, in addition, $T(\lambda V_x) = \mu' V_{y'}$ for some $(\mu', y') \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, then we obtain $(\mu, y) = (\mu', y')$ by Lemma 3.4. Therefore, if we define $\alpha(\lambda, x) = \mu$ and $\phi(\lambda, x) = y$, then $\alpha\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathbb{T}$ and $\phi\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathcal{M} \times \mathbb{T}$ are well defined maps with $T(\lambda V_x) = \alpha(\lambda, x) V_{\phi(\lambda, x)}$. $\square$

LEMMA 3.9. *The maps* α *and* ϕ *from Lemma 3.8 are both surjective maps satisfying*

$$\alpha(-\lambda, x) = -\alpha(\lambda, x) \quad \text{and} \quad \phi(-\lambda, x) = \phi(\lambda, x)$$

for all $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$.

PROOF. Take any $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, and then λV_x is a maximal convex subset of $S(B)$ by Proposition 3.7. We get $T(-\lambda V_x) = -T(\lambda V_x)$, which was proved by Mori [**56**, Proposition 2.3] in a general setting. Lemma 3.8 shows that $\alpha(-\lambda, x) V_{\phi(-\lambda, x)} = T(-\lambda V_x) = -T(\lambda V_x) = -\alpha(\lambda, x) V_{\phi(\lambda, x)}$. Applying Lemma 3.4, we obtain $\alpha(-\lambda, x) = -\alpha(\lambda, x)$ and $\phi(-\lambda, x) = \phi(\lambda, x)$.

There exist well defined maps $\beta\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathbb{T}$ and $\psi\colon \mathbb{T} \times (\mathcal{M} \times \mathbb{T}) \to \mathcal{M} \times \mathbb{T}$ such that

$$T^{-1}(\mu V_y) = \beta(\mu, y) V_{\psi(\mu, y)} \qquad ((\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})),$$

since T^{-1} has the same property as T. For each $(\mu, y) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$, we have, by (2.5),

$$\mu V_y = T(T^{-1}(\mu V_y)) = T(\beta(\mu, y) V_{\psi(\mu, y)}) = \alpha(\beta(\mu, y), \psi(\mu, y)) V_{\phi(\beta(\mu, y), \psi(\mu, y))}.$$

We derive from Lemma 3.4 that $\mu = \alpha(\beta(\mu, y), \psi(\mu, y))$ and $y = \phi(\beta(\mu, y), \psi(\mu, y))$. These prove that both α and ϕ are surjective. $\square$

By definition, $\phi(\lambda, x) \in \mathcal{M} \times \mathbb{T}$ for each $(\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T})$. There exist $\phi_1(\lambda, x) \in \mathcal{M}$ and $\phi_2(\lambda, x) \in \mathbb{T}$ such that

$$\phi(\lambda, x) = (\phi_1(\lambda, x), \phi_2(\lambda, x)).$$

We shall regard ϕ_1 and ϕ_2 as maps defined on $\mathbb{T} \times (\mathcal{M} \times \mathbb{T})$ to $\mathcal{M}$ and $\mathbb{T}$, respectively. By Lemma 3.9, both ϕ_1 and ϕ_2 are surjective maps with

$$\phi_j(-\lambda, x) = \phi_j(\lambda, x) \qquad ((\lambda, x) \in \mathbb{T} \times (\mathcal{M} \times \mathbb{T}),\ j = 1, 2). \tag{2.6}$$

Lemma 3.10. *Let $\lambda_j \in \mathbb{T}$ and $(\eta_j, z_j) \in \mathcal{M} \times \mathbb{T}$ for $j = 1, 2$. If $\eta_1 \neq \eta_2$, then there exist $\widetilde{f_j} \in S(B)$ such that $\widetilde{f_j} \in \lambda_j V_{(\eta_j, z_j)}$ for $j = 1, 2$ and $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1$.*

Proof. Take $j \in \{1, 2\}$ and open sets O_j in $\mathcal{M}$ with $\eta_j \in O_j$ and $O_1 \cap O_2 = \emptyset$. By Urysohn's lemma, there exists $u_j \in S(C(\mathcal{M}))$ such that $u_j(\eta_j) = 1$ and $u_j = 0$ on $\mathcal{M} \setminus O_j$. Let $f_j = \mathcal{I}(\lambda_j \overline{z_j} u_j)$, and then we see that $\widetilde{f_j}(\eta, z) = \lambda_j \overline{z_j}\, u_j(\eta) z$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ by (2.1) and (2.3). It follows from $\widetilde{f_j} \in \lambda_j V_{(\eta_j, z_j)}$ for $j = 1, 2$ that $1 = |\widetilde{f_1}(\eta_1, z_1) - \widetilde{f_2}(\eta_1, z_1)| \leq \|\widetilde{f_1} - \widetilde{f_2}\|_\infty$. Hence, it is enough to prove that $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty \leq 1$. We shall prove $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Fix an arbitrary $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. If $\eta \in O_1$, then $u_2(\eta) = 0$, since $O_1 \cap O_2 = \emptyset$, and thus

$$|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| = |\lambda_1 \overline{z_1} u_1(\eta) - \lambda_2 \overline{z_2} u_2(\eta)| \leq |u_1(\eta)| + |u_2(\eta)| \leq 1.$$

If $\eta \in \mathcal{M} \setminus O_1$, then $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ by the choice of u_1. We conclude that $|\widetilde{f_1}(\eta, z) - \widetilde{f_2}(\eta, z)| \leq 1$ for all $(\eta, z) \in \mathcal{M} \times \mathbb{T}$, which yields $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty \leq 1$. $\square$

Lemma 3.11. *If $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, then $\phi_1(\lambda, x) = \phi_1(1, x)$; we shall write $\phi_1(\lambda, x) = \phi_1(x)$ for simplicity.*

Proof. Take any $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. Then $T(V_x) = \alpha(1, x) V_{\phi(1,x)}$ and $T(\lambda V_x) = \alpha(\lambda, x) V_{\phi(\lambda,x)}$ by (2.5). Suppose, on the contrary, that $\phi_1(\lambda, x) \neq \phi_1(1, x)$. There exist $\widetilde{f_1} \in \alpha(1, x) V_{\phi(1,x)} = T(V_x)$ and $\widetilde{f_2} \in \alpha(\lambda, x) V_{\phi(\lambda,x)} = T(\lambda V_x)$ such that $\|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1$ by Lemma 3.10. We infer from the choice of $\widetilde{f_1}$ and $\widetilde{f_2}$ that $T^{-1}(\widetilde{f_1}) \in V_x$ and $T^{-1}(\widetilde{f_2}) \in \lambda V_x$, which implies that $T^{-1}(\widetilde{f_1})(x) = 1$ and $T^{-1}(\widetilde{f_2})(x) = \lambda$. If $\operatorname{Re} \lambda \leq 0$, then $|1 - \lambda| \geq \sqrt{2}$, and thus

$$\begin{aligned}\sqrt{2} &\le |1-\lambda| = |T^{-1}(\widetilde{f_1})(x) - T^{-1}(\widetilde{f_2})(x)| \\ &\le \|T^{-1}(\widetilde{f_1}) - T^{-1}(\widetilde{f_2})\|_\infty = \|\widetilde{f_1} - \widetilde{f_2}\|_\infty = 1,\end{aligned}$$

where we have used that T is an isometry on $S(B)$. We arrive at a contradiction, which shows $\phi_1(\lambda, x) = \phi_1(1, x)$, provided that $\operatorname{Re}\lambda \le 0$. Now we consider the case when $\operatorname{Re}\lambda > 0$. Then $\phi_1(-\lambda, x) = \phi_1(1, x)$, since $\operatorname{Re}(-\lambda) < 0$. By (2.6), $\phi_1(\lambda, x) = \phi_1(-\lambda, x) = \phi_1(1, x)$, even if $\operatorname{Re}\lambda > 0$. $\square$

LEMMA 3.12. *For each $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$, the following inequality holds:*

$$|\lambda_1 - \lambda_2| \le |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|. \tag{2.7}$$

PROOF. Fix $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. We set $f_j = \alpha(\lambda_j, x)\mathbf{1}_I \in S(\operatorname{Lip}(I))$ for each $j \in \{1, 2\}$. We see that $\widetilde{f_j} \in \alpha(\lambda_j, x)V_{\phi(\lambda_j, x)} = T(\lambda_j V_x)$ by (2.5). Then $T^{-1}(\widetilde{f_j}) \in \lambda_j V_x$, and hence $T^{-1}(\widetilde{f_j})(x) = \lambda_j$. We obtain

$$\begin{aligned}|\lambda_1 - \lambda_2| = |T^{-1}(\widetilde{f_1})(x) - T^{-1}(\widetilde{f_2})(x)| &\le \|T^{-1}(\widetilde{f_1}) - T^{-1}(\widetilde{f_2})\|_\infty = \|\widetilde{f_1} - \widetilde{f_2}\|_\infty \\ &= |\alpha(\lambda_1, x) - \alpha(\lambda_2, x)|\, \|\widetilde{\mathbf{1}_I}\|_\infty = |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|.\end{aligned}$$

Thus, $|\lambda_1 - \lambda_2| \le |1 - \overline{\alpha(\lambda_1, x)}\alpha(\lambda_2, x)|$ holds for all $\lambda_1, \lambda_2 \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. $\square$

LEMMA 3.13. *For each $x \in \mathcal{M} \times \mathbb{T}$, there exists $\varepsilon_0(x) \in \{\pm 1\}$ such that $\alpha(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(1, x)$ for all $\lambda \in \mathbb{T}$; for simplicity, we shall write $\alpha(1, x) = \alpha(x)$.*

PROOF. Let $\lambda \in \mathbb{T} \setminus \{\pm 1\}$ and $x \in \mathcal{M} \times \mathbb{T}$. Taking $\lambda_1 = 1$ and $\lambda_2 = \pm\lambda$ in (2.7), we obtain

$$|1 - \lambda| \le |1 - \overline{\alpha(1, x)}\alpha(\lambda, x)| \quad \text{and} \quad |1 + \lambda| \le |1 + \overline{\alpha(1, x)}\alpha(\lambda, x)|,$$

where we have used Lemma 3.9. Since $\overline{\alpha(1, x)}\alpha(\lambda, x) \in \mathbb{T}$, we conclude that

$$\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, \overline{\lambda}\}.$$

If we consider the case when $\lambda = i$, then we have $\overline{\alpha(1, x)}\alpha(i, x) \in \{\pm i\}$. This implies that $\alpha(i, x) = i\varepsilon_0(x)\alpha(1, x)$ for some $\varepsilon_0(x) \in \{\pm 1\}$. Entering $\lambda_1 = i$ and $\lambda_2 = \lambda$ into (2.7) to get

$$|i - \lambda| \le |1 - \overline{\alpha(i, x)}\alpha(\lambda, x)| = |1 + i\varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)| = |i - \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|,$$

and thus $|i - \lambda| \le |i - \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|$. Because $\alpha(-\lambda, x) = -\alpha(\lambda, x)$ by Lemma 3.9, we get $|i + \lambda| \le |i + \varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x)|$. These inequalities imply $\varepsilon_0(x)\overline{\alpha(1, x)}\alpha(\lambda, x) \in \{\lambda, -\overline{\lambda}\}$, since $\varepsilon_0(x)\overline{\alpha(1,x)}\alpha(\lambda,x) \in \mathbb{T}$. Then

$$\overline{\alpha(1,x)}\alpha(\lambda,x) \in \{\lambda, \overline{\lambda}\} \cap \{\varepsilon_0(x)\lambda, -\varepsilon_0(x)\overline{\lambda}\}.$$

We have two possible cases to consider. If $\varepsilon_0(x) = 1$, then we obtain $\overline{\alpha(1,x)}\alpha(\lambda,x) \in \{\lambda,\overline{\lambda}\} \cap \{\lambda, -\overline{\lambda}\}$. Since $\lambda \neq \pm 1$, we conclude that $\overline{\alpha(1,x)}\alpha(\lambda,x) = \lambda$, and hence $\alpha(\lambda,x) = \lambda^{\varepsilon_0(x)}\alpha(1,x)$. If $\varepsilon_0(x) = -1$, then $\overline{\alpha(1,x)}\alpha(\lambda,x) \in \{\lambda,\overline{\lambda}\} \cap \{-\lambda,\overline{\lambda}\}$, which yields $\overline{\alpha(1,x)}\alpha(\lambda,x) = \overline{\lambda}$. Thus, $\alpha(\lambda,x) = \lambda^{\varepsilon_0(x)}\alpha(1,x)$. These identities are valid even for $\lambda = \pm 1$. By the liberty of the choice of $\lambda \in \mathbb{T}$, we conclude that $\alpha(\lambda,x) = \lambda^{\varepsilon_0(x)}\alpha(1,x)$ for all $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. □

By Lemmas 3.11 and 3.13, we can rewrite (2.5) as

$$T(\lambda V_x) = \lambda^{\varepsilon_0(x)}\alpha(x)V_{(\phi_1(x),\phi_2(\lambda,x))} \tag{2.8}$$

for all $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$.

Definition 3.14. Let λV_x and μV_y be maximal convex subsets of $S(B)$, where $\lambda, \mu \in \mathbb{T}$ and $x, y \in \mathcal{M} \times \mathbb{T}$. We denote by $d_H(\lambda V_x, \mu V_y)$ the Hausdorff distance of λV_x and μV_y, that is,

$$d_H(\lambda V_x, \mu V_y) = \max\left\{\sup_{\widetilde{F}\in\lambda V_x} d(\widetilde{F}, \mu V_y), \sup_{\widetilde{g}\in\mu V_y} d(\lambda V_x, \widetilde{g})\right\}, \tag{2.9}$$

where $d(\widetilde{F}, \mu V_y) = \inf_{\widetilde{h}\in\mu V_y} \|\widetilde{F} - \widetilde{h}\|_\infty$ and $d(\lambda V_x, \widetilde{g}) = \inf_{\widetilde{h}\in\lambda V_x} \|\widetilde{h} - \widetilde{g}\|_\infty$.

Since T is a surjective isometry on $S(B)$, we obtain

$$d(T(\widetilde{F}), T(\mu V_y)) = \inf_{\widetilde{h}\in T(\mu V_y)} \|T(\widetilde{F}) - \widetilde{h}\|_\infty = \inf_{T^{-1}(\widetilde{h})\in\mu V_y} \|\widetilde{F} - T^{-1}(\widetilde{h})\|_\infty = d(\widetilde{F}, \mu V_y)$$

for every $\widetilde{F} \in \lambda V_x$. Hence, $\sup_{T(\widetilde{F})\in T(\lambda V_x)} d(T(\widetilde{F}), T(\mu V_y)) = \sup_{\widetilde{F}\in\lambda V_x} d(\widetilde{F}, \mu V_y)$. By the same reasoning, we get $\sup_{T(\widetilde{g})\in T(\mu V_y)} d(T(\lambda V_x), T(\widetilde{g})) = \sup_{\widetilde{g}\in\mu V_y} d(\lambda V_x, \widetilde{g})$, and thus

$$d_H(T(\lambda V_x), T(\mu V_y)) = d_H(\lambda V_x, \mu V_y) \qquad (\lambda, \mu \in \mathbb{T}, x, y \in \mathcal{M} \times \mathbb{T}). \tag{2.10}$$

Remark 3.15. Let $\lambda \in \mathbb{T}$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. For each $\widetilde{F} \in \lambda V_{(\eta,z)}$, we observe that

$$\overline{\lambda}f(0) \in [0,1] \quad \text{and} \quad \widehat{f'}(\eta)\overline{\lambda}z = \|\widehat{f'}\|_\infty.$$

In fact, $f(0) + \widehat{f'}(\eta)z = \lambda$ by the definition of $\lambda V_{(\eta,z)}$. Then

$$1 = \overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\} = |\overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\}| \leq |\overline{\lambda}f(0)| + |\widehat{f'}(\eta)\overline{\lambda}z| \leq \|f\|_\sigma = 1,$$

and thus, $|\overline{\lambda}f(0) + \widehat{f'}(\eta)\overline{\lambda}z| = |\overline{\lambda}f(0)| + |\widehat{f'}(\eta)\overline{\lambda}z|$. This implies that $\overline{\lambda}f(0) = t\widehat{f'}(\eta)\overline{\lambda}z$ for some $t \geq 0$, provided $\widehat{f'}(\eta) \neq 0$. Since $\overline{\lambda}\{f(0) + \widehat{f'}(\eta)z\} = 1$, we have $\widehat{f'}(\eta)\overline{\lambda}z = 1/(1+t)$ and $\overline{\lambda}f(0) = t/(1+t) \in [0,1]$. If $\widehat{f'}(\eta) = 0$, then $\overline{\lambda}f(0) = 1$, and hence $\overline{\lambda}f(0) \in [0,1]$ as well. In particular, $\overline{\lambda} f(0) = |f(0)|$. We infer from $\widehat{f'}(\eta)\overline{\lambda} z = 1 - \overline{\lambda} f(0)$ and $\|\widehat{f'}\|_\infty = 1 - |f(0)|$ that $\widehat{f'}(\eta)\overline{\lambda} z = \|\widehat{f'}\|_\infty$.

LEMMA 3.16. *For each $\eta \in \mathcal{M}$, $z \in \mathbb{T}$ and $k \in \{\pm 1\}$, the following equalities hold:*

$$\sup_{\widetilde{F} \in kV_{(\eta,k)}} d(\widetilde{F}, kV_{(\eta,z)}) = \sup_{\widetilde{g} \in kV_{(\eta,z)}} d(kV_{(\eta,k)}, \widetilde{g}) = |1 - kz|. \tag{2.11}$$

In particular, $d_H(kV_{(\eta,k)}, kV_{(\eta,z)}) = |1 - kz|$ for all $\eta \in \mathcal{M}$, $z \in \mathbb{T}$ and $k = \pm 1$.

PROOF. Fix an arbitrary $\widetilde{F} \in kV_{(\eta,k)}$ and $\widetilde{g} \in kV_{(\eta,z)}$, and then

$$f(0) + \widehat{f'}(\eta)k = k \quad \text{and} \quad g(0) + \widehat{g'}(\eta)z = k. \tag{2.12}$$

We notice that $kf(0), kg(0) \in [0,1]$, $\widehat{f'}(\eta) = \|\widehat{f'}\|_\infty$ and $\widehat{g'}(\eta)kz = \|\widehat{g'}\|_\infty$ by *Remark* 3.15. We deduce from the choice of $\widetilde{f}$ and $\widetilde{g}$ that

$$\begin{aligned}
|(1 - kz)(kf(0) - 1)| &\leq |kf(0) - kg(0)| + |kg(0) - 1 - kz(kf(0) - 1)| \\
&= |f(0) - g(0)| + |\overline{z}(g(0) - k) - (kf(0) - 1)| \\
&= |f(0) - g(0)| + |\widehat{g'}(\eta) - \widehat{f'}(\eta)| && \text{by (2.12)} \\
&\leq |f(0) - g(0)| + \|\widehat{f'} - \widehat{g'}\|_\infty = \|f - g\|_\sigma = \|\widetilde{f} - \widetilde{g}\|_\infty.
\end{aligned}$$

That is, $|1 - kz|(1 - kf(0)) \leq \|\widetilde{f} - \widetilde{g}\|_\infty$. We also have $|(1 - k\overline{z})(kg(0) - 1)| \leq \|\widetilde{f} - \widetilde{g}\|_\infty$ by a similar calculation, and thus, $|1 - kz|(1 - kg(0)) \leq \|\widetilde{f} - \widetilde{g}\|_\infty$. By the liberty of the choice of $\widetilde{f} \in kV_{(\eta,k)}$ and $\widetilde{g} \in kV_{(\eta,z)}$, we obtain

$$|1 - kz|(1 - kf(0)) \leq d(\widetilde{f}, kV_{(\eta,z)}) \quad \text{and} \quad |1 - kz|(1 - kg(0)) \leq d(kV_{(\eta,k)}, \widetilde{g}).$$

Setting $f_1 = f(0) + \mathcal{I}(k\overline{z}\widehat{f'})$ and $g_1 = g(0) + \mathcal{I}(kz\widehat{g'})$, we see that $\widetilde{f_1}(\eta, z) = f(0) + k\widehat{f'}(\eta) = k$ and $\widetilde{g_1}(\eta, k) = g(0) + z\widehat{g'}(\eta) = k$ by (2.12), where we have used that $\mathcal{I}(u)(0) = 0$ for $u \in \mathcal{A}$. Consequently, $\widetilde{f_1} \in kV_{(\eta,z)}$ and $\widetilde{g_1} \in kV_{(\eta,k)}$. By the choice of f_1, we have

$$\begin{aligned}
\|\widetilde{f} - \widetilde{f_1}\|_\infty &= \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |\widetilde{f}(\zeta, \nu) - \widetilde{f_1}(\zeta, \nu)| = \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |(1 - k\overline{z})\widehat{f'}(\zeta)\nu| \\
&= |1 - k\overline{z}| \, \|\widehat{f'}\|_\infty = |1 - kz| \, \widehat{f'}(\eta) = |1 - kz|(1 - kf(0))
\end{aligned}$$

by (2.12). In the same way, we get

$$\|\widetilde{g_1} - \widetilde{g}\|_\infty = \sup_{(\zeta,\nu) \in \mathcal{M} \times \mathbb{T}} |(kz - 1)\widehat{g'}(\zeta)\nu| = |kz - 1| \, \|\widehat{g'}\|_\infty = |1 - kz|(1 - kg(0)),$$

which yields $d(\widetilde{f}, kV_{(\eta,z)}) = |1-kz|(1-kf(0))$ and $d(kV_{(\eta,k)}, \widetilde{g}) = |1-kz|(1-kg(0))$. Having in mind that $kf(0), kg(0) \in [0,1]$, we conclude that $\sup_{\widetilde{f}\in kV_{(\eta,k)}} d(\widetilde{f}, kV_{(\eta,z)}) = |1-kz| = \sup_{\widetilde{g}\in kV_{(\eta,z)}} d(kV_{(\eta,k)}, \widetilde{g})$. $\square$

LEMMA 3.17. *The identity $\phi_1(\eta, z) = \phi_1(\eta, 1)$ holds for all $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$; we shall write $\phi_1(\eta, z) = \phi_1(\eta)$ for the sake of simplicity of notation.*

PROOF. Fix arbitrary $k \in \{\pm 1\}$, $\eta \in \mathcal{M}$ and $z \in \mathbb{T} \setminus \{\pm 1\}$. We assume that $\phi_1(\eta, z) \neq \phi_1(\eta, k)$. There exists $u_k \in S(C(\mathcal{M}))$ such that

$$u_k(\phi_1(\eta, z)) = k\alpha(\eta, z)\overline{\phi_2(k, (\eta, z))} \quad \text{and} \quad u_k(\phi_1(\eta, k)) = -k\alpha(\eta, k)\overline{\phi_2(k, (\eta, k))}.$$

Setting $g_k = \mathcal{I}(u_k)$, we see that $\widetilde{g_k} \in k\alpha(\eta, z)V_{\phi(k,(\eta,z))} \cap (-k\alpha(\eta, k))V_{\phi(k,(\eta,k))}$, where we have used $\phi_1(\lambda, x) = \phi_1(x)$ by Lemma 3.11. For any $\widetilde{f} \in k\alpha(\eta, k)V_{\phi(k,(\eta,k))}$, we obtain

$$2 = |k\alpha(\eta, k) + k\alpha(\eta, k)| = |\widetilde{f}(\phi(k, (\eta, k))) - \widetilde{g_k}(\phi(k, (\eta, k)))| \leq \|\widetilde{f} - \widetilde{g_k}\|_\infty \leq 2,$$

which shows $d(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, \widetilde{g_k}) = 2$. Combining (2.8), (2.9), (2.10) and (2.11), we get

$$\begin{aligned} 2 &\leq \sup_{\widetilde{g}\in k\alpha(\eta,z)V_{\phi(k,(\eta,z))}} d(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, \widetilde{g}) \\ &\leq d_H(k\alpha(\eta, k)V_{\phi(k,(\eta,k))}, k\alpha(\eta, z)V_{\phi(k,(\eta,z))}) = d_H(T(kV_{(\eta,k)}), T(kV_{(\eta,z)})) \\ &= d_H(kV_{(\eta,k)}, kV_{(\eta,z)}) = |1 - kz|, \end{aligned}$$

which implies $z = -k$. This contradicts $z \neq \pm 1$, and thus $\phi_1(\eta, z) = \phi_1(\eta, k)$ for $z \neq \pm 1$. Entering $z = i$ and $k = \pm 1$ into the last equality, we get $\phi_1(\eta, 1) = \phi_1(\eta, i) = \phi_1(\eta, -1)$. Therefore, we conclude $\phi_1(\eta, z) = \phi_1(\eta, 1)$ for all $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$. $\square$

LEMMA 3.18. *The following inequalities hold for all $\lambda, \mu \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$;*

$$|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) - \mu^{\varepsilon_0(x)}| \leq |\lambda - \mu|, \tag{2.13}$$

$$\textit{and} \quad |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) + \mu^{\varepsilon_0(x)}| \leq |\lambda + \mu|.$$

PROOF. Take any $\lambda, \mu \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. For each $\widetilde{f} \in \lambda V_x$ and $\widetilde{g} \in \mu V_x$, we obtain $|\lambda - \mu| = |\widetilde{f}(x) - \widetilde{g}(x)| \leq \|\widetilde{f} - \widetilde{g}\|_\infty$, which yields $|\lambda - \mu| \leq d(\widetilde{f}, \mu V_x)$. Set $f_0 = \overline{\lambda}\mu f$, and then we see that $\widetilde{f_0} \in \mu V_x$ with $\|\widetilde{f} - \widetilde{f_0}\|_\infty = \|(1 - \overline{\lambda}\mu)\widetilde{f}\|_\infty = |\lambda - \mu|$. This implies $d(\widetilde{f}, \mu V_x) = |\lambda - \mu|$. By the same argument, we see that $d(\lambda V_x, \widetilde{g}) = |\lambda - \mu|$. Consequently, $d_H(\lambda V_x, \mu V_x) = |\lambda - \mu|$ by (2.9).

Let us define $f_1 = \alpha(\lambda, x)\overline{\phi_2(\lambda, x)}\mathcal{I}(\mathbf{1}_\mathcal{M})$, and then we see that $\widetilde{f_1} \in \alpha(\lambda, x)V_{\phi(\lambda,x)} = T(\lambda V_x)$ by (2.3) and (2.5). Set $\widetilde{g_1} = T(\widetilde{g})$ for each $\widetilde{g} \in \mu V_x$. Then $\widetilde{g_1} \in T(\mu V_x) = \alpha(\mu, x)V_{\phi(\mu,x)}$. By the definition of the set νV_y, we have $\widehat{f_1'}(\phi_1(x))\phi_2(\lambda, x) = \lambda^{\varepsilon_0(x)}\alpha(x)$ and $g_1(0)+\widehat{g_1'}(\phi_1(x))\phi_2(\mu, x) = \mu^{\varepsilon_0(x)}\alpha(x)$, where we have used (2.8). We deduce from $\alpha(x), \phi_2(\lambda, x), \phi_2(\mu, x) \in \mathbb{T}$ that

$$\begin{aligned} |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu, x)}| &\leq |\widehat{f_1'}(\phi_1(x)) - \widehat{g_1'}(\phi_1(x))| + |g_1(0)| \\ &\leq |f_1(0) - g_1(0)| + \|\widehat{f_1'} - \widehat{g_1'}\|_\infty = \|f_1 - g_1\|_\sigma = \|\widetilde{f_1} - \widetilde{g_1}\|_\infty, \end{aligned}$$

which shows $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu, x)}| \leq d(\widetilde{f_1}, T(\mu V_x))$. We infer from (2.9) and (2.10) that

$$\begin{aligned} |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)} - \mu^{\varepsilon_0(x)}\overline{\phi_2(\mu, x)}| &\leq \sup_{T(\widetilde{f}) \in T(\lambda V_x)} d(T(\widetilde{f}), T(\mu V_x)) \\ &\leq d_H(T(\lambda V_x), T(\mu V_x)) = d_H(\lambda V_x, \mu V_x) = |\lambda - \mu|. \end{aligned}$$

Thus, $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) - \mu^{\varepsilon_0(x)}| \leq |\lambda - \mu|$. Noting that $\phi_2(-\mu, x) = \phi_2(\mu, x)$ by (2.6), we obtain $|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(\mu, x) + \mu^{\varepsilon_0(x)}| \leq |\lambda + \mu|$. □

Lemma 3.19. *For each $x \in \mathcal{M} \times \mathbb{T}$, there exists $\varepsilon_1(x) \in \{\pm 1\}$ such that $\phi_2(\lambda, x) = \lambda^{\varepsilon(x)-\varepsilon_1(x)}\phi_2(1, x)$ for all $\lambda \in \mathbb{T}$.*

Proof. Fix arbitrary $x \in \mathcal{M} \times \mathbb{T}$ and $\lambda \in \mathbb{T} \setminus \{\pm 1\}$. We obtain

$$|\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(1, x) \pm 1| \leq |\lambda \pm 1|$$

by (2.13) with $\mu = 1$, which implies $\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(1, x) \in \{\lambda, \overline{\lambda}\}$. Hence,

$$\overline{\phi_2(\lambda, x)}\phi_2(1, x) \in \{\lambda^{1-\varepsilon_0(x)}, \lambda^{-1-\varepsilon_0(x)}\}.$$

In particular, $\overline{\phi_2(i, x)}\phi_2(1, x) \in \{\pm\varepsilon_0(x)\}$, and thus $\phi_2(i, x) = \varepsilon_1(x)\varepsilon_0(x)\phi_2(1, x)$ for some $\varepsilon_1(x) \in \{\pm 1\}$. Entering $\mu = i$ into (2.13), we get

$$|\lambda - i| \geq |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\phi_2(i, x) - \varepsilon_0(x)i| = |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\varepsilon_1(x)\phi_2(1, x) - i|.$$

By the same reasoning, we have $|\lambda + i| \geq |\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\varepsilon_1(x)\phi_2(1, x) + i|$. Then we derive from these two inequalities that $\lambda^{\varepsilon_0(x)}\overline{\phi_2(\lambda, x)}\varepsilon_1(x)\phi_2(1, x) \in \{\lambda, -\overline{\lambda}\}$. Thus, $\varepsilon_1(x)\overline{\phi_2(\lambda, x)}\phi_2(1, x) \in \{\lambda^{1-\varepsilon_0(x)}, -\lambda^{-1-\varepsilon_0(x)}\}$. Now we obtain

$$\overline{\phi_2(\lambda, x)}\phi_2(1, x) \in \{\lambda^{1-\varepsilon_0(x)}, \lambda^{-1-\varepsilon_0(x)}\} \cap \{\varepsilon_1(x)\lambda^{1-\varepsilon_0(x)}, -\varepsilon_1(x)\lambda^{-1-\varepsilon_0(x)}\}.$$

Note that $\lambda \neq \pm 1$. If $\varepsilon_1(x) = 1$, then we get $\overline{\phi_2(\lambda, x)}\phi_2(1, x) = \lambda^{1-\varepsilon_0(x)}$, and if $\varepsilon_1(x) = -1$, then $\overline{\phi_2(\lambda, x)}\phi_2(1, x) = \lambda^{-1-\varepsilon_0(x)}$. These imply that $\overline{\phi_2(\lambda, x)}\phi_2(1, x) = \lambda^{\varepsilon_1(x)-\varepsilon_0(x)}$ for $\lambda \in \mathbb{T} \setminus \{\pm 1\}$. The last identity is valid even for $\lambda \in \{\pm 1\}$ by (2.6). Therefore, we conclude that $\phi_2(\lambda, x) = \lambda^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(1, x)$ for all $\lambda \in \mathbb{T}$. □

We shall write $\phi_2(1,x) = \phi_2(x)$ for $x \in \mathcal{M} \times \mathbb{T}$. Let $\lambda \in \mathbb{T}$ and $x \in \mathcal{M} \times \mathbb{T}$. By (2.8), $T(\widetilde{f})(\phi_1(x), \phi_2(\lambda, x)) = \lambda^{\varepsilon_0(x)}\alpha(x) = \alpha(\lambda, x)$ for $f \in S(\mathrm{Lip}(I))$ with $\widetilde{f} \in \lambda V_x$. Noting that $T(\widetilde{f}) = \widetilde{\Delta(f)}$ by (2.4), we infer from Lemma 3.17 that

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda, x) = \alpha(\lambda, x) \tag{2.14}$$

for all $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ with $\widetilde{f} \in \lambda V_x$. If we apply Lemma 3.19, then we can rewrite the last equality as

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = \lambda^{\varepsilon_0(x)}\alpha(x) \tag{2.15}$$

for $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ satisfying $\widetilde{f} \in \lambda V_x$.

Lemma 3.20. *Suppose that $\Delta(\lambda_0 \mathbf{1}_I)(0) = 0$ for some $\lambda_0 \in \mathbb{T}$. Then $\widehat{\Delta(\lambda_0 id_I)'} = 0$ on $\mathcal{M}$ for the identity function id_I on I.*

Proof. Fix arbitrary $\eta \in \mathcal{M}$ and $z \in \mathbb{T}$, and we set $x = (\eta, z)$. We note $\lambda_0\widetilde{\mathbf{1}_I} \in \lambda_0 V_x$, and then equality (2.15) shows that $\widehat{\Delta(\lambda_0 \mathbf{1}_I)'}(\phi_1(\eta))\lambda_0^{-\varepsilon_1(x)}\phi_2(x) = \alpha(x)$. We set $e(\eta) = \widehat{\Delta(\lambda_0 \mathbf{1}_I)'}(\phi_1(\eta))$ for the sake of simplicity of notation. Then we can rewrite the above equality as

$$e(\eta)\lambda_0^{-\varepsilon_1(x)}\phi_2(x) = \alpha(x). \tag{2.16}$$

Since $\lambda_0 id_I \in \lambda_0 z V_{(\eta,z)}$, we get, by (2.15),

$$\Delta(\lambda_0 id_I)(0) + \widehat{\Delta(\lambda_0 id_I)'}(\phi_1(\eta))(\lambda_0 z)^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = (\lambda_0 z)^{\varepsilon_0(x)}\alpha(x).$$

Combining (2.16) with the last equality, we obtain

$$\Delta(\lambda_0 id_I)(0) + \widehat{\Delta(\lambda_0 id_I)'}(\phi_1(\eta))(\lambda_0 z)^{\varepsilon_0(x)-\varepsilon_1(x)}\phi_2(x) = (\lambda_0 z)^{\varepsilon_0(x)}e(\eta)\lambda_0^{-\varepsilon_1(x)}\phi_2(x),$$

which leads to

$$\Delta(\lambda_0 id_I)(0) = (\lambda_0 z)^{\varepsilon_0(x)}\left\{e(\eta)z^{\varepsilon_1(x)} - \widehat{\Delta(\lambda_0 id_I)'}(\phi_1(\eta))\right\}(\lambda_0 z)^{-\varepsilon_1(x)}\phi_2(x).$$

Note that $|e(\eta)| = 1$ by (2.16). Taking the modulus of the above equality, we get $|\Delta(\lambda_0 id_I)(0)| = |z^{\varepsilon_1(x)} - \overline{e(\eta)}\,\widehat{\Delta(\lambda_0 id_I)'}(\phi_1(\eta))|$. Since $z \in \mathbb{T}$ is arbitrary, the last equality holds for $z = \pm 1, i$. Then we have $\widehat{\Delta(\lambda_0 id_I)'}(\phi_1(\eta)) = 0$. Having in mind that $\eta \in \mathcal{M}$ is arbitrarily fixed, we obtain $\widehat{\Delta(\lambda_0 id_I)'} = 0$ on $\mathcal{M}$, where we have used $\phi_1(\mathcal{M}) = \mathcal{M}$ by Lemmas 3.9, 3.11 and 3.17. $\square$

Lemma 3.21. *For each $\lambda \in \mathbb{T}$, the value $\Delta(\lambda \mathbf{1}_I)(0)$ is nonzero.*

Proof. Suppose, on the contrary, that $\Delta(\lambda_0 \mathbf{1}_I)(0) = 0$ for some $\lambda_0 \in \mathbb{T}$. Then $\widehat{\Delta(\lambda_0 id_I)}' = 0$ on $\mathcal{M}$ by Lemma 3.20. We define a function $f_0 \in S(\mathrm{Lip}(I))$ by $f_0 = \lambda_0(2id_I + id_I^2)/4$. We shall prove that $\widehat{f_0'}(\eta_0) = \lambda_0$ for some $\eta_0 \in \mathcal{M}$. Let $\mathcal{R}(id_I)$ be the *essential range* of $id_I \in \mathrm{Lip}(I)$, that is, $\mathcal{R}(id_I)$ is the set of all $\zeta \in \mathbb{C}$ for which $\{t \in I : |id_I(t) - \zeta| < \epsilon\}$ has positive measure for all $\epsilon > 0$. By definition, we see that $\mathcal{R}(id_I) = id_I(I) = I$. For the spectrum $\sigma(id_I)$ of id_I, we observe that $\mathcal{R}(id_I) = \sigma(id_I) = \widehat{id_I}(\mathcal{M})$ (see, for example, [**23**, Lemma 2.63]). Thus, there exists $\eta_0 \in \mathcal{M}$ such that $\widehat{id_I}(\eta_0) = 1$, which yields $\widehat{f_0'}(\eta_0) = \lambda_0(2 + 2\widehat{id_I}(\eta_0))/4 = \lambda_0$ as is claimed. Fix an arbitrary $z \in \mathbb{T}$, and then we see that $\lambda_0 \widetilde{id_I} \in \lambda_0 z V_{(\eta_0, z)}$ with $\widehat{\Delta(\lambda_0 id_I)}' = 0$ on $\mathcal{M}$. Applying (2.14) to $f = \lambda_0 id_I$, we have $\Delta(\lambda_0 id_I)(0) = \alpha(\lambda_0 z, (\eta_0, z))$. Having in mind that $z \in \mathbb{T}$ is arbitrary, we may enter $z = \pm 1$ into the last equality. Then we get

$$\alpha(\lambda_0, (\eta_0, 1)) = \alpha(-\lambda_0, (\eta_0, -1)). \tag{2.17}$$

Note also that $\widetilde{f_0} \in \lambda_0 z V_{(\eta_0, z)}$, and thus

$$\Delta(f_0)(0) + \widehat{\Delta(f_0)}'(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z)) = \alpha(\lambda_0 z, (\eta_0, z))$$

by (2.14). Since $\Delta(\lambda_0 id_I)(0) = \alpha(\lambda_0 z, (\eta_0, z))$, we can rewrite the above equality as

$$\Delta(f_0)(0) + \widehat{\Delta(f_0)}'(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z)) = \Delta(\lambda_0 id_I)(0), \tag{2.18}$$

which yields $|\Delta(\lambda_0 id_I)(0) - \Delta(f_0)(0)| = |\widehat{\Delta(f_0)}'(\phi_1(\eta_0))| \leq \|\widehat{\Delta(f_0)}'\|_\infty$. We thus obtain

$$\begin{aligned} 2\|\widehat{\Delta(f_0)}'\|_\infty &\geq |\Delta(\lambda_0 id_I)(0) - \Delta(f_0)(0)| + \|\widehat{\Delta(f_0)}'\|_\infty \\ &= |\Delta(\lambda_0 id_I)(0) - \Delta(f_0)(0)| + \|\widehat{\Delta(\lambda_0 id_I)}' - \widehat{\Delta(f_0)}'\|_\infty \\ &= \|\Delta(\lambda_0 id_I) - \Delta(f_0)\|_\sigma = \|\lambda_0 id_I - f_0\|_\sigma = \frac{1}{2}\|\widehat{\mathbf{1}}_I - \widehat{id_I}\|_\infty = \frac{1}{2}. \end{aligned}$$

Hence, we have $\|\widehat{\Delta(f_0)}'\|_\infty \geq 1/4$, which implies $|\Delta(f_0)(0)| \leq 3/4$, since $\|\Delta(f_0)\|_\sigma = 1$. It follows from (2.18) that

$$1 = |\alpha(\lambda_0 z, (\eta_0, z))| = |\Delta(\lambda_0 id_I)(0)| = |\Delta(f_0)(0) + \widehat{\Delta(f_0)}'(\phi_1(\eta_0))\phi_2(\lambda_0 z, (\eta_0, z))|.$$

Since $|\Delta(f_0)(0)| \leq 3/4$, we see that $\widehat{\Delta(f_0)}'(\phi_1(\eta_0)) \neq 0$. By the liberty of the choice of $z \in \mathbb{T}$, we deduce from (2.18) that $\phi_2(\lambda_0 z, (\eta_0, z))$ is invariant with respect to $z \in \mathbb{T}$. Entering $z = \pm 1$ into $\phi_2(\lambda_0 z, (\eta_0, z))$, we get

$$\phi_2(\lambda_0, (\eta_0, 1)) = \phi_2(-\lambda_0, (\eta_0, -1)). \tag{2.19}$$

Set $f_1 = \lambda_0(2 + id_I^2)/4 \in S(\mathrm{Lip}(I))$, and then we have $\widetilde{f_1} \in \lambda_0 V_{(\eta_0,1)}$, because $\widehat{id_I}(\eta_0) = 1$. We deduce from (2.14) that

$$\Delta(f_1)(0) + \widehat{\Delta(f_1)'}(\phi_1(\eta_0))\phi_2(\lambda_0, (\eta_0, 1)) = \alpha(\lambda_0, (\eta_0, 1)). \tag{2.20}$$

Combining (2.17) and (2.19) with (2.20), we have

$$\Delta(f_1)(0) + \widehat{\Delta(f_1)'}(\phi_1(\eta_0))\phi_2(-\lambda_0, (\eta_0, -1)) = \alpha(-\lambda_0, (\eta_0, -1)).$$

Here, we recall that $T(\widetilde{f_1}) = \widetilde{\Delta(f_1)}$ by (2.4). Then the above equality with (2.5) and (2.14) implies that $T(\widetilde{f_1}) \in \alpha(-\lambda_0, (\eta_0, -1))V_{\phi(-\lambda_0,(\eta_0,-1))} = T(-\lambda_0 V_{(\eta_0,-1)})$, which shows $\widetilde{f_1} \in (-\lambda_0)V_{(\eta_0,-1)}$. Consequently, $\widetilde{f_1} \in (-\lambda_0)V_{(\eta_0,-1)} \cap \lambda_0 V_{(\eta_0,1)}$, and therefore, we obtain

$$f_1(0) - \widehat{f_1'}(\eta_0) = -\lambda_0 = -\{f_1(0) + \widehat{f_1'}(\eta_0)\}.$$

This leads to $f_1(0) = -f_1(0)$, which yields $f_1(0) = 0$. On the other hand, $f_1(0) = \lambda_0(2 + id_I^2(0))/4 = \lambda_0/2 \neq 0$. This is a contradiction. We conclude that $\Delta(\lambda \mathbf{1}_I)(0) \neq 0$ for all $\lambda \in \mathbb{T}$. □

LEMMA 3.22. *The values $\alpha(x)$ and $\varepsilon_0(x)$ are both independent from the variable $x \in \mathcal{M} \times \mathbb{T}$; we shall write $\alpha(x) = \alpha$ and $\varepsilon_0(x) = \varepsilon_0$.*

PROOF. Take any $\lambda \in \mathbb{T}$ and $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$. According to (2.14), applied to $f = \lambda \mathbf{1}_I$, we have

$$\begin{aligned} 1 = |\lambda^{\varepsilon_0(x)}\alpha(x)| &= |\Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x)| \\ &\leq |\Delta(\lambda \mathbf{1}_I)(0)| + |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))| \leq \|\Delta(\lambda \mathbf{1}_I)\|_\sigma = 1. \end{aligned}$$

The above inequalities show that

$$|\Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x)| = 1 = |\Delta(\lambda \mathbf{1}_I)(0)| + |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))|.$$

Note that $\Delta(\lambda \mathbf{1}_I)(0) \neq 0$ by Lemma 3.21. By the above equality, there exists $t \geq 0$ such that $\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x) = t\Delta(\lambda \mathbf{1}_I)(0)$. We thus obtain

$$|t\Delta(\lambda \mathbf{1}_I)(0)| = |\widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))| = 1 - |\Delta(\lambda \mathbf{1}_I)(0)|,$$

which yields $(1 + t)|\Delta(\lambda \mathbf{1}_I)(0)| = 1$. Consequently,

$$\lambda^{\varepsilon_0(x)}\alpha(x) = \Delta(\lambda \mathbf{1}_I)(0) + \widehat{\Delta(\lambda \mathbf{1}_I)}'(\phi_1(\eta))\phi_2(\lambda, x) = (1 + t)\Delta(\lambda \mathbf{1}_I)(0) = \frac{\Delta(\lambda \mathbf{1}_I)(0)}{|\Delta(\lambda \mathbf{1}_I)(0)|}$$

by (2.14). Then $\alpha(x) = \Delta(\mathbf{1}_I)(0)/|\Delta(\mathbf{1}_I)(0)|$ is independent from $x \in \mathcal{M} \times \mathbb{T}$. Letting $\lambda = i$ in the above equality, we get $i\varepsilon_0(x)\alpha(x) = \Delta(i\mathbf{1}_I)(0)/|\Delta(i\mathbf{1}_I)(0)|$. Thus, ε_0 is constant on $\mathcal{M} \times \mathbb{T}$. $\square$

By Lemma 3.22, we can rewrite (2.15) as

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0 - \varepsilon_1(x)}\phi_2(x) = \lambda^{\varepsilon_0}\alpha \tag{2.21}$$

for all $\lambda \in \mathbb{T}$, $x = (\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ with $\widetilde{f} \in \lambda V_x$.

LEMMA 3.23. *Let $\eta \in \mathcal{M}$, $\lambda \in \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ be such that $\widehat{f'}(\eta) = \lambda$. Then $\Delta(f)$ satisfies $\Delta(f)(0) = 0$ and*

$$\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha \tag{2.22}$$

for all $z \in \mathbb{T}$.

PROOF. Fix an arbitrary $z \in \mathbb{T}$. By the choice of f, we have $\widetilde{f} \in \lambda z V_{(\eta,z)}$. By (2.21) with $\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0 - \varepsilon_1(\eta,z)}\phi_2(\eta, z)$, we obtain

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha. \tag{2.23}$$

We observe that $\|\widehat{\Delta(f)'}\|_\infty \neq 0$; for if $\|\widehat{\Delta(f)'}\|_\infty = 0$, then we would have $\Delta(f)(0) = (\lambda z)^{\varepsilon_0}\alpha$ for all $z \in \mathbb{T}$, which is impossible. Equality (2.23) shows that

$$\begin{aligned} 1 &= |\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z))| \\ &\leq |\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| \leq \|\Delta(f)\|_\sigma = 1, \end{aligned}$$

and hence, $|\widehat{\Delta(f)'}(\phi_1(\eta))| = \|\widehat{\Delta(f)'}\|_\infty \neq 0$. Then there exists $s \geq 0$ such that

$$\Delta(f)(0) = s\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)). \tag{2.24}$$

It follows from (2.23) that

$$(1 + s)\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha,$$

which yields $(1+s)\|\widehat{\Delta(f)'}\|_\infty = 1$, or equivalently, $s\|\widehat{\Delta(f)'}\|_\infty = 1-\|\widehat{\Delta(f)'}\|_\infty$. These equalities show that

$$\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = \|\widehat{\Delta(f)'}\|_\infty(\lambda z)^{\varepsilon_0}\alpha.$$

We deduce from the last equality with (2.24) that $\Delta(f)(0) = s\|\widehat{\Delta(f)'}\|_\infty(\lambda z)^{\varepsilon_0}\alpha = (1 - \|\widehat{\Delta(f)'}\|_\infty)(\lambda z)^{\varepsilon_0}\alpha$, that is,

$$\Delta(f)(0) = (1 - \|\widehat{\Delta(f)'}\|_\infty)(\lambda z)^{\varepsilon_0}\alpha.$$

By the liberty of the choice of $z \in \mathbb{T}$, we get $1 - \|\widehat{\Delta(f)'}\|_\infty = 0 = \Delta(f)(0)$. Thus, by (2.23), $\widehat{\Delta(f)'}(\phi_1(\eta))\phi_2(\lambda z, (\eta, z)) = (\lambda z)^{\varepsilon_0}\alpha$ for all $z \in \mathbb{T}$. $\square$

Lemma 3.24. *For each $\lambda, z \in \mathbb{T}$ and $\eta \in \mathcal{M}$,*

$$\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)},$$

where $\varepsilon_1(\eta) = \varepsilon_1(\eta, 1)$.

Proof. Fix arbitrary $\lambda, z \in \mathbb{T}$ and $\eta \in \mathcal{M}$. Setting $\mu = \lambda\overline{z}$ and $v = \mu\mathbf{1}_\mathcal{M} \in S(C(\mathcal{M}))$, we see that $\mathcal{I}(v) \in S(\mathrm{Lip}(I))$ satisfies $\widehat{\mathcal{I}(v)}'(\eta) = \mu$ by (2.3). We may apply (2.22) to $f = \mathcal{I}(v)$, and we get $\widehat{\Delta(\mathcal{I}(v))}'(\phi_1(\eta))\phi_2(\mu z, (\eta, z)) = (\mu z)^{\varepsilon_0}\alpha$. Therefore, we obtain

$$\widehat{\Delta(\mathcal{I}(v))}'(\phi_1(\eta))\phi_2(\mu z, (\eta, z)) = \mu^{\varepsilon_0}\alpha \cdot z^{\varepsilon_0} = \widehat{\Delta(\mathcal{I}(v)}'(\phi_1(\eta))\phi_2(\mu, (\eta, 1))z^{\varepsilon_0}.$$

Then $\widehat{\Delta(\mathcal{I}(v))}'(\phi_1(\eta)) \neq 0$, and hence $\phi_2(\mu z, (\eta, z)) = \phi_2(\mu, (\eta, 1))z^{\varepsilon_0}$. This implies

$$\phi_2(\lambda, (\eta, z)) = \phi_2(\lambda\overline{z}, (\eta, 1))z^{\varepsilon_0}.$$

Applying Lemmas 3.19 and 3.22 to the last equality, we now get

$$\begin{aligned}\phi_2(\lambda, (\eta, z)) &= \phi_2(\lambda\overline{z}, (\eta, 1))z^{\varepsilon_0} = (\lambda\overline{z})^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_0}\\ &= \lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)}.\end{aligned}$$

Consequently, $\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(1, (\eta, 1))z^{\varepsilon_1(\eta)}$. $\square$

We shall write $\phi_2(1, (\eta, 1)) = \phi_2(\eta)$ for simplicity. According to Lemma 3.24, we can write

$$\phi_2(\lambda, (\eta, z)) = \lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} \tag{2.25}$$

for all $\lambda \in \mathbb{T}$ and $(\eta, z) \in \mathcal{M} \times \mathbb{T}$. Combining (2.21) and (2.25), with $\phi_2(\lambda, x) = \lambda^{\varepsilon_0 - \varepsilon_1(x)}\phi_2(x)$, we obtain

$$\Delta(f)(0) + \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0 - \varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} = \lambda^{\varepsilon_0}\alpha \tag{2.26}$$

for all $\lambda \in \mathbb{T}$, $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ with $\widetilde{f} \in \lambda V_{(\eta,z)}$.

Lemma 3.25. *Let $\lambda \in \mathbb{T}$, $(\eta, z) \in \mathcal{M} \times \mathbb{T}$ and $f \in S(\mathrm{Lip}(I))$ be such that $\widetilde{f} \in \lambda V_{(\eta,z)}$. Then*

$$\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha \quad \textit{and} \quad \widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty \lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}.$$

In particular,

$$|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| = |f(0)| + |\widehat{f'}(\eta)| \tag{2.27}$$

for all $f \in S(\mathrm{Lip}(I))$ with $\widetilde{f} \in \lambda V_{(\eta,z)}$.

Proof. By assumption, (2.26) holds. Taking the modulus of (2.26) to get

$$
\begin{aligned}
1 &\leq |\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)}| \\
&\leq |\Delta(f)(0)| + \|\widehat{\Delta(f)'}\|_\infty = \|\Delta(f)\|_\sigma = 1.
\end{aligned}
\tag{2.28}
$$

We derive from the last inequalities that $|\widehat{\Delta(f)'}(\phi_1(\eta))| = \|\widehat{\Delta(f)'}\|_\infty$.

If $\Delta(f)(0) = 0$, then the identity $\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha$ is obvious; in addition, $\|\widehat{\Delta(f)'}\|_\infty = \|\Delta(f)\|_\sigma = 1$, and hence $\widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}$ by (2.26). We next consider the case when $\Delta(f)(0) \neq 0$. There exists $s \geq 0$ such that $\widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)} = s\Delta(f)(0)$ by (2.28). Entering the last equality into (2.26) to get $(1+s)\Delta(f)(0) = \lambda^{\varepsilon_0}\alpha$. We thus obtain $(1+s)|\Delta(f)(0)| = 1$, and consequently, $\Delta(f)(0) = |\Delta(f)(0)|\lambda^{\varepsilon_0}\alpha$ holds even if $\Delta(f)(0) \neq 0$. Having in mind that $|\Delta(f)(0)| + \|\widehat{\Delta(f)'}\|_\infty = 1$, we infer from (2.26) that

$$
\begin{aligned}
\|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_0}\alpha &= (1 - |\Delta(f)(0)|)\lambda^{\varepsilon_0}\alpha = \lambda^{\varepsilon_0}\alpha - \Delta(f)(0) \\
&= \widehat{\Delta(f)'}(\phi_1(\eta))\lambda^{\varepsilon_0-\varepsilon_1(\eta)}\phi_2(\eta)z^{\varepsilon_1(\eta)}.
\end{aligned}
$$

This shows that $\widehat{\Delta(f)'}(\phi_1(\eta)) = \|\widehat{\Delta(f)'}\|_\infty\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}z^{-\varepsilon_1(\eta)}$. Since $\widetilde{f} \in \lambda V_{(\eta,z)}$, we get

$$
1 = |\lambda| = |f(0) + \widehat{f'}(\eta)z| \leq |f(0)| + |\widehat{f'}(\eta)| \leq \|f\|_\sigma = 1,
$$

and hence $|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta))| = 1 = |f(0)| + |\widehat{f'}(\eta)|$. □

For each $\lambda \in \mathbb{T}$ and $\eta \in \mathcal{M}$, we define λP_η by

$$
\lambda P_\eta = \{u \in S(C(\mathcal{M})) : u(\eta) = \lambda\}.
$$

Lemma 3.26. *Let $\eta_0 \in \mathcal{M}$ and $f \in S(\mathrm{Lip}(I))$. We set $\lambda = \widehat{f'}(\eta_0)/|\widehat{f'}(\eta_0)|$ if $\widehat{f'}(\eta_0) \neq 0$, and $\lambda = 1$ if $\widehat{f'}(\eta_0) = 0$. For each $t \in \mathbb{R}$ with $0 < t < 1$, there exists $u_t \in P_{\eta_0}$ such that*

$$
|tf(0)|\lambda + t\widehat{f'} + \left\{1 - |tf(0)| - |t\widehat{f'}(\eta_0)|\right\}\lambda u_t \in \lambda P_{\eta_0}.
$$

Proof. Note first that $1 - |tf(0)| - |t\widehat{f'}(\eta_0)| > 0$, since $|tf(0)| + |t\widehat{f'}(\eta_0)| \leq \|tf\|_\sigma < 1$. We set $r = 1 - |tf(0)| - |t\widehat{f'}(\eta_0)|$,

$$
G_0 = \left\{\eta \in \mathcal{M} : |t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \geq \frac{r}{4}\right\},
$$

$$
\text{and} \quad G_m = \left\{\eta \in \mathcal{M} : \frac{r}{2^{m+2}} \leq |t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \leq \frac{r}{2^{m+1}}\right\}
$$

for each $m \in \mathbb{N}$. We see that G_n is a closed subset of $\mathcal{M}$ with $\eta_0 \notin G_n$ for all $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N} \cup \{0\}$, there exists $v_n \in P_{\eta_0}$ such that

$$
v_n = 0 \quad \text{on } G_n \tag{2.29}
$$

by Urysohn's lemma. Setting $u_t = v_0 \sum_{n=1}^{\infty} v_n/2^n$, we see that u_t converges in $C(\mathcal{M})$, since $\|v_n\|_\infty = 1$ for all $n \in \mathbb{N}$. We observe that

$$1 = u_t(\eta_0) \le \|u_t\|_\infty \le \|v_0\|_\infty \sum_{n=1}^{\infty} \frac{\|v_n\|_\infty}{2^n} = 1,$$

and hence $u_t \in P_{\eta_0}$. Here, we define

$$w_t = |tf(0)|\lambda + t\widehat{f'} + r\lambda u_t \in C(\mathcal{M}).$$

We shall prove that $w_t \in \lambda P_{\eta_0}$. Since $u_t(\eta_0) = 1$ and $t\widehat{f'}(\eta_0) = |t\widehat{f'}(\eta_0)|\lambda$, we have

$$w_t(\eta_0) = |tf(0)|\lambda + t\widehat{f'}(\eta_0) + \left\{1 - |tf(0)| - |t\widehat{f'}(\eta_0)|\right\}\lambda = \lambda.$$

Fix an arbitrary $\eta \in \mathcal{M}$. To prove that $|w_t(\eta)| \le 1$, we shall consider three cases. First, we consider the case when $\eta \in G_0$. Then $v_0(\eta) = 0$ by (2.29), and hence $u_t(\eta) = 0$ by definition. We thus obtain $|w_t(\eta)| \le ||tf(0)|\lambda + t\widehat{f'}(\eta)| \le \|tf\|_\sigma < 1$, and consequently, $|w_t(\eta)| < 1$ if $\eta \in G_0$.

We next consider the case when $\eta \in \cup_{n=1}^{\infty} G_n$, and then $\eta \in G_m$ for some $m \in \mathbb{N}$. By the choice of G_m, we get $|t\widehat{f'}(\eta) - t\widehat{f'}(\eta_0)| \le r/2^{m+1}$. Thus, $|t\widehat{f'}(\eta)| \le |t\widehat{f'}(\eta_0)| + r/2^{m+1}$. We derive from (2.29) that $|r\lambda u_t(\eta)| \le r|v_0(\eta)| \sum_{n \ne m} |v_n(\eta)|/2^n \le r(1 - 2^{-m})$. Since $|tf(0)| + |\widehat{tf'}(\eta_0)| = 1 - r$, we obtain

$$\begin{aligned} |w_t(\eta)| &\le |tf(0)| + |t\widehat{f'}(\eta)| + |r\lambda u_t(\eta)| \le |tf(0)| + |t\widehat{f'}(\eta_0)| + \frac{r}{2^{m+1}} + r\left(1 - \frac{1}{2^m}\right) \\ &= (1 - r) - \frac{r}{2^{m+1}} + r = 1 - \frac{r}{2^{m+1}} < 1. \end{aligned}$$

Hence, $|w_t(\eta)| < 1$ for $\eta \in \cup_{n=1}^{\infty} G_n$.

Finally we consider the case when $\eta \notin \cup_{n=0}^{\infty} G_n$. Then $\widehat{f'}(\eta) = \widehat{f'}(\eta_0)$, and hence $|w_t(\eta)| \le |tf(0)| + |t\widehat{f'}(\eta_0)| + r = 1$. We thus conclude that $|w_t(\eta)| \le 1$ for all $\eta \in \mathcal{M}$, and consequently, $w_t \in \lambda P_{\eta_0}$. $\square$

3. Proof of Main results

Proof of Theorem 3.1. Fix arbitrary $f \in S(\mathrm{Lip}(I))$ and $\eta \in \mathcal{M}$. Set $\zeta = \phi_1(\eta)$ and $\lambda = \widehat{f'}(\eta)/|\widehat{f'}(\eta)|$ if $\widehat{f'}(\eta) \ne 0$, and $\lambda = 1$ if $\widehat{f'}(\eta) = 0$. Thus, $\widehat{f'}(\eta) = |\widehat{f'}(\eta)|\lambda$. For each $t \in \mathbb{R}$ with $0 < t < 1$, we define $r = 1 - |tf(0)| - |t\widehat{f'}(\eta)|$, and then $r > 0$. By Lemma 3.26, there exists $u_t \in P_\eta$ such that $w_t = |tf(0)|\lambda + t\widehat{f'} + r\lambda u_t \in \lambda P_\eta$. We obtain

$$\begin{aligned} \|w_t - \widehat{f'}\|_\infty &= \||tf(0)|\lambda + (t-1)\widehat{f'} + r\lambda u_t\|_\infty \\ &\le |tf(0)| + (1-t)\|\widehat{f'}\|_\infty + 1 - |tf(0)| - |t\widehat{f'}(\eta)| \\ &= (1-t)\|\widehat{f'}\|_\infty + 1 - |t\widehat{f'}(\eta)|. \end{aligned}$$

Since $w_t \in \lambda P_\eta$, we see that $\widehat{\mathcal{I}(w_t)'}(\eta) = w_t(\eta) = \lambda$, that is, $\widetilde{\mathcal{I}(w_t)} \in \lambda V_{(\eta,1)}$. Then $\Delta(\mathcal{I}(w_t))(0) = 0$ and $\Delta(\widehat{\mathcal{I}(w_t)})'(\zeta) = \Delta(\widehat{\mathcal{I}(w_t)})'(\phi_1(\eta)) = \lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$ by Lemma 3.25. We get

$$\begin{aligned} 1 - |\widehat{\Delta(f)'}(\zeta)| &= |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}| - |\widehat{\Delta(f)'}(\zeta)| \le |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| \\ &= |\Delta(\widehat{\mathcal{I}(w_t)})'(\zeta) - \widehat{\Delta(f)'}(\zeta)| \le \|\Delta(\widehat{\mathcal{I}(w_t)})' - \widehat{\Delta(f)'}\|_\infty \\ &= \|\Delta(\mathcal{I}(w_t)) - \Delta(f)\|_\sigma - |\Delta(f)(0)| \\ &= \|\mathcal{I}(w_t) - f\|_\sigma - |\Delta(f)(0)| = |f(0)| + \|w_t - \widehat{f'}\|_\infty - |\Delta(f)(0)| \\ &\le |f(0)| + (1-t)\|\widehat{f'}\|_\infty + 1 - |t\widehat{f'}(\eta)| - |\Delta(f)(0)|, \end{aligned}$$

where we have used that $\Delta(\mathcal{I}(w_t))(0) = 0 = \mathcal{I}(w_t)(0)$ and Δ is an isometry. Letting $t \nearrow 1$ in the above inequalities, we have

$$1 - |\widehat{\Delta(f)'}(\zeta)| \le |\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| \le |f(0)| + 1 - |\widehat{f'}(\eta)| - |\Delta(f)(0)|. \tag{3.1}$$

In particular, we obtain $|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\zeta)| \le |f(0)| - |\widehat{f'}(\eta)|$, that is,

$$|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\phi_1(\eta))| \le |f(0)| - |\widehat{f'}(\eta)|. \tag{3.2}$$

Let $\eta_0 \in \mathcal{M}$ be such that $|\widehat{f'}(\eta_0)| = \|\widehat{f'}\|_\infty$. There exist $\mu, z \in \mathbb{T}$ such that $f(0) = |f(0)|\mu$ and $\widehat{f'}(\eta_0) = |\widehat{f'}(\eta_0)|z = \|\widehat{f'}\|_\infty z$. Thus,

$$f(0) + \widehat{f'}(\eta_0)\overline{z}\mu = (|f(0)| + \|\widehat{f'}\|_\infty)\mu = \|f\|_\sigma\mu = \mu,$$

and hence $\widetilde{f} \in \mu V_{(\eta_0,\overline{z}\mu)}$. Equality (2.27) shows that

$$|\Delta(f)(0)| + |\widehat{\Delta(f)'}(\phi_1(\eta_0))| = |f(0)| + |\widehat{f'}(\eta_0)|. \tag{3.3}$$

Note that $|\Delta(f)(0)| - |\widehat{\Delta(f)'}(\phi_1(\eta_0))| \le |f(0)| - |\widehat{f'}(\eta_0)|$ holds by (3.2). If we add the last inequality to (3.3), we get $|\Delta(f)(0)| \le |f(0)|$. We may apply the above arguments to Δ^{-1}, then we obtain $|\Delta^{-1}(g)(0)| \le |g(0)|$ for all $g \in S(\mathrm{Lip}(I))$. Entering $g = \Delta(f)$ into the last inequality to get $|f(0)| \le |\Delta(f)(0)|$, and thus

$$|\Delta(f)(0)| = |f(0)|.$$

It follows from (3.2) that $|\widehat{f}'(\eta)| \leq |\widehat{\Delta(f)'}(\phi_1(\eta))|$. Having in mind that $\widetilde{f} \in \mu V_{(\eta_0,\overline{z}\mu)}$ and $f(0) = |f(0)|\mu$, we derive from Lemma 3.25 that

$$\Delta(f)(0) = |\Delta(f)(0)|\mu^{\varepsilon_0}\alpha = |f(0)|\mu^{\varepsilon_0}\alpha = [f(0)]^{\varepsilon_0}\alpha, \tag{3.4}$$

where $[\nu]^{\varepsilon_0} = \nu$ if $\varepsilon_0 = 1$ and $[\nu]^{\varepsilon_0} = \overline{\nu}$ if $\varepsilon_0 = -1$ for $\nu \in \mathbb{C}$.

Now we shall prove that ϕ_1 is injective. Suppose that $\phi_1(\eta_1) = \phi_1(\eta_2)$ for $\eta_1, \eta_2 \in \mathcal{M}$. Set $f_1 = \mathcal{I}(\mathbf{1}_{\mathcal{M}})$, and thus $\widehat{f}_1'(\eta_j) = 1$ for $j = 1, 2$ by (2.3). Equalities (2.22) and (2.25) show that $\widehat{\Delta(f_1)'}(\phi_1(\eta_j))\phi_2(\eta_j) = \alpha$ for $j = 1, 2$. Since $\phi_1(\eta_1) = \phi_1(\eta_2)$, we have $\phi_2(\eta_1) = \phi_2(\eta_2)$. Applying Lemmas 3.17, 3.22 and 3.24 to (2.8) with $\lambda = 1$, we obtain $T(V_{(1,(\eta,1))}) = \alpha V_{(\phi_1(\eta),\phi_2(\eta))}$. Therefore, we get $T(V_{(1,(\eta_1,1))}) = T(V_{(1,(\eta_2,1))})$, and consequently, $V_{(1,(\eta_1,1))} = V_{(1,(\eta_2,1))}$. Lemma 3.4 shows that $\eta_1 = \eta_2$, which proves that ϕ_1 is injective. Now, we may apply the arguments in the last paragraph to Δ^{-1} and ϕ_1^{-1}, and then we obtain $|\widehat{\Delta(f)'}(\zeta)| \leq |(\widehat{\Delta^{-1}(\Delta(f))})'(\phi_1^{-1}(\zeta))|$, which shows $|\widehat{\Delta(f)'}(\phi_1(\eta))| \leq |\widehat{f}'(\eta)|$. We thus conclude that $|\widehat{\Delta(f)'}(\zeta)| = |\widehat{\Delta(f)'}(\phi_1(\eta))| = |\widehat{f}'(\eta)|$. By inequalities (3.1) and $|\Delta(f)(0)| = |f(0)|$, we obtain

$$|\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)} - \widehat{\Delta(f)'}(\zeta)| + |\widehat{\Delta(f)'}(\zeta)| = 1.$$

The above equality implies that $\widehat{\Delta(f)'}(\zeta) = s\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$ for some $s \geq 0$. Then

$$s = |s\lambda^{\varepsilon_1(z)}\alpha\overline{\phi_2(\eta)}| = |\widehat{\Delta(f)'}(\zeta)| = |\widehat{f}'(\eta)|,$$

and thus, $s\lambda^{\varepsilon_1(\eta)} = |\widehat{f}'(\eta)|\lambda^{\varepsilon_1(\eta)} = [\widehat{f}'(\eta)]^{\varepsilon_1(\eta)}$, since $\widehat{f}'(\eta) = |\widehat{f}'(\eta)|\lambda$. We infer from $\widehat{\Delta(f)'}(\zeta) = s\lambda^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$ that $\widehat{\Delta(f)'}(\zeta) = [\widehat{f}'(\eta)]^{\varepsilon_1(\eta)}\alpha\overline{\phi_2(\eta)}$. Hence,

$$\widehat{\Delta(f)'}(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}\,[\widehat{f}'(\eta)]^{\varepsilon_1(\eta)} \tag{3.5}$$

for all $f \in S(\mathrm{Lip}(I))$ and $\eta \in \mathcal{M}$.

We now define $T : \mathrm{Lip}(I) \to \mathrm{Lip}(I)$ by

$$T(g) = \begin{cases} \|g\|_\sigma \Delta\left(\dfrac{g}{\|g\|_\sigma}\right) & \text{if} \quad g \in \mathrm{Lip}(I) \setminus \{0\}, \\ 0 & \text{if} \quad g = 0. \end{cases}$$

By the definition of T with (3.4) and (3.5), we observe that

$$T(g)(0) = \alpha[g(0)]^{\varepsilon_0} \quad \text{and} \quad \widehat{T(g)'}(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}[\widehat{g'}(\eta)]^{\varepsilon_1(\eta)} \tag{3.6}$$

for all $g \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$. We thus obtain

$$\begin{aligned}\|T(g_1) - T(g_2)\|_\sigma &= |T(g_1)(0) - T(g_2)(0)| + \sup_{\eta\in\mathcal{M}} |\widehat{T(g_1)'}(\phi_1(\eta)) - \widehat{T(g_2)'}(\phi_1(\eta))| \\ &= |g_1(0) - g_2(0)| + \sup_{\eta\in\mathcal{M}} |\widehat{g_1'}(\eta) - \widehat{g_2'}(\eta)| = \|g_1 - g_2\|_\sigma\end{aligned}$$

for all $g_1, g_2 \in \mathrm{Lip}(I)$, where we have used $\phi_1(\mathcal{M}) = \mathcal{M}$. Hence T is an isometry on $\mathrm{Lip}(I)$. We infer from (3.6) that T is real linear. We deduce that T is surjective, since so is Δ. Therefore, T is a surjective, real linear isometry on $\mathrm{Lip}(I)$ that extends Δ to $\mathrm{Lip}(I)$. $\square$

Proof of Corollary 3.2. Let T_1 be a surjective isometry on $\mathrm{Lip}(I)$. By the Mazur–Ulam theorem [**52**], $T_1 - T_1(0)$ is a surjective, real linear isometry. Without loss of generality, we may and do assume that T_1 is a surjective real linear isometry. Since T_1^{-1} has the same property as T_1, we see that T_1 maps $S(\mathrm{Lip}(I))$ onto itself. Now we may apply (3.4) and (3.5) to T_1, and then we obtain

$$T_1(f)(0) = \alpha[f(0)]^{\varepsilon_0} \quad \text{and} \quad \widehat{T_1(f)'}(\phi_1(\eta)) = \alpha\overline{\phi_2(\eta)}[\widehat{f'}(\eta)]^{\varepsilon_1(\eta)}$$

for all $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$, where $\alpha \in \mathbb{T}$, $\varepsilon_0 \in \{\pm 1\}$, $\phi_1 \colon \mathcal{M} \to \mathcal{M}$, $\phi_2 \colon \mathcal{M} \to \mathbb{T}$ and $\varepsilon_1 \colon \mathcal{M} \to \{\pm 1\}$ are from proof of Theorem 3.1. As we proved in the second paragraph of Proof of Theorem 3.1, we know that ϕ_1 is injective. By Lemma 3.9, $\psi_1 = \phi_1^{-1}$ is well defined, and then we have

$$\widehat{T_1(f)'}(\eta) = \alpha\overline{\phi_2(\psi_1(\eta))}[\widehat{f'}(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} \tag{3.7}$$

for $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$. We shall prove that ψ_1 and ϕ_2 are both continuous. Let $\{\eta_a\}$ be a net in $\mathcal{M}$ converging to $\eta \in \mathcal{M}$. By the continuity of $\widehat{T_1(f)'}$, we see that $|\widehat{T_1(f)'}(\eta_a)|$ converges to $|\widehat{T_1(f)'}(\eta)|$ for each $f \in \mathrm{Lip}(I)$. This implies that $|\widehat{f'}(\psi_1(\eta_a))|$ converges to $|\widehat{f'}(\psi_1(\eta))|$ for every $f \in \mathrm{Lip}(I)$ by (3.7). Since the weak topology of $\mathcal{M}$ induced by the family $\{|\widehat{f'}| : f \in \mathrm{Lip}(I)\}$ is Hausdorff, we observe that the identity map from $\mathcal{M}$ with the original topology onto $\mathcal{M}$ with the weak topology is a homeomorphism. Hence, $\psi_1(\eta_a)$ converges to $\psi_1(\eta)$ with respect to the original topology of $\mathcal{M}$, and thus ψ_1 is continuous on $\mathcal{M}$. Since ψ_1 is a bijective continuous map on the compact Hausdorff space $\mathcal{M}$, it must be a homeomorphism. Let id_I be the identity function on I. Then we have $\widehat{T_1(id_I)'} = \alpha\overline{\phi_2 \circ \psi_1}$ by (3.7), which implies the continuity of ϕ_2 on $\mathcal{M}$. Moreover, the identity $\widehat{T_1(i(id_I))'} = \alpha\overline{\phi_2 \circ \psi_1}\, i(\varepsilon_1 \circ \psi_1)$ shows that $\varepsilon_1 \circ \psi_1$ is continuous on $\mathcal{M}$. Since ψ_1 is a homeomorphism, we have $\varepsilon_1 = (\varepsilon_1 \circ \psi_1) \circ \psi_1^{-1}$ is continuous on $\mathcal{M}$ as well. Then $\mathcal{M}_1 = \{\eta \in \mathcal{M} : \varepsilon_1(\psi_1(\eta)) = 1\}$ is a closed and open subset of $\mathcal{M}$ with $\varepsilon_1(\psi_1(\eta)) = -1$ for all $\eta \in \mathcal{M} \setminus \mathcal{M}_1$.

We define a map $\Phi\colon C(\mathcal{M}) \to C(\mathcal{M})$ by $\Phi(u)(\eta) = [u(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))}$ for $u \in C(\mathcal{M})$ and $\eta \in \mathcal{M}$. We see that Φ is a well defined real linear map on $C(\mathcal{M})$. For each $v_0 \in C(\mathcal{M})$, we set $u_0(\eta) = [v_0(\psi_1^{-1}(\eta))]^{\varepsilon_1(\eta)}$ for $\eta \in \mathcal{M}$. Then we have $\Phi(u_0)(\eta) = [u_0(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} = [v_0(\eta)]^{\varepsilon_1(\psi_1(\eta))\varepsilon_1(\psi_1(\eta))} = v_0(\eta)$, which shows that Φ is surjective. It is routine to check that Φ is an injective homomorphism, and consequently, Φ is a real algebra automorphism on $C(\mathcal{M})$. Let Γ be the Gelfand transformation from $L^\infty(I)$ onto $C(\mathcal{M})$, that is, $\Gamma(h) = \widehat{h}$ for $h \in L^\infty(I)$. We define a real algebra automorphism $\Psi = \Gamma^{-1} \circ \Phi \circ \Gamma$ on $L^\infty(I)$. For each $f \in \mathrm{Lip}(I)$ and $\eta \in \mathcal{M}$, we obtain

$$[\widehat{f'}(\psi_1(\eta))]^{\varepsilon_1(\psi_1(\eta))} = \Phi(\widehat{f'})(\eta) = (\Phi \circ \Gamma)(f')(\eta) = (\Gamma \circ \Psi)(f')(\eta) = \Gamma(\Psi(f'))(\eta).$$

By the continuity of ϕ_2 and ψ_1, we may set $h_0 = \Gamma^{-1}(\alpha\overline{\phi_2 \circ \psi_1}) \in L^\infty(I)$. We derive from (3.7) that

$$\widehat{T_1(f)'}(\eta) = \Gamma(h_0)(\eta)\Gamma(\Psi(f'))(\eta) = \Gamma(h_0\Psi(f'))(\eta) = \widehat{h_0\Psi(f')}(\eta)$$

for all $\eta \in \mathcal{M}$. Therefore, we conclude $T_1(f)' = h_0\Psi(f')$ for every $f \in \mathrm{Lip}(I)$. According to (2.2), we have

$$T_1(f)(t) = T_1(f)(0) + \int_0^t T_1(f)'\,dm = \alpha[f(0)]^{\varepsilon_0} + \int_0^t h_0\Psi(f')\,dm$$

for every $t \in I$ and $f \in \mathrm{Lip}(I)$. $\square$

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