Stability of approximate solutions constructed by the wave front tracking method for conservation laws

Yoshimasa Sasaki

Doctoral Program in Fundamental Science
Graduate School of Science and Technology
Niigata University

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Contents

1 Introduction

This dissertation is concerned with the L^1_{loc} stability with respect to the initial conditions and flux functions, of the sequence of piecewise constant solutions constructed by the wave front tracking method for the Cauchy problem

$$\begin{aligned} &u_t + f_n(u)_x = 0 \quad (x \in \mathbb{R},\ t > 0), \\ &u(x,0) = \bar{u}_n(x) \quad (x \in \mathbb{R}). \end{aligned} \tag{E$_\mathrm{n}$}$$

Here, u is an unknown function of x and t, and f_n is a piecewise linear approximation of $f \in C^2(\mathbb{R})$ defined by, for each $u \in \big[2^{-n}i, 2^{-n}(i+1)\big]$ $(i \in \mathbb{Z})$,

$$f_n(u) = \frac{u - 2^{-n}i}{2^{-n}} f\big(2^{-n}(i+1)\big) + \frac{2^{-n}(i+1) - u}{2^{-n}} f\big(2^{-n}i\big)$$

with $n \in \mathbb{N}$, while $\bar{u}_n$ is a piecewise constant approximation of $\bar{u} \in BV(\mathbb{R})$.

Dafermos [4] (cf. Bressan [1]) proved that a limit function of subsequence of solutions constructed by the wave front tracking method is an entropy solution of the Cauchy problem

$$u_t + f(u)_x = 0 \quad (x \in \mathbb{R},\ t > 0), \tag{1}$$

$$u(x,0) = \bar{u}(x) \quad (x \in \mathbb{R}), \tag{2}$$

with $\bar{u} \in BV(\mathbb{R})$. Note that $BV(\mathbb{R})$ is a set of all functions defined on $\mathbb{R}$ satisfying

$$\begin{aligned} \text{T.V.}\,\{\psi\} := \sup &\left\{ \sum_{j=1}^{N} \big|\psi(x_j) - \psi(x_{j-1})\big| \;\middle|\; N \geq 1,\, x_0 < x_1 < \cdots < x_N \right\} \\ &< \infty. \end{aligned}$$

In general, it is not easy to directly estimate a difference between solutions constructed by the wave front tracking method. For this reason, we need some compactness theorems such as Helly's theorem to argue the convergence of the sequence of solutions constructed by the wave front tracking method.

By using the result of the uniqueness theorem to the entropy solution given by Kružkov [8] (cf. Dafermos [5], and Godlewski and Raviart [6]), Lucier [12], and Holden and Risebro [7] proved the L^1 stability of the sequence of solutions constructed by the wave front tracking method for the Cauchy problem $(\mathrm{E_n})$ with $\bar{u} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$. In general, it is not easy to argue the L^1 stability of the sequence of solutions to the Cauchy problem $(\mathrm{E_n})$ without using the result of the uniqueness theorem to the entropy solution. Because of that, Liu and Yang [9] introduced the direct method for arguing the L^1 stability without using the result of the uniqueness theorem to the entropy solutions. Since, using this method, we can study the structure of the sequence of solutions constructed by the wave front tracking method, we prove the L^1 stability of the sequence of solutions to the Cauchy problem $(\mathrm{E_n})$ with $\bar{u} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$. However, since the proof deeply depends on the assumption $\bar{u} \in L^1(\mathbb{R})$, we are not able to apply the method to an argument about the L^1_{loc} stability of the sequence of solutions constructed by the wave front tracking method for the Cauchy problem $(\mathrm{E_n})$ with $\bar{u} \in BV(\mathbb{R})$.

Ohwa and Sasaki [13] extended the result of [7] and [12] to the case where $\bar{u} \in BV(\mathbb{R})$. More precisely, locally generalizing the method introduced by Liu and Yang, and applying the proof technique of the uniqueness theorem to the entropy solution given by Kružkov, they proved the L^1_{loc} stability of the sequence of solutions constructed by the wave front tracking method for the Cauchy problem $(\mathrm{E_n})$ with $\bar{u} \in BV(\mathbb{R})$.

L^1 stability for system of conservation laws can be seen in [3, 10, 11]. More precisely, they proved the L^1 stability with respect to the initial conditions for hyperbolic systems of conservation laws. However, these results deeply depend on the assumption that $\bar{u} \in L^1$ has a sufficiently small total variation.

This dissertation is organized as follows. In section 2, we review the construction of the solution for the Cauchy problem $(\mathrm{E_n})$ by using the wave front tracking method. Moreover, reviewing some important properties of the solution, we prove the existence of solution for the Cauchy problem (1)–(2). In section 3, following [13], we calculate a certain amount, which is a key estimate for our result. In section 4, we prove the L^1_{loc} stability with respect to the initial conditions and flux functions, of the sequence of piecewise constant solutions constructed by the wave front tracking method for the Cauchy problem $(\mathrm{E_n})$.

2 The wave front tracking method

In this section, we prove the existence of solution constructed by the wave front tracking method for the Cauchy problem (1)–(2). Following [1, 3, 13], we construct an entropy solution for the Cauchy problem ($\mathrm{E_n}$) and review some properties of the solution. Here, it is said that $u \in C\big([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})\big)$ is an entropy solution of the Cauchy problem (1)–(2) if u satisfies (2) and

$$\iint |u-k|\varphi_t + \operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\varphi_x \, dxdt \geq 0$$

holds for every $k \in \mathbb{R}$ and every non-negative function $\varphi \in C^1_{\mathrm{c}}\big(\mathbb{R} \times (0,\infty)\big)$. Note that

$$\operatorname{sgn}(w) = \begin{cases} 1 & (\text{if } w > 0) \\ 0 & (\text{if } w = 0) \\ -1 & (\text{if } w < 0) \end{cases}.$$

Since, $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, we may assume that f is a Lipschitz continuous with Lipschitz constant $L > 0$.

The wave front tracking method is an approximation theory for proving the existence of a solution for the Cauchy problem (1)–(2) by considering the Cauchy problem ($\mathrm{E_n}$). For this reason, we need to approximate an initial condition $\bar{u} \in BV(\mathbb{R})$. We denote by $2^{-n}\mathbb{Z}$ the set $\{2^{-n}j \mid j \in \mathbb{Z}\}$. For any $\bar{u} \in BV(\mathbb{R})$, we use the following lemma to approximate $\bar{u}$ by a piecewise constant function $\bar{u}_n$ satisfying

$$\bar{u}_n(x) \in 2^{-n}\mathbb{Z} \quad (x \in \mathbb{R}),$$
$$\|\bar{u}_n - \bar{u}\|_{L^\infty} \leq 2^{-n+1} \to 0 \quad (n \to \infty),$$
$$\mathrm{T.V.}\,\{\bar{u}_n\} \leq 2\,\mathrm{T.V.}\,\{\bar{u}\}.$$

Lemma 1. *Let $n \in \mathbb{N}$. If $\bar{u} \in BV(\mathbb{R})$, then there exists a piecewise constant function $\bar{u}_n$ with $\bar{u}_n(x) \in 2^{-n}\mathbb{Z}$ ($x \in \mathbb{R}$) satisfying*

$$\mathrm{T.V.}\,\{\bar{u}_n\} \leq 2\,\mathrm{T.V.}\,\{\bar{u}\} \tag{3}$$

and

$$\|\bar{u}_n - \bar{u}\|_{L^\infty} \leq 2^{-n+1} \to 0 \quad (n \to \infty). \tag{4}$$

Proof. Since it is only interesting in the L^1_{loc} equivalence class of BV functions, we may assume that $\bar{u}$ is right continuous. Then, the function

$$U(x) = \sup \left\{ \sum_{j=1}^{N} \left| \bar{u}(x_j) - \bar{u}(x_{j-1}) \right| \,\middle|\, N \geq 1,\, x_0 < x_1 < \cdots < x_N = x \right\}$$

defined on $(-\infty, x]$ is right continuous and nondecreasing and satisfies

$$U(-\infty) = 0, \quad U(\infty) = \mathrm{T.V.}\left\{\bar{u}\right\},$$
$$\left|\bar{u}(y) - \bar{u}(x)\right| \leq U(y) - U(x) \quad (x < y).$$

Given $n \in \mathbb{N}$, let N be the largest integer that does not exceed $2^n\,\mathrm{T.V.}\left\{\bar{u}\right\}$ and we define

$$x_0 = -\infty, \quad x_N = \infty, \quad x_j = \min\left\{ x \,\middle|\, U(x) \geq j2^{-n} \right\} \quad (j = 1, ..., N-1).$$

Then, setting $\bar{u}_n(x) = \bar{u}(x_j)$ $(x \in [x_j, x_{j+1}))$, we have

$$\mathrm{T.V.}\left\{\bar{u}_n\right\} \leq \mathrm{T.V.}\left\{\bar{u}\right\}, \quad \|\bar{u}_n - \bar{u}\|_{L^\infty} \leq 2^{-n}.$$

Moreover, appropriately and slightly changing the value of $\bar{u}_n(x)$ so that $\bar{u}_n(x) \in 2^{-n}\mathbb{Z}$, since the number of the points of discontinuity of $\bar{u}_n$ is at most $2^n\,\mathrm{T.V.}\left\{\bar{u}\right\}$, we have

$$\mathrm{T.V.}\left\{\bar{u}_n\right\} \leq \mathrm{T.V.}\left\{\bar{u}\right\} + 2^{-n} \cdot 2^n\,\mathrm{T.V.}\left\{\bar{u}\right\} = 2\,\mathrm{T.V.}\left\{\bar{u}\right\},$$
$$\|\bar{u}_n - \bar{u}\|_{L^\infty} \leq 2^{-n+1} \to 0 \quad (n \to \infty).$$

Thus, Lemma 1 is proved. □

Since the initial condition $\bar{u} \in BV(\mathbb{R})$ is approximated by a piecewise constant function, we consider a piecewise constant solution for the initial value problem ($\mathrm{E_n}$) with

$$u(x,0) = \begin{cases} u_L & (x < 0) \\ u_R & (x > 0) \end{cases} \quad (u_L, u_R \in 2^{-n}\mathbb{Z}), \tag{5}$$

which is said to be the Riemann problem ($\mathrm{E_n}$). To solve the Riemann problem ($\mathrm{E_n}$), we review two conditions which is said to be the Rankine–Hugoniot condition and the shock admissibility condition (cf. Oleinik [14]).

Assume that u_n is a piecewise constant solution of the Riemann problem ($\mathrm{E_n}$). Then, each shock wave $\alpha = \langle u_-, u_+ \rangle$ of u_n travels with a speed

$$\lambda(\alpha) = \frac{f_n(u_+) - f_n(u_-)}{u_+ - u_-}.$$

This condition is said to be the Rankine–Hugoniot condition. For all $w_\pm \in \mathbb{R}$ with $w_+ \neq w_-$, for simplicity, we set

$$\lambda(w_-, w_+) = \frac{f_n(w_+) - f_n(w_-)}{w_+ - w_-}.$$

Next, assume that u_n be the piecewise constant entropy solution of the Riemann problem ($\mathrm{E_n}$). Namely, we assume that u_n satisfies (5) and

$$\iint |u_n - k|\varphi_t + \operatorname{sgn}(u_n - k)\big(f_n(u) - f_n(k)\big)\varphi_x \, dxdt \geq 0$$

holds for every $k \in \mathbb{R}$ and every non-negative function $\varphi \in C^1_{\mathrm{c}}\big(\mathbb{R} \times (0, \infty)\big)$. Then, u_n is the piecewise constant entropy solution of the Riemann problem ($\mathrm{E_n}$) if and only if, for all waves $\alpha = \langle u_-, u_+ \rangle$, the Rankine–Hugoniot condition holds and

$$\begin{cases} f_n\big(\theta u_+ + (1-\theta)u_-\big) \geq \theta f_n(u_+) + (1-\theta) f_n(u_-) & (\text{if } u_- < u_+) \\ f_n\big(\theta u_+ + (1-\theta)u_-\big) \leq \theta f_n(u_+) + (1-\theta) f_n(u_-) & (\text{if } u_- > u_+) \end{cases}.$$

holds for all $\theta \in [0, 1]$. Hence, for all $u \in \big(\min\{u_-, u_+\}, \max\{u_-, u_+\}\big)$, we have

$$\lambda(u, u_+) \leq \lambda(u_-, u_+), \tag{6}$$

$$\lambda(u_-, u_+) \leq \lambda(u_-, u). \tag{7}$$

Inequalities (6) and (7) are said to be the shock admissibility condition. Note that inequalities (6) and (7) are equivalent.

Now, we construct the piecewise constant entropy solution for the Riemann problem ($\mathrm{E_n}$).

- In case of $u_L < u_R$.

Then, by taking $f_*(u)$ to be the largest convex function of $f_n(u)$ such that

$$f_*(u) \le f_n(u) \quad \big(u \in [u_L, u_R]\big),$$

the derivative of $f_*(u)$ is a non-decreasing piecewise constant function. Hence, denoting by $u_L = u_0 < u_1 < \cdots < u_N = u_R$ the points of discontinuity of derivative of $f_*(u)$ and setting

$$\lambda_j = \frac{f_n(u_j) - f_n(u_{j-1})}{u_j - u_{j-1}} \quad (j = 1, 2, ..., N),$$

we see that

$$u_n(x,t) = \begin{cases} u_L & \big(x < \lambda_1 t\big) \\ u_j & \big(\lambda_j t < x < \lambda_{j+1} t\big) \quad (j = 1, 2, ..., N-1) \\ u_R & \big(\lambda_N t < x\big) \end{cases}$$

is the piecewise constant entropy solution to the Riemann problem ($\mathrm{E_n}$). Indeed, for all $k \in \mathbb{R}$ and all non-negative function $\varphi \in C_{\mathrm{c}}^1\big(\mathbb{R} \times (0,\infty)\big)$, we have

$$\begin{aligned}
&\iint |u_n - k|\varphi_t + \mathrm{sgn}(u_n - k)\big(f_n(u_n) - f_n(k)\big)\varphi_x \, dxdt \\
&= \sum_{j=1}^{N} \iint \big\{\lambda_j\big(|u_j - k| - |u_{j-1} - k|\big) \\
&\qquad - \mathrm{sgn}(u_j - k)\big(f_n(u_j) - f_n(k)\big) \\
&\qquad + \mathrm{sgn}(u_{j-1} - k)\big(f_n(u_{j-1}) - f_n(k)\big)\big\}\varphi(\lambda_j t, t)\, dt \\
&= \sum_{j=1}^{N} \int \big\{\lambda_j(u_j + u_{j-1} - 2k) + 2f_n(k) - f_n(u_j) - f_n(u_{j-1})\big\} \\
&\qquad \cdot \chi_{[u_{j-1}, u_j]}(k)\varphi(\lambda_j t, t)\, dt \\
&\ge 0.
\end{aligned}$$

Here, χ_I is a characteristic function of the interval I. Namely,

$$\chi_I(x) = \begin{cases} 1 & (\text{if } x \in I) \\ 0 & (\text{if } x \notin I) \end{cases}.$$

Since the piecewise constant entropy solution u_n is monotonic, we have

$$\mathrm{T.V.}\,\bigl\{u_n(\cdot,t)\bigr\} = \mathrm{T.V.}\,\bigl\{u_n(\cdot,0)\bigr\} \qquad (t>0).$$

Moreover, we have

$$\int_{\mathbb{R}} |u_n(x,t) - u_n(x,s)|\,dx \le tL\,\mathrm{T.V.}\,\bigl\{u_n(\cdot,0)\bigr\} \quad (t,s\ge 0).$$

Note that, by the construction of the solution u_n, we have

$$u_n(x,t) \in 2^{-n}\mathbb{Z}$$

for all $(x,t) \in \mathbb{R}\times[0,\infty)$.

- In case of $u_L > u_R$.

  Then, by taking $f^*(u)$ to be the smallest concave function of $f_n(u)$ such that

  $$f^*(u) \ge f_n(u) \quad \bigl(u \in [u_R, u_L]\bigr),$$

  the derivative of $f^*(u)$ is a non-increasing piecewise constant function. Hence, denoting by $u_R = u_0 < u_1 < \cdots < u_N = u_L$ the points of discontinuity of derivative of $f^*(u)$ and setting λ_j in the same way in case of $u_L < u_R$, we see that

  $$u_n(x,t) = \begin{cases} u_L & \bigl(x < \lambda_N t\bigr) \\ u_j & \bigl(\lambda_{j+1}t < x < \lambda_j t\bigr) \quad (j = 1,2,...,N-1) \\ u_R & \bigl(\lambda_1 t < x\bigr) \end{cases}$$

  is the piecewise constant entropy solution to the Riemann problem $(\mathrm{E_n})$. Indeed, for all $k\in\mathbb{R}$ and all non-negative function $\varphi \in C^1_{\mathrm{c}}\bigl(\mathbb{R}\times(0,\infty)\bigr)$, we have

  $$\begin{aligned} &\iint |u_n - k|\varphi_t + \mathrm{sgn}(u_n - k)\bigl(f_n(u_n) - f_n(k)\bigr)\varphi_x\,dxdt \\ &= \sum_{j=1}^{N} \iint \bigl\{\lambda_j\bigl(-|u_j - k| + |u_{j-1} - k|\bigr) \\ &\qquad + \mathrm{sgn}(u_j - k)\bigl(f_n(u_j) - f_n(k)\bigr) \end{aligned}$$

$$
\begin{aligned}
&- \operatorname{sgn}(u_{j-1}-k)\bigl(f_n(u_{j-1})-f_n(k)\bigr)\bigr\}\varphi(\lambda_j t,t)\,dt \\
&= \sum_{j=1}^{N} \int \bigl\{\lambda_j(-u_j-u_{j-1}+2k)-2f_n(k)+f_n(u_j)+f_n(u_{j-1})\bigr\} \\
&\qquad \cdot \chi_{[u_j,u_{j-1}]}(k)\varphi(\lambda_j t,t)\,dt \\
&\geq 0.
\end{aligned}
$$

Since the piecewise constant entropy solution u_n is monotonic, we have

$$\text{T.V.}\,\bigl\{u_n(\cdot,t)\bigr\} = \text{T.V.}\,\bigl\{u_n(\cdot,0)\bigr\} \qquad (t>0).$$

Moreover, we have

$$\int_{\mathbb{R}} |u_n(x,t)-u_n(x,s)|\,dx \leq tL\,\text{T.V.}\,\bigl\{u_n(\cdot,0)\bigr\} \quad (t,s\geq 0).$$

Note that, by the construction of the solution u_n, we have

$$u_n(x,t)\in 2^{-n}\mathbb{Z}$$

for all $(x,t)\in\mathbb{R}\times[0,\infty)$.

Next, we generate a piecewise constant solution u_n to the Cauchy problem ($\mathrm{E_n}$) with $u(x,0)=\bar{u}_n(x)$ in the following algorithm:

Step 1. We set $u_n(x,0)=\bar{u}_n(x)$. Then, we denote by $x_1<x_2<\cdots<x_N$ the points of discontinuity of $u_n(x,0)$.
Step 2. For each $\alpha=1,2,...,N$, we solve the Riemann problem ($\mathrm{E_n}$) generated by the jump $\bigl(u_n(x_\alpha-0,0),u_n(x_\alpha+0,0)\bigr)$ and construct a piecewise constant entropy solution u_n on a forward neighbourhood with respect to time.
Step 3. We prolong the solution until a time t_1 where the first collision of fronts takes place. At time t_1, we solve the corresponding Riemann problem ($\mathrm{E_n}$) in the way above.
Step 4. We repeat the operation of Step 3 until no collision of fronts takes place.

We have a global solution u_n to the Cauchy problem ($\mathrm{E_n}$) by using this algorithm. However, to ensure that the operation of Step 3 only occurs finitely many times, we need to estimate a total variation of u_n. Let (ξ, τ) be a point where a collision of m fronts $\xi_1(t) < \cdots < \xi_m(t)$ $(t < \tau)$ takes place. We denote by $u_i - u_{i-1}$ $(i = 1, \ldots, m)$ the jump $u(\xi_i(t)+, t) - u(\xi_i(t)-, t)$. Namely,

$$u_i - u_{i-1} = u(\xi_i(t)+, t) - u(\xi_i(t)-, t) \quad (i = 1, \ldots, m)$$

To estimate a total variation of u_n, we discuss the following two cases.

In case that all jumps has the same sign(Case 1).

- In case of $u_0 < \cdots < u_m$.
  By the construction of the entropy solution, we have

$$\frac{d}{dt}\xi_i(t) = \frac{f_n(u_i) - f_n(u_{i-1})}{u_i - u_{i-1}} \quad (i = 1, \ldots, m)$$

  and, for all $k \in [u_{i-1}, u_i]$, we have

$$f_n(k) \geq \frac{k - u_{i-1}}{u_i - u_{i-1}} f_n(u_i) + \frac{u_i - k}{u_i - u_{i-1}} f_n(u_{i-1}).$$

  Then, noting that

$$\frac{d}{dt}\xi_1(t) > \cdots > \frac{d}{dt}\xi_m(t),$$

  we have

$$\begin{aligned} f_n(k) &\geq \frac{f_n(u_m) - f_n(u_0)}{u_m - u_0}(k - u_0) + f_n(u_0) \\ &\geq \frac{k - u_0}{u_m - u_0} f_n(u_m) + \frac{u_m - k}{u_m - u_0} f_n(u_0) \end{aligned}$$

  for all $k \in [u_0, u_m]$. Hence, we prolong the solution u_n which has a single jump traveling with a speed

$$\lambda(u_0, u_m) = \frac{f_n(u_m) - f_n(u_0)}{u_m - u_0}.$$

  In this case, we have

$$\text{T.V.}\left\{u_n(\cdot, \tau-)\right\} = \text{T.V.}\left\{u_n(\cdot, \tau+)\right\}.$$

  Moreover, we see that the number of fronts decreases at least by one.

- In case of $u_0 > \cdots > u_m$.
  By the construction of the entropy solution, we have

$$\frac{d}{dt}\xi_i(t) = \frac{f_n(u_i) - f_n(u_{i-1})}{u_i - u_{i-1}} \quad (i = 1, \dots, m)$$

  and, for all $k \in [u_{i-1}, u_i]$, we have

$$f_n(k) \leq \frac{k - u_{i-1}}{u_i - u_{i-1}} f_n(u_i) + \frac{u_i - k}{u_i - u_{i-1}} f_n(u_{i-1}).$$

  Then, noting that

$$\frac{d}{dt}\xi_1(t) > \cdots > \frac{d}{dt}\xi_m(t)$$

  we have

$$\begin{aligned} f_n(k) &\leq \frac{f_n(u_m) - f_n(u_0)}{u_m - u_0}(k - u_0) + f_n(u_0) \\ &\leq \frac{k - u_0}{u_m - u_0} f_n(u_m) + \frac{u_m - k}{u_m - u_0} f_n(u_0) \end{aligned}$$

  for all $k \in [u_m, u_0]$. Hence, we prolong the solution u_n which has a single jump traveling with a speed

$$\lambda(u_0, u_m) = \frac{f_n(u_m) - f_n(u_0)}{u_m - u_0}.$$

  In this case, we have

$$\text{T.V.}\left\{u_n(\cdot, \tau-)\right\} = \text{T.V.}\left\{u_n(\cdot, \tau+)\right\}.$$

  Moreover, we see that the number of fronts decreases at least by one.

In case that at least two of jumps have opposite signs(Case 2).
In this case, we have

$$\text{T.V.}\left\{u_n(\cdot, \tau+)\right\} - \text{T.V.}\left\{u_n(\cdot, \tau-)\right\} \leq -2 \cdot 2^{-n}.$$

We see that the total variation of the solution u_n decrease in time when collision of fronts takes place.

By the discussion in Cases 1 and 2, we see that T.V. $\{\bar{u}_n(\cdot,t)\}$ does not increase in time and is uniformly bounded at $t=0$. Since T.V. $\{\bar{u}_n(\cdot,t)\}$ decreases when collision of fronts takes place, Case 2 can occur finitely many times. Moreover, since the number of fronts decreases when collision of fronts takes place, Case 1 can occur finitely many times. Hence, we see that the operation of Step 3 only occurs finitely many times.

In addition, we see that the solution u_n is an L^1 continuous with respect to the time. Indeed, for all $s,t>0$ ($t>\tau>s$) which are sufficiently close to τ, we have

$$\begin{aligned}\int_{\mathbb{R}}|u_n(x,t)-u_n(x,s)|\,dx &\le \int_{\mathbb{R}}|u_n(x,t)-u_n(x,\tau)|\,dx \\ &\qquad + \int_{\mathbb{R}}|u_n(x,\tau)-u_n(x,s)|\,dx \\ &\le (t-\tau)L\,\text{T.V.}\,\{u_n(\cdot,\tau)\} \\ &\qquad + (\tau-s)L\,\text{T.V.}\,\{u_n(\cdot,\tau)\} \\ &= (t-s)L\,\text{T.V.}\,\{u_n(\cdot,\tau)\}.\end{aligned}$$

Properties of the piecewise constant solution u_n constructed by the wave front tracking method are summarized as follows:

- There exists a bounded closed interval $K\subset\mathbb{R}$ such that, for all $n\in\mathbb{N}$, the components of u_n belong to $2^{-n}\mathbb{Z}\cap K$.
- The total variation of u_n does not increase in time.
- For all $s,t\in[0,\infty)$, the solution u_n satisfies

$$\int_{\mathbb{R}}|u_n(x,s)-u_n(x,t)|\,dx \le L\,\text{T.V.}\,\{\bar{u}_n\}\,|s-t|. \tag{8}$$

Remark. By Helly's theorem, we see that there exists a subsequence u_{n_l} which converges to some function u in $L^1_{\text{loc}}([0,\infty)\times\mathbb{R})$. In fact, this limit function u is an entropy solution for the Cauchy problem (1)–(2). Indeed, since u_{n_l} is an entropy solution for the Cauchy problem ($\mathrm{E_{n_l}}$) and f_n converges to f uniformly on a compact interval, we have

$$\iint |u-k|\varphi_t+\operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\varphi_x\,dxdt$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \iint |u_{n_l} - k|\varphi_t + \operatorname{sgn}(u_{n_l} - k)\big(f_{n_l}(u_{n_l}) - f_{n_l}(k)\big)\varphi_x \, dxdt \\
&\geq 0
\end{aligned}
$$

for every $k \in \mathbb{R}$ and every non-negative function $\varphi \in C^1_{\mathrm{c}}\big(\mathbb{R} \times (0, \infty)\big)$.

3 Basic estimates

For a bounded closed interval $K \subset \mathbb{R}$, we define

$$\|f\|_{\mathrm{Lip}(K)} = \sup_{u \in K} |f(u)| + \sup_{u,v \in K,\ u \neq v} \frac{|f(u) - f(v)|}{|u - v|}.$$

For a piecewise linear continuous function f_n, we have the following lemma:

Lemma 2. *For any bounded closed interval $K \subset \mathbb{R}$, there exist $L, M > 0$ such that, for all $n \in \mathbb{N}$ and all $u, v \in K$,*

$$|f_n(u) - f_n(v)| \leq L|u - v| \tag{9}$$

and

$$\|f_n - f\|_{\mathrm{Lip}(K)} \leq 2^{-n+1} M \to 0 \quad (n \to \infty). \tag{10}$$

Proof. Since $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, we denote by $L > 0$ the Lipschitz constant of f. Without loss of generality, we may assume that $u < v$ ($u, v \in K$).

We first prove (9). If $u, v \in \big[2^{-n} i, 2^{-n}(i+1)\big]$, then we have

$$\big|f_n(u) - f_n(v)\big| = \left|\frac{u - v}{2^{-n}}\Big(f\big(2^{-n}(i+1)\big) - f\big(2^{-n} i\big)\Big)\right| \leq L|u - v|.$$

If not, then, for $u \in \big[2^{-n} i, 2^{-n}(i+1)\big]$, $v \in \big[2^{-n} j, 2^{-n}(j+1)\big]$ with $i + 1 \leq j$, we have

$$\begin{aligned}
&\big|f_n(u) - f_n(v)\big| \\
&\quad = \left|\big(u - 2^{-n}(i+1)\big)\frac{f\big(2^{-n}(i+1)\big) - f\big(2^{-n} i\big)}{2^{-n}} + f\big(2^{-n}(i+1)\big)\right. \\
&\qquad \left. - \big(v - 2^{-n} j\big)\frac{f\big(2^{-n}(j+1)\big) - f\big(2^{-n} j\big)}{2^{-n}} - f\big(2^{-n} j\big)\right| \\
&\quad \leq L\big(2^{-n}(i+1) - u\big) + L\big(v - 2^{-n} j\big) + L\big(2^{-n} j - 2^{-n}(i+1)\big) \\
&\quad = L|u - v|.
\end{aligned}$$

Thus, (9) is proved.

Next, we prove (10). Since, for any $u \in K$, there exists $i \in \mathbb{Z}$ such that $u \in \big[2^{-n}i, 2^{-n}(i+1)\big]$, we have

$$\begin{aligned}\big|f_n(u) - f(u)\big| &= \left|\big(u - 2^{-n}i\big)\frac{f\big(2^{-n}(i+1)\big) - f\big(2^{-n}i\big)}{2^{-n}} + f\big(2^{-n}i\big) - f(u)\right| \\ &\leq 2L\big(u - 2^{-n}i\big) \leq 2^{-n+1}L.\end{aligned}$$

Moreover, if $u, v \in \big[2^{-n}i, 2^{-n}(i+1)\big]$, then, by the mean value theorem, for some constants $w_1 \in (u, v)$ and $w_2 \in \big(2^{-n}i, 2^{-n}(i+1)\big)$, we have

$$\begin{aligned}\left|\frac{(f_n - f)(u) - (f_n - f)(v)}{u - v}\right| &= \big|(f_n - f)^{'}(w_1)\big| \\ &= \left|\frac{f\big(2^{-n}(i+1)\big) - f\big(2^{-n}i\big)}{2^{-n}} - f^{'}(w_1)\right| \\ &= \big|f^{'}(w_2) - f^{'}(w_1)\big| \\ &\leq \max_{w \in K}|f^{''}(w)||w_2 - w_1| \\ &\leq 2^{-n}\max_{w \in K}|f^{''}(w)|.\end{aligned}$$

If not, then, for $u \in \big[2^{-n}i, 2^{-n}(i+1)\big]$, $v \in \big[2^{-n}j, 2^{-n}(j+1)\big]$ with $i + 1 \leq j$, using

$$(f_n - f)\big(2^{-n}(i+1)\big) = 0 = (f_n - f)\big(2^{-n}j\big),$$

we have

$$\begin{aligned}\left|\frac{(f_n - f)(u) - (f_n - f)(v)}{u - v}\right| &= \left|\frac{(f_n - f)(u) - (f_n - f)\big(2^{-n}(i+1)\big)}{u - v}\right. \\ &\qquad \left. + \frac{(f_n - f)\big(2^{-n}j\big) - (f_n - f)(v)}{u - v}\right| \\ &\leq \left|\frac{(f_n - f)(u) - (f_n - f)\big(2^{-n}(i+1)\big)}{u - 2^{-n}(i+1)}\right| \\ &\qquad + \left|\frac{(f_n - f)\big(2^{-n}j\big) - (f_n - f)(v)}{2^{-n}j - v}\right|\end{aligned}$$

$$\leq 2^{-n+1} \max_{w \in K} |f''(w)|.$$

Hence, setting $M = L + \max_{w \in K} |f''(w)|$, we have

$$\begin{aligned} \|f_n - f\|_{\mathrm{Lip}(K)} &\leq 2^{-n+1} L + 2^{-n+1} \max_{w \in K} |f''(w)| \\ &= 2^{-n+1} \Big(L + \max_{w \in K} |f''(w)| \Big) \\ &= 2^{-n+1} M \to 0 \quad (n \to \infty). \end{aligned}$$

Thus, (10) is proved. □

Remark. We see that, by inequality (9), the Lipschitz constant $L > 0$ of $f_n \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ does not depend on $n \in \mathbb{N}$.

Let $\bar{u}, \bar{v} \in BV(\mathbb{R})$. We consider a difference between solutions to the Cauchy problems ($\mathrm{E_n}$) and ($\mathrm{E_m}$) with piecewise constant initial functions $\bar{u}_n$ and $\bar{v}_m$ satisfying

$$\bar{u}_n(x) \in 2^{-n}\mathbb{Z} \quad (x \in \mathbb{R}), \quad \mathrm{T.V.}\big\{\bar{u}_n\big\} \leq 2\,\mathrm{T.V.}\big\{\bar{u}\big\},$$

$$\|\bar{u}_n - \bar{u}\|_{L^\infty} \leq 2^{-n+1} \to 0 \quad (n \to \infty),$$

and

$$\bar{v}_m(x) \in 2^{-m}\mathbb{Z} \quad (x \in \mathbb{R}), \quad \mathrm{T.V.}\big\{\bar{v}_m\big\} \leq 2\,\mathrm{T.V.}\big\{\bar{v}\big\},$$

$$\|\bar{v}_m - \bar{v}\|_{L^\infty} \leq 2^{-m+1} \to 0 \quad (m \to \infty),$$

respectively. In the following, u_n and v_m ($n \leq m$) are piecewise constant solutions constructed by the wave front method for the Cauchy problems ($\mathrm{E_n}$) with $u(x,0) = \bar{u}_n(x)$ and ($\mathrm{E_m}$) with $u(x,0) = \bar{v}_m(x)$, respectively.

Since $\bar{u}_n(x) \in 2^{-n}\mathbb{Z} \subset 2^{-m}\mathbb{Z}$, by the construction of solution, we have

$$f_n\big(u_n(x,t)\big) = f_m\big(u_n(x,t)\big). \tag{11}$$

Also, using inequality (8), we see that for all $s, t \in [0, \infty)$

$$\left| \int_{\mathbb{R}} |u_n(x,s) - v_m(x,s)|\, dx - \int_{\mathbb{R}} |u_n(x,t) - v_m(x,t)|\, dx \right| = O(1)|s - t|,$$

which means that $u_n - v_m \in C\big([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})\big)$. Here, we remark that $O(1)$ is an uniform constant which is independent of n, m, s, and t.

Definition. It is said to be an interaction of u_n and v_m when either the fronts of u_n or the fronts of v_m intersects with each other, or the front of u_n and the front of v_m intersect with a different speed (namely, it is the situation in which the fronts of u_n and the fronts of v_m never overlap). Then, a time when the interaction takes place is said to be an interaction time of u_n and v_m. The number of interaction times is finite.

Let $\lambda(\alpha)$ and x_α denote the speed and the location of wave α of solutions u_n and v_m. Let $T > 0$. For each α, setting

$$q^{\pm}(\alpha) = u_n\big(x_\alpha \pm 0\big) - v_m\big(x_\alpha \pm 0\big),$$

$$Q^{\pm}(\alpha) = \operatorname{sgn} q^{\pm}(\alpha) \Big(f_m\big(u_n(x_\alpha \pm 0)\big) - f_m\big(v_m(x_\alpha \pm 0)\big)\Big),$$

we have, for all $\phi \in C_c^\infty\big(\mathbb{R} \times (0,\infty)\big)$ with $\phi \geq 0$ which vanishes outside of $T > 0$,

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}} \Big(|u_n - v_m|\phi_t + \operatorname{sgn}(u_n - v_m)\,\big(f_m(u_n) - f_m(v_m)\big)\phi_x \Big)\, dxdt \\
&= \int_0^T \sum_\alpha \Big(-\lambda(\alpha)\big(|q^-(\alpha)| - |q^+(\alpha)|\big) + \big(Q^-(\alpha) - Q^+(\alpha)\big)\Big)\phi(x_\alpha, t)\, dt \\
&= \int_0^T \sum_\alpha \Big(J^-(\alpha) + J^+(\alpha)\Big)\phi(x_\alpha, t)\, dt \\
&= \int_0^T \sum_\alpha J(\alpha)\phi(x_\alpha, t)\, dt,
\end{aligned}$$

where

$$\begin{aligned}
J^-(\alpha) &= -\big(\lambda(\alpha)|q^-(\alpha)| - Q^-(\alpha)\big), \\
J^+(\alpha) &= \lambda(\alpha)|q^+(\alpha)| - Q^+(\alpha), \\
J(\alpha) &= J^-(\alpha) + J^+(\alpha).
\end{aligned}$$

Note that $J^{\pm}(\alpha)$ and $J(\alpha)$ are defined for all t when no interaction takes place. In the following, we denote by W^+ the set of all waves α of u_n and v_m with $q^-(\alpha) \cdot q^+(\alpha) \geq 0$ and by W^- the set of all waves α of u_n and v_m with $q^-(\alpha) \cdot q^+(\alpha) < 0$.

We need to consider the following three cases to estimate $J(\alpha)$:

(i) At x_α, u_n has a shock wave and v_m is continuous.

(ii) At x_α, u_n is continuous and v_m has a shock wave.

(iii) At x_α, u_n and v_m have shock waves propagating with the same speed.

First, we have the following result for $J(\alpha)$ $(\alpha \in W^+)$:

Lemma 3. *For each $\alpha \in W^+$, we have*

$$J(\alpha) = 0. \tag{12}$$

Proof.

• In case of $q^-(\alpha) \cdot q^+(\alpha) > 0$.

(i) In case that u_n has a shock wave and v_m is continuous.

In this case, by equation (11), we have

$$\begin{aligned}
&\lambda(\alpha)\big(q^-(\alpha) - q^+(\alpha)\big) - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(v_m(x_\alpha - 0)\big) \\
&\qquad + f_m\big(u_n(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha + 0)\big) \\
&= \lambda(\alpha)\big(u_n(x_\alpha - 0) - u_n(x_\alpha + 0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(u_n(x_\alpha + 0)\big) \\
&= f_n\big(u_n(x_\alpha - 0)\big) - f_n\big(u_n(x_\alpha + 0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(u_n(x_\alpha + 0)\big) \\
&= 0
\end{aligned}$$

so that

$$J(\alpha) = 0.$$

(ii) In case that u_n is continuous and v_m has a shock wave.

In this case, we have

$$\lambda(\alpha)\big(q^-(\alpha) - q^+(\alpha)\big) - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(v_m(x_\alpha - 0)\big)$$

$$
\begin{aligned}
&+ f_m\big(u_n(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha + 0)\big) \\
&= \lambda(\alpha)\big(v_m(x_\alpha + 0) - v_m(x_\alpha - 0)\big) \\
&\quad - \Big(f_m\big(v_m(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha - 0)\big)\Big) \\
&= 0
\end{aligned}
$$

so that

$$J(\alpha) = 0.$$

(iii) In case that u_n and v_m have shock waves propagating with the same speed:

In this case, by equation (11), we have

$$
\begin{aligned}
&\lambda(\alpha)\big(q^-(\alpha) - q^+(\alpha)\big) - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(v_m(x_\alpha - 0)\big) \\
&\qquad + f_m\big(u_n(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha + 0)\big) \\
&= \lambda(\alpha)\big(u_n(x_\alpha - 0) - u_n(x_\alpha + 0)\big) - f_m\big(u_n(x_\alpha - 0)\big) \\
&\qquad + f_m\big(u_n(x_\alpha + 0)\big) + \lambda(\alpha)\big(v_m(x_\alpha + 0) - v_m(x_\alpha - 0)\big) \\
&\qquad - \Big(f_m\big(v_m(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha - 0)\big)\Big) \\
&= f_n\big(u_n(x_\alpha - 0)\big) - f_n\big(u_n(x_\alpha + 0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(u_n(x_\alpha + 0)\big) \\
&= 0
\end{aligned}
$$

so that

$$J(\alpha) = 0.$$

• In case of $q^-(\alpha) \cdot q^+(\alpha) = 0$.

First, we prove the case of $q^-(\alpha) = 0$ and $q^+(\alpha) \neq 0$. Then, noting that $u_n(x_\alpha - 0) = v_m(x_\alpha - 0)$, we have $J^-(\alpha) = 0$.

(i) In case that u_n has a shock wave and v_m is continuous.

In this case, noting that $u_n(x_\alpha - 0) = v_m(x_\alpha - 0) = v_m(x_\alpha + 0)$, by equation (11), we have

$$\begin{aligned}
&\lambda(\alpha)q^+(\alpha) - f_m\big(u_n(x_\alpha+0)\big) + f_m\big(v_m(x_\alpha+0)\big) \\
&= \lambda(\alpha)\big(u_n(x_\alpha+0) - u_n(x_\alpha-0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha+0)\big) + f_m\big(u_n(x_\alpha-0)\big) \\
&= f_n\big(u_n(x_\alpha+0)\big) - f_n\big(u_n(x_\alpha-0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha+0)\big) + f_m\big(u_n(x_\alpha-0)\big) \\
&= 0
\end{aligned}$$

so that

$$J(\alpha) = 0.$$

(ii) In case that u_n is continuous and v_m has a shock wave.

In this case, noting that $u_n(x_\alpha + 0) = u_n(x_\alpha - 0) = v_m(x_\alpha - 0)$, we have

$$\begin{aligned}
&\lambda(\alpha)q^+(\alpha) - f_m\big(u_n(x_\alpha+0)\big) + f_m\big(v_m(x_\alpha+0)\big) \\
&= \lambda(\alpha)\big(v_m(x_\alpha-0) - v_m(x_\alpha+0)\big) \\
&\qquad - \Big(f_m\big(v_m(x_\alpha-0)\big) - f_m\big(v_m(x_\alpha+0)\big)\Big) \\
&= 0
\end{aligned}$$

so that

$$J(\alpha) = 0.$$

(iii) In case that u_n and v_m have shock waves propagating with the same speed.

In this case, by equation (11), we have

$$\lambda(\alpha)q^+(\alpha) - f_m\big(u_n(x_\alpha+0)\big) + f_m\big(v_m(x_\alpha+0)\big)$$

$$
\begin{aligned}
&= \lambda(\alpha)\big(u_n(x_\alpha+0)-u_n(x_\alpha-0)\big)\\
&\qquad - f_m\big(u_n(x_\alpha+0)\big)+f_m\big(u_n(x_\alpha-0)\big)\\
&\qquad + \lambda(\alpha)\big(v_m(x_\alpha-0)-v_m(x_\alpha+0)\big)\\
&\qquad - \Big(f_m\big(v_m(x_\alpha-0)\big)-f_m\big(v_m(x_\alpha+0)\big)\Big)\\
&= f_n\big(u_n(x_\alpha+0)\big)-f_n\big(u_n(x_\alpha-0)\big)\\
&\qquad - f_m\big(u_n(x_\alpha+0)\big)+f_m\big(u_n(x_\alpha-0)\big)\\
&= 0
\end{aligned}
$$

so that

$$J(\alpha)=0.$$

On the other hand, we prove the case of $q^-(\alpha)\neq 0$ and $q^+(\alpha)=0$. Then, noting that $u_n(x_\alpha+0)=v_m(x_\alpha+0)$, we have $J^+(\alpha)=0$.

(i) In case that u_n has a shock wave and v_m is continuous.

In this case, noting that $u_n(x_\alpha+0)=v_m(x_\alpha+0)=v_m(x_\alpha-0)$, by equation (11), we have

$$
\begin{aligned}
&\lambda(\alpha)q^-(\alpha)-f_m\big(u_n(x_\alpha-0)\big)+f_m\big(v_m(x_\alpha-0)\big)\\
&= \lambda(\alpha)\big(u_n(x_\alpha-0)-u_n(x_\alpha+0)\big)\\
&\qquad - f_m\big(u_n(x_\alpha-0)\big)+f_m\big(u_n(x_\alpha+0)\big)\\
&= f_n\big(u_n(x_\alpha-0)\big)-f_n\big(u_n(x_\alpha+0)\big)\\
&\qquad - f_m\big(u_n(x_\alpha-0)\big)+f_m\big(u_n(x_\alpha+0)\big)\\
&= 0
\end{aligned}
$$

so that

$$J(\alpha)=0.$$

(ii) In case that u_n is continuous and v_m has a shock wave.

In this case, noting that $u_n(x_\alpha - 0) = u_n(x_\alpha + 0) = v_m(x_\alpha + 0)$, we have

$$\begin{aligned}
&\lambda(\alpha)q^-(\alpha) - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(v_m(x_\alpha - 0)\big) \\
&= \lambda(\alpha)\big(v_m(x_\alpha + 0) - v_m(x_\alpha - 0)\big) \\
&\qquad - \Big(f_m\big(v_m(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha - 0)\big)\Big) \\
&= 0
\end{aligned}$$

so that

$$J(\alpha) = 0.$$

(iii) In case that u_n and v_m have shock waves propagating with the same speed.

In this case, by equation (11), we have

$$\begin{aligned}
&\lambda(\alpha)q^-(\alpha) - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(v_m(x_\alpha - 0)\big) \\
&= \lambda(\alpha)\big(u_n(x_\alpha - 0) - u_n(x_\alpha + 0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(u_n(x_\alpha + 0)\big) \\
&\qquad + \lambda(\alpha)\big(v_m(x_\alpha + 0) - v_m(x_\alpha - 0)\big) \\
&\qquad - \Big(f_m\big(v_m(x_\alpha + 0)\big) - f_m\big(v_m(x_\alpha - 0)\big)\Big) \\
&= f_n\big(u_n(x_\alpha - 0)\big) - f_n\big(u_n(x_\alpha + 0)\big) \\
&\qquad - f_m\big(u_n(x_\alpha - 0)\big) + f_m\big(u_n(x_\alpha + 0)\big) \\
&= 0
\end{aligned}$$

so that

$$J(\alpha) = 0.$$

□

Next, we have the following result for $J(\alpha)$ $(\alpha \in W^-)$:

Lemma 4. *For each $\alpha \in W^-$, we have*

$$J(\alpha) \geq \begin{cases} 0 & (\text{if } n = m) \\ -\|f_n - f_m\|_{\mathrm{Lip}(K)} \left|u_n(x_\alpha + 0) - u_n(x_\alpha - 0)\right| & (\text{if } n < m) \end{cases}. \tag{13}$$

Proof. Setting

$$J^-(\alpha) = -\operatorname{sgn} q^-(\alpha) \left(\left(\lambda(\alpha) - \lambda\big(q^-(\alpha)\big) q^-(\alpha) \right),\right.$$
$$J^+(\alpha) = \operatorname{sgn} q^+(\alpha) \left(\left(\lambda(\alpha) - \lambda\big(q^+(\alpha)\big) q^+(\alpha) \right),\right.$$

where

$$\lambda\big(q^\pm(\alpha)\big) = \frac{f_m\big(u_n\big(x_\alpha \pm 0\big)\big) - f_m\big(v_m\big(x_\alpha \pm 0\big)\big)}{u_n\big(x_\alpha \pm 0\big) - v_m\big(x_\alpha \pm 0\big)},$$

by an argument similar to the proof in Lemma 3, we have

$$\Big(\lambda(\alpha) - \lambda\big(q^-(\alpha)\big)\Big) q^-(\alpha) - \Big(\lambda(\alpha) - \lambda\big(q^+(\alpha)\big)\Big) q^+(\alpha) = 0,$$

which means that

$$\Big(\lambda(\alpha) - \lambda\big(q^-(\alpha)\big)\Big) |q^-(\alpha)| + \Big(\lambda(\alpha) - \lambda\big(q^+(\alpha)\big)\Big) |q^+(\alpha)| = 0.$$

Hence we have

$$J(\alpha) = -2\Big(\lambda(\alpha) - \lambda\big(q^-(\alpha)\big)\Big) |q^-(\alpha)| = 2\Big(\lambda(\alpha) - \lambda\big(q^+(\alpha)\big)\Big) |q^+(\alpha)|. \tag{14}$$

In the following, for simplicity, we set $u_\pm = u_n(x_\alpha \pm 0)$ and $v_\pm = v_m(x_\alpha \pm 0)$.

(i) In case that u_n has a shock wave and v_m is continuous.

(a) In case of $u_+ < v_\pm < u_-$.

By inequalities (6) and (7), we have

$$\lambda(\alpha) - \lambda\big(q^-(\alpha)\big) = \frac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm}$$

$$
\begin{aligned}
&= \lambda(u_-, u_+) - \lambda(u_-, v_\pm) \\
&\quad + \frac{f_n(u_-) - f_n(v_\pm)}{u_- - v_\pm} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm} \\
&\leq \frac{f_n(u_-) - f_n(v_\pm)}{u_- - v_\pm} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm},
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\alpha) - \lambda\big(q^+(\alpha)\big) &= \frac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm} \\
&= \lambda(u_-, u_+) - \lambda(v_\pm, u_+) \\
&\quad + \frac{f_n(u_+) - f_n(v_\pm)}{u_+ - v_\pm} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm} \\
&\geq \frac{f_n(u_+) - f_n(v_\pm)}{u_+ - v_\pm} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm}
\end{aligned}
$$

so that

$$
\begin{aligned}
J(\alpha) \geq \Big(&- f_n(u_-) + f_n(v_\pm) + f_m(u_-) - f_m(v_\pm)\Big) \\
&- \Big(f_n(u_+) - f_n(v_\pm) - f_m(u_+) + f_m(v_\pm)\Big).
\end{aligned}
$$

Hence, if $n = m$, then we have

$$
J(\alpha) \geq 0,
$$

while if $n < m$, then we have

$$
\begin{aligned}
J(\alpha) &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)}\,(u_- - v_\pm) \\
&\quad - \|f_n - f_m\|_{\mathrm{Lip}(K)}\,(v_\pm - u_+) \\
&= -\|f_n - f_m\|_{\mathrm{Lip}(K)}\,(u_- - u_+).
\end{aligned}
$$

(b) In case of $u_- < v_\pm < u_+$.

By inequalities (6) and (7), we have

$$
\lambda(\alpha) - \lambda\big(q^-(\alpha)\big) = \frac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm}
$$

$$
\begin{aligned}
&= \lambda(u_-, u_+) - \lambda(u_-, v_\pm) \\
&\qquad + \frac{f_n(u_-) - f_n(v_\pm)}{u_- - v_\pm} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm} \\
&\le \frac{f_n(u_-) - f_n(v_\pm)}{u_- - v_\pm} - \frac{f_m(u_-) - f_m(v_\pm)}{u_- - v_\pm},
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\alpha) - \lambda\big(q^+(\alpha)\big) &= \frac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm} \\
&= \lambda(u_-, u_+) - \lambda(v_\pm, u_+) \\
&\qquad + \frac{f_n(u_+) - f_n(v_\pm)}{u_+ - v_\pm} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm} \\
&\ge \frac{f_n(u_+) - f_n(v_\pm)}{u_+ - v_\pm} - \frac{f_m(u_+) - f_m(v_\pm)}{u_+ - v_\pm}
\end{aligned}
$$

so that

$$
\begin{aligned}
J(\alpha) \ge &\Big(f_n(u_-) - f_n(v_\pm) - f_m(u_-) + f_m(v_\pm)\Big) \\
&- \Big(- f_n(u_+) + f_n(v_\pm) + f_m(u_+) - f_m(v_\pm)\Big).
\end{aligned}
$$

Hence, if $n = m$, then we have

$$
J(\alpha) \ge 0,
$$

while if $n < m$, then we have

$$
\begin{aligned}
J(\alpha) &\ge -\|f_n - f_m\|_{\mathrm{Lip}(K)} \, (v_\pm - u_-) \\
&\qquad - \|f_n - f_m\|_{\mathrm{Lip}(K)} \, (u_+ - v_\pm) \\
&= -\|f_n - f_m\|_{\mathrm{Lip}(K)} \, (u_+ - u_-).
\end{aligned}
$$

(ii) In case that u_n is continuous and v_m has a shock wave.

(a) In case of $v_+ < u_\pm < v_-$.

By inequalities (6) and (7), we have

$$\lambda(\alpha) - \lambda\big(q^-(\alpha)\big) = \frac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \frac{f_m(u_\pm) - f_m(v_-)}{u_\pm - v_-}$$
$$= \lambda(v_-, v_+) - \lambda(v_-, u_\pm)$$
$$\leq 0,$$

$$\lambda(\alpha) - \lambda\big(q^+(\alpha)\big) = \frac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \frac{f_m(u_\pm) - f_m(v_+)}{u_\pm - v_+}$$
$$= \lambda(v_-, v_+) - \lambda(u_\pm, v_+)$$
$$\geq 0$$

so that

$$J(\alpha) \geq 0.$$

(b) In case of $v_- < u_\pm < v_+$.

By inequalities (6) and (7), we have

$$\lambda(\alpha) - \lambda\big(q^-(\alpha)\big) = \frac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \frac{f_m(u_\pm) - f_m(v_-)}{u_\pm - v_-}$$
$$= \lambda(v_-, v_+) - \lambda(v_-, u_\pm)$$
$$\leq 0,$$

$$\lambda(\alpha) - \lambda\big(q^+(\alpha)\big) = \frac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \frac{f_m(u_\pm) - f_m(v_+)}{u_\pm - v_+}$$
$$= \lambda(v_-, v_+) - \lambda(u_\pm, v_+)$$
$$\geq 0$$

so that

$$J(\alpha) \geq 0.$$

(iii) In case that u_n and v_m have shock waves propagating with a same speed.

In this case, we have

$$\lambda(\alpha) - \lambda\big(q^-(\alpha)\big)$$

$$= \begin{cases} \dfrac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \dfrac{f_m(u_-) - f_m(v_-)}{u_- - v_-} \\ \dfrac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \dfrac{f_m(u_-) - f_m(v_-)}{u_- - v_-} \end{cases}$$

$$= \begin{cases} \lambda(u_-, u_+) - \lambda(u_-, v_-) \\ \qquad + \dfrac{f_n(u_-) - f_n(v_-)}{u_- - v_-} - \dfrac{f_m(u_-) - f_m(v_-)}{u_- - v_-} \\ \lambda(v_-, v_+) - \lambda(v_-, u_-) \end{cases},$$

$$\lambda(\alpha) - \lambda\big(q^+(\alpha)\big)$$

$$= \begin{cases} \dfrac{f_n(u_+) - f_n(u_-)}{u_+ - u_-} - \dfrac{f_m(u_+) - f_m(v_+)}{u_+ - v_+} \\ \dfrac{f_m(v_+) - f_m(v_-)}{v_+ - v_-} - \dfrac{f_m(u_+) - f_m(v_+)}{u_+ - v_+} \end{cases}$$

$$= \begin{cases} \lambda(u_-, u_+) - \lambda(v_+, u_+) \\ \qquad + \dfrac{f_n(u_+) - f_n(v_+)}{u_+ - v_+} - \dfrac{f_m(u_+) - f_m(v_+)}{u_+ - v_+} \\ \lambda(v_-, v_+) - \lambda(u_+, v_+) \end{cases}.$$

First, we prove the case of $u_- < v_-$ and $v_+ < u_+$.

(a) In case of $u_- < v_- < v_+ < u_+$.

By inequalities (6) and (7), we have

$$J(\alpha) \geq \Big(f_n(u_-) - f_n(v_-) - f_m(u_-) + f_m(v_-)\Big)$$

$$- \Big(- f_n(u_+) + f_n(v_+) + f_m(u_+) - f_m(v_+) \Big).$$

Hence, if $n = m$, then we have

$$J(\alpha) \geq 0,$$

while if $n < m$, then we have

$$\begin{aligned} J(\alpha) &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (v_- - u_-) \\ &\quad - \|f_n - f_m\|_{\mathrm{Lip}(K)} (u_+ - v_+) \\ &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_+ - u_-). \end{aligned}$$

(b) In case of $u_- \leq v_+ < v_- \leq u_+$.

By inequalities (6) and (7), we have

$$\begin{aligned} J(\alpha) \geq &\Big(f_n(u_-) - f_n(v_-) - f_m(u_-) + f_m(v_-) \Big) \\ &- \Big(- f_n(u_+) + f_n(v_+) + f_m(u_+) - f_m(v_+) \Big). \end{aligned}$$

Hence, if $n = m$, then we have

$$J(\alpha) \geq 0,$$

while if $n < m$, then we have

$$\begin{aligned} J(\alpha) &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (v_+ - u_-) \\ &\quad - \|f_n - f_m\|_{\mathrm{Lip}(K)} (u_+ - v_-) \\ &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_+ - u_-). \end{aligned}$$

(c) In case of $u_- \leq v_+ < u_+ \leq v_-$.

By inequality (6) and equation (14), we have

$$J(\alpha) \geq 0.$$

(d) In case of $v_+ < u_+ < u_- < v_-$.

By inequalities (6) and (7), we have

$$J(\alpha) \geq 0.$$

(e) In case of $v_+ \le u_- < u_+ \le v_-$.

By inequalities (6) and (7), we have

$$J(\alpha) \ge 0.$$

(f) In case of $v_+ \le u_- < v_- \le u_+$.

By inequality (7) and equation (14), we have

$$J(\alpha) \ge 0.$$

On the other hand, we prove the case $v_- < u_-$ and $u_+ < v_+$.

(a) In case of $u_+ < v_+ < v_- < u_-$.

By inequalities (6) and (7), we have

$$\begin{aligned} J(\alpha) \ge -&\Big(f_n(u_-) - f_n(v_-) - f_m(u_-) + f_m(v_-)\Big) \\ &- \Big(f_n(u_+) - f_n(v_+) - f_m(u_+) + f_m(v_+)\Big). \end{aligned}$$

Hence, if $n = m$, then we have

$$J(\alpha) \ge 0,$$

while if $n < m$, then we have

$$\begin{aligned} J(\alpha) &\ge -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_- - v_-) \\ &\quad - \|f_n - f_m\|_{\mathrm{Lip}(K)} (v_+ - u_+) \\ &\ge -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_- - u_+). \end{aligned}$$

(b) In case of $u_+ \le v_- < v_+ \le u_-$.

By inequalities (6) and (7), we have

$$\begin{aligned} J(\alpha) \ge -&\Big(f_n(u_-) - f_n(v_-) - f_m(u_-) + f_m(v_-)\Big) \\ &- \Big(f_n(u_+) - f_n(v_+) - f_m(u_+) + f_m(v_+)\Big). \end{aligned}$$

Hence, if $n = m$, then we have

$$J(\alpha) \geq 0,$$

while if $n < m$, then we have

$$\begin{aligned} J(\alpha) &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_- - v_+) \\ &\quad - \|f_n - f_m\|_{\mathrm{Lip}(K)} (v_- - u_+) \\ &\geq -\|f_n - f_m\|_{\mathrm{Lip}(K)} (u_- - u_+). \end{aligned}$$

(c) In case of $v_- \leq u_+ < v_+ \leq u_-$.

By inequality (6) and equation (14), we have

$$J(\alpha) \geq 0.$$

(d) In case of $v_- < u_- < u_+ < v_+$.

By inequalities (6) and (7), we have

$$J(\alpha) \geq 0.$$

(e) In case of $v_- \leq u_+ < u_- \leq v_+$.

By inequalities (6) and (7), we have

$$J(\alpha) \geq 0.$$

(f) In case of $u_+ \leq v_- < u_- \leq v_+$.

By inequality (7) and equation (14), we have

$$J(\alpha) \geq 0.$$

□

Using Lemmas 3 and 4, we have the following:

Proposition 1. *Let $T > 0$. For piecewise constant solutions u_n and v_m ($n \leq m$) constructed by the wave front tracking method for the Cauchy problems (E_n) with $u(x, 0) = \bar{u}_n(x)$ and ($\mathrm{E_m}$) with $u(x, 0) = \bar{v}_m(x)$,* *respectively, and all* $\phi \in C_c^\infty\big(\mathbb{R}\times(0,\infty)\big)$ *with* $\phi \geq 0$ *which vanishes outside of* $T > 0$*, we have*

$$\int_0^\infty\!\!\int_{\mathbb{R}} \Big(|u_n - v_m|\phi_t + \operatorname{sign}(u_n - v_m)\,\big(f_m(u_n) - f_m(v_m)\big)\phi_x\Big)\,dxdt$$
$$\geq -\big(\max_{x,t}\phi\big)\,T\,\|f_n - f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}. \tag{15}$$

4 L^1_{loc} stability

In this section, we prove the L^1_{loc} stability of sequence of piecewise constant solutions constructed by the wave front tracking method for the Cauchy problem $(\mathrm{E_n})$.

Theorem 1. *Let $\bar{u}, \bar{v} \in BV(\mathbb{R})$. For piecewise constant solutions u_n and v_m $(n \leq m)$ constructed by the wave front tracking method for the Cauchy problems (E_n) with $u(x,0) = \bar{u}_n(x)$ and $(\mathrm{E_m})$ with $u(x,0) = \bar{v}_m(x)$, respectively, each $t \geq 0$, and all $R > 0$, we have*

$$\int_{|x|\leq R} |u_n(x,t) - v_m(x,t)|\, dx \leq \int_{|x|\leq R+Lt} |u_n(x,0) - v_m(x,0)|\, dx + t\, \|f_n - f_m\|_{\mathrm{Lip}(K)}\, \mathrm{T.V.}\left\{\bar{u}_n\right\}. \quad (16)$$

Proof. We define $\delta \in C^\infty_{\mathrm{c}}(\mathbb{R})$ by

$$\delta(x) = \begin{cases} C \exp\left(\dfrac{1}{|x|^2 - 1}\right) & (|x| < 1) \\ 0 & (|x| \geq 1) \end{cases},$$

where C is a constant which is chosen so that $\displaystyle\int_{\mathbb{R}} \delta(x)\, dx = 1$. Using this δ, for each $h > 0$, we define

$$\delta_h(x) = h\delta(hx), \quad \zeta_h(x) = \int_{-\infty}^{x} \delta_h(s)\, ds.$$

An elementary calculation yields

$$\delta_h(x) = 0 \quad \left(|x| \geq \frac{1}{h}\right), \quad \int_{-\frac{1}{h}}^{\frac{1}{h}} \delta_h(x)\, dx = 1,$$

$$\begin{cases} \zeta_h(x) = 0 & \left(x \leq -\dfrac{1}{h}\right) \\ 0 < \zeta_h(x) < 1 & \left(|x| < \dfrac{1}{h}\right) \\ \zeta_h(x) = 1 & \left(x \geq \dfrac{1}{h}\right) \end{cases}.$$

For convenience, we set

$$q(u_n,v_m)=|u_n-v_m|,\quad Q(u_n,v_m)=\operatorname{sgn}(u_n-v_m)\big(f_m(u_n)-f_m(v_m)\big).$$

Now, let $0<\tau_0<\tau<T$ and, for sufficiently large $h>0$, we take

$$\phi(x,t)=\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)\Big(\zeta_h(t-\tau_0)-\zeta_h(t-\tau)\Big).$$

Then, since $\phi\in C_{\mathrm{c}}^\infty\big(\mathbb{R}\times(0,\infty)\big)$ with $\phi\geq 0$ vanishes outside of $T>0$ and satisfies

$$\begin{aligned}\phi_t&=-L\delta_h\big(|x|-R-L(\tau-t)\big)\Big(\zeta_h(t-\tau_0)-\zeta_h(t-\tau)\Big)\\&\qquad+\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big),\\\phi_x&=-\frac{x}{|x|}\delta_h\big(|x|-R-L(\tau-t)\big)\Big(\zeta_h(t-\tau_0)-\zeta_h(t-\tau)\Big),\end{aligned}$$

inserting this ϕ into inequality (15), we have

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}}q(u_n,v_m)\Big(1-\zeta_h(|x|-R-L(\tau-t))\Big)\\&\qquad\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt\\&\geq\int_0^\infty\!\!\int_{\mathbb{R}}\left(Lq(u_n,v_m)+\frac{x}{|x|}Q(u_n,v_m)\right)\\&\qquad\cdot\delta_h\big(|x|-R-L(\tau-t)\big)\Big(\zeta_h(t-\tau_0)-\zeta_h(t-\tau)\Big)\,dxdt\\&\qquad-\left(\tau+\frac{1}{h}\right)\|f_n-f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}.\end{aligned}$$

Hence, by

$$\big|Q(u_n,v_m)\big|\leq Lq(u_n,v_m),\quad\zeta_h(t-\tau_0)-\zeta_h(t-\tau)\geq 0,$$

we have

$$\int_0^\infty\!\!\int_{\mathbb{R}}q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$\geq -\Big(\tau+\frac{1}{h}\Big)\|f_n-f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}.$$

More precisely, by the construction of ϕ, we have

$$\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{|x|-R-L(\tau-t)\leq\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$\geq -\Big(\tau+\frac{1}{h}\Big)\|f_n-f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}.$$

Here, to estimate the first term, we write the following:

$$\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{|x|-R-L(\tau-t)\leq\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$=\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{|x|-R-L(\tau-t)\leq-\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$+\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{-\frac{1}{h}\leq|x|-R-L(\tau-t)\leq\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$=\int_{\tau_0-\frac{1}{h}}^{\tau_0+\frac{1}{h}}\int_{|x|\leq R+L(\tau-t)-\frac{1}{h}} q(u_n,v_m)\delta_h(t-\tau_0)\,dxdt$$

$$-\int_{\tau-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{|x|\leq R+L(\tau-t)-\frac{1}{h}} q(u_n,v_m)\delta_h(t-\tau)\,dxdt$$

$$+\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{-\frac{1}{h}\le|x|-R-L(\tau-t)\le\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt.$$

Then, in terms of the first and second terms, setting $s=\tau_0,\tau$, for a.e. s, we have

$$\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{|x|\le R+L(\tau-t)-\frac{1}{h}} q\big(u_n(x,t),v_m(x,t)\big)\delta_h(t-s)\,dxdt$$

$$=\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{|x|\le R+L(\tau-t)} q\big(u_n(x,t),v_m(x,t)\big)\delta_h(t-s)\,dxdt$$

$$-\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{R+L(\tau-t)-\frac{1}{h}\le|x|\le R+L(\tau-t)} q\big(u_n(x,t),v_m(x,t)\big)\delta_h(t-s)\,dxdt$$

$$=\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{|x|\le R+L(\tau-t)} q\big(u_n(x,t),v_m(x,t)\big)\delta_h(t-s)\,dxdt$$

$$+O(1)h\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{R+L(\tau-t)-\frac{1}{h}\le|x|\le R+L(\tau-t)} dxdt$$

$$=\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}\int_{|x|\le R+L(\tau-t)} q\big(u_n(x,t),v_m(x,t)\big)\delta_h(t-s)\,dxdt+O(1)/h$$

$$\to\int_{|x|\le R+L(\tau-s)} q\big(u_n(x,s),v_m(x,s)\big)\,dx\quad(h\to\infty),$$

and, in terms of third term,

$$\int_{\tau_0-\frac{1}{h}}^{\tau+\frac{1}{h}}\int_{-\frac{1}{h}\le|x|-R-L(\tau-t)\le\frac{1}{h}} q(u_n,v_m)\Big(1-\zeta_h\big(|x|-R-L(\tau-t)\big)\Big)$$

$$\cdot\Big(\delta_h(t-\tau_0)-\delta_h(t-\tau)\Big)\,dxdt$$

$$\to 0\quad(h\to\infty).$$

Therefore, noting that $u_n - v_m \in C\big([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})\big)$, for any τ_0, τ with $\tau_0 < \tau$, we have

$$\int_{|x|\le R} |u_n(x,\tau) - v_m(x,\tau)|\,dx \le \int_{|x|\le R+L(\tau-\tau_0)} |u_n(x,\tau_0) - v_m(x,\tau_0)|\,dx$$
$$+ \tau \,\|f_n - f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}.$$

Thus, especially, for all $t \ge 0$, we have

$$\int_{|x|\le R} |u_n(x,t) - v_m(x,t)|\,dx \le \int_{|x|\le R+Lt} |u_n(x,0) - v_m(x,0)|\,dx$$
$$+ t \,\|f_n - f_m\|_{\mathrm{Lip}(K)}\,\mathrm{T.V.}\,\big\{\bar{u}_n\big\}.$$

□

For each $R > 0$, we define

$$p_R(w) = \int_{|x|\le R} |w|\,dx \quad \big(w \in L^1_{\mathrm{loc}}(\mathbb{R})\big)$$

and

$$d(w_1,w_2) = \sum_{R=1}^{\infty} 2^{-R}\frac{p_R(w_1-w_2)}{1+p_R(w_1-w_2)} \quad \big(w_1,w_2 \in L^1_{\mathrm{loc}}(\mathbb{R})\big).$$

Then we see that $\big(L^1_{\mathrm{loc}}(\mathbb{R}), d\big)$ is the Fréchet space, that is, a complete metric space (see [2]).

Theorem 2. *Let $\bar{u} \in BV(\mathbb{R})$. The sequence of piecewise constant solutions $\{u_n\}$ constructed by the wave front tracking method for the Cauchy problem (E_n) with $u(x,0) = \bar{u}_n(x)$ converges to a unique limit function in the sense of $L^1_{\mathrm{loc}}\big(\mathbb{R}\times[0,\infty)\big)$, and, for all $T > 0$ and all $R > 0$, the rate of convergence to the limit u is given by*

$$\int_0^T\int_{|x|\le R} |u_n(x,t) - u(x,t)|\,dxdt \le 2^{-n+1}T\Big(2R + T\big(L + M\,\mathrm{T.V.}\,\big\{\bar{u}\big\}\big)\Big). \tag{17}$$

Proof. Using inequalities (3), (4), (10) and (16), for each $t \geq 0$ and all $R > 0$, we have

$$p_R\big(u_n(\cdot,t) - u_m(\cdot,t)\big) \leq 2^{-n+3}(R + Lt) + 2^{-n+2} Mt \,\text{T.V.}\,\{\bar{u}\}$$
$$= 2^{-n+2}\Big(2R + t\big(2L + M\,\text{T.V.}\,\{\bar{u}\}\big)\Big)$$

so that, for an arbitrarily fixed $N \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{n\to\infty} d\big(u_n(\cdot,t), u_m(\cdot,t)\big) \\
&\quad \leq \limsup_{n\to\infty} \sum_{R=1}^{N} 2^{-R} \frac{p_R\big(u_n(\cdot,t) - u_m(\cdot,t)\big)}{1 + p_R\big(u_n(\cdot,t) - u_m(\cdot,t)\big)} \\
&\qquad + \limsup_{n\to\infty} \sum_{R=N+1}^{\infty} 2^{-R} \frac{p_R\big(u_n(\cdot,t) - u_m(\cdot,t)\big)}{1 + p_R\big(u_n(\cdot,t) - u_m(\cdot,t)\big)} \\
&\quad \leq \limsup_{n\to\infty} 2^{-n+1} N\Big(2R + t\big(2L + M\,\text{T.V.}\,\{\bar{u}\}\big)\Big) + 2^{-N} \\
&\quad = 2^{-N}.
\end{aligned}$$

Hence, by the arbitrariness of $N \in \mathbb{N}$, $\{u_n(\cdot,t)\}$ is a Cauchy sequence in $\big(L^1_{\text{loc}}(\mathbb{R}), d\big)$ which means that $\{u_n(\cdot,t)\}$ converges to some $u(\cdot,t) \in L^1_{\text{loc}}(\mathbb{R})$. Then, for all $R \geq 1$, we have

$$2^{-R} \frac{p_R\big(u_n(\cdot,t) - u(\cdot,t)\big)}{1 + p_R\big(u_n(\cdot,t) - u(\cdot,t)\big)} \leq d\big(u_n(\cdot,t), u(\cdot,t)\big) \to 0 \quad (n \to \infty).$$

Hence, noting that $p_R\big(u_n(\cdot,t) - u(\cdot,t)\big)$ is bounded with respect to $n \in \mathbb{N}$, we have

$$\int_{|x|\leq R} |u_n(\cdot,t) - u(\cdot,t)|\,dx = p_R\big(u_n(\cdot,t) - u(\cdot,t)\big) \to 0 \quad (n \to \infty).$$

Therefore, by inequality (8), for all $s, t \in [0,\infty)$, we have

$$\int_{|x|\leq R} |u(x,s) - u(x,t)|\,dx \leq 2L\,\text{T.V.}\,\{\bar{u}\}\,|s - t|$$

so that $u \in L^1_{\text{loc}}\big(\mathbb{R} \times [0, \infty)\big)$. Thus, using once again inequalities (3), (4), (10) and (16), for all $t \geq 0$ and all $R > 0$, we have

$$\begin{aligned}\int_{|x|\leq R} |u_n(x,t) - u(x,t)|\, dx &\leq \int_{|x|\leq R+Lt} |u_n(x,0) - u(x,0)|\, dx \\ &\qquad + t\, \|f_n - f\|_{\text{Lip}(K)}\, \text{T.V.}\, \{\bar{u}_n\} \\ &\leq 2^{-n+2}(R + Lt) + 2^{-n+2} M t\, \text{T.V.}\, \{\bar{u}\} \\ &= 2^{-n+2}\Big(R + t\big(L + M\, \text{T.V.}\, \{\bar{u}\}\big)\Big)\end{aligned}$$

so that, for all $T > 0$,

$$\begin{aligned}\int_0^T \int_{|x|\leq R} |u_n(x,t) - u(x,t)|\, dxdt &\leq 2^{-n+1} T\Big(2R + T\big(L + M\, \text{T.V.}\, \{\bar{u}\}\big)\Big) \\ &\to 0 \quad (n \to \infty),\end{aligned}$$

which means that $\{u_n\}$ converges to the unique limit u in the sense of $L^1_{\text{loc}}\big(\mathbb{R} \times [0, \infty)\big)$. □

The limit u of the sequence of piecewise constant solutions $\{u_n\}$ constructed by the wave front tracking method for the Cauchy problem (E_n) with $u(x,0) = \bar{u}_n(x)$ is an entropy solution to the Cauchy problem (1)–(2) with $u(x,0) = \bar{u}(x)$ (see [1] and [4]). Therefore, by Theorem 1, we can obtain the following result similar to the uniqueness theorem (cf. Kružkov [8]) of the entropy solution to the Cauchy problem (1)–(2):

Corollary 1. *Let $\bar{u}, \bar{v} \in BV(\mathbb{R})$. We denote by u and v entropy solutions which are limits of sequences of piecewise constant solutions constructed by the wave front tracking method for the Cauchy problems (1)–(2) with $u(x,0) = \bar{u}(x)$ and $u(x,0) = \bar{v}(x)$, respectively. Then, for all $t \geq 0$ and all $R > 0$, we have*

$$\int_{|x|\leq R} |u(x,t) - v(x,t)|\, dx \leq \int_{|x|\leq R+Lt} |u(x,0) - v(x,0)|\, dx. \tag{18}$$

Especially, if $\int_{\mathbb{R}} |u(x,0) - v(x,0)|\, dx < \infty$, then we have

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)|\, dx \leq \int_{\mathbb{R}} |u(x,0) - v(x,0)|\, dx. \tag{19}$$

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