Further Study on Possibility-Theoretical Indices for Comparing Fuzzy Sets

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Contents

Chapter 1

Introduction

The concept of fuzzy sets was introduced by Zadeh in his most famous paper [38] published in 1965 as a generalization of ordinary sets. In natural language, we commonly use indefinite expressions such as "young people," "old people," "numbers around 10," and "numbers much greater than 10." Within the framework of ordinary set theory, we cannot define the sets corresponding to these expressions because modifiers such as "young," "old," "around 10," "much greater than 10" are subjective and have vague boundaries. Fuzzy set theory enables us to mathematically quantify them as fuzzy sets reflecting individual subjective views. In [38], the concept of fuzzy relations was also introduced as a generalization of ordinary relations. Fuzzy theory, which uses fuzzy sets, fuzzy relations, fuzzy logic, and other fuzzy concepts, has been developed by numerous researchers and has been applied to a wide range of scientific fields (see [2] and the references therein).

In various decision-making situations, comparing and ranking multiple objects are essentially important. As individuals, companies, and even countries, we often compare and rank multiple alternatives on the basis of some kind of criterion and choose the best or a better one from them. In mathematical optimization, we need to compare the values of an objective function and sometimes to check whether each element of its domain fulfills constraint inequalities. If the set of objects to be compared is totally ordered, the comparison is simple for the reason that we can completely rank the objects. If not, the comparison is complicated. Especially if the set of objects is not even partially ordered, it becomes more complicated.

For example, the set of all real numbers is totally ordered by the usual "less than or equal to" relation. Vectors in the n-dimensional Euclidean space with $n \geq 2$ are usually compared by the componentwise order, which is not a total order but a partial order on the space. In general, any real vector space is preordered by a convex cone containing the origin. If the convex cone is pointed, then the space is partially ordered. When we consider comparing ordinary sets in a real vector space, we notice that there inconveniently does not exist on the power set of the space any total order or partial order which is naturally defined. In the area of set optimization, certain kinds of binary relations called "set relations" are used to compere the values of a set-valued objective function (see [8, 18, 19, 23]). The most famous set relations are two types called "l-type" and "u-type" which both are preorders. Six types of set relations proposed by Kuroiwa, Tanaka, and Ha in [24] include these two types and have a systematic structure: They are defined based on all combinations of the universality and existence of elements. The set relations except for the above two types are not preorders, but it is known that they satisfy certain order-like properties.

Next, let us turn our eyes to comparing fuzzy sets. There are a large number of studies discussing how to compare and rank multiple fuzzy sets, particularly fuzzy numbers (i.e., fuzzy sets in the real line with certain restrictions). Some of the comparison criteria and ranking methods proposed in these studies are reviewed in [1, 33, 34]. As the large number of studies implicitly tell us, it is difficult to have a definitive criterion for comparing fuzzy sets. This is because every criterion focuses only on a particular aspect of fuzzy sets and hence cannot fit various situations and demands. In [3], Dubois and Prade proposed four comparison indices for fuzzy numbers based on possibility theory. These indices jointly describe all of the relative positions of two fuzzy numbers and thus constitute a candidate for a definitive set of criteria for the comparison. In [14], Inuiguchi, Ichihashi, and Kume extended the four indices to define six types of fuzzy relations between fuzzy sets in a general setting. Some properties of these fuzzy relations were investigated in [14, 16]. In addition, they were applied to certain fuzzy mathematical programming problems in [13, 15, 16].

This thesis presents a further study on the above six types of fuzzy relations of Inuiguchi and others. Its main part is based on my recent paper [10]. We here consider the fuzzy relations specifically in a preordered vector space. It is shown that the six types of fuzzy relations are completely related to the six types of set relations of Kuroiwa and others. In fact, this relationship was hinted before in [16]. We state the relationship explicitly as three theorems with different assumptions. Furthermore, one of the theorems is applied to the area of fuzzy optimization. As a result, it is demonstrated that solving a general fuzzy optimization problem with respect to each of the fuzzy relations is equivalent to solving a certain set optimization problem with respect to the corresponding set relation or its extension.

The rest of the thesis is organized as follows. Chapter 2 is devoted to the preliminary knowledge which is necessary for the subsequent parts of the thesis. Chapter 3 is divided into three sections. In Section 3.1, we define six types of possibility-theoretical fuzzy relations between fuzzy sets and provide some basic properties of them. In Section 3.2, we describe a relationship of the fuzzy relations to six types of set relations. In Section 3.3, we apply the described relationship to fuzzy optimization. Chapter 4 finally concludes the thesis.

Chapter 2

Preliminaries

We begin by recalling some fundamental concepts and their properties related to the areas of topological vector space theory, set optimization, fuzzy set theory, and possibility theory.

2.1 Preordered topological vector space

Let Z be a real vector space. The set of all subsets of Z is called its *power set* and is denoted by $\mathcal{P}(Z)$. On $\mathcal{P}(Z)$, the *addition* and the *scalar multiplication* are defined respectively by

$$A + B := \{a + b \mid a \in A,\, b \in B\}, \quad \lambda A := \{\lambda a \mid a \in A\}$$

for $A, B \in \mathcal{P}(Z)$ and $\lambda \in \mathbb{R}$. In particular, we use the notations $z + A = A + z := A + \{z\}$ for $z \in Z$, $-A := (-1)A$, and $A - B := A + (-B)$. Note that $A - B$ is different from the *relative complement* $A \setminus B := \{z \mid z \in A,\, z \notin B\}$.

A subset C of Z is called a *cone* if $\lambda C \subset C$ for all $\lambda > 0$ and a *convex cone* if it is a cone and $C + C \subset C$. Any convex cone C containing the origin 0_Z defines a *preorder* (i.e., reflexive and transitive relation) on Z by

$$z \leq_C z' \;:\Longleftrightarrow\; z' - z \in C$$

for $z, z' \in Z$. Thus the space Z is *preordered* by the convex cone C. The preorder $\leq_C$ is *compatible with the linear structure* of Z in the sense that

$$z \leq_C z' \implies z + z'' \leq_C z' + z'', \quad \lambda z \leq_C \lambda z'$$

for every $z, z', z'' \in Z$ and $\lambda > 0$. If C is *pointed* (i.e., $C \cap (-C) = \{0_Z\}$), then $\leq_C$ is antisymmetric and hence becomes a *partial order*. Given $z \in Z$, two *order intervals* are defined as

$$[z, +\infty)_C := \{z' \in Z \mid z \leq_C z'\}, \quad (-\infty, z]_C := \{z' \in Z \mid z' \leq_C z\}.$$

Next, let Z be a *real topological vector space*. This means the vector space Z is equipped with a topology such that the addition $Z \times Z \ni (z, z') \mapsto z + z' \in Z$ and the scalar multiplication $\mathbb{R} \times Z \ni (\lambda, z) \mapsto \lambda z \in Z$ are both continuous. The topological interior and the topological closure of a set A are denoted by $\operatorname{int} A$ and $\operatorname{cl} A$, respectively. It is known that there exists a neighborhood base $\mathcal{B}$ of 0_Z satisfying the following properties (see [9, p. 47]):

(i) Every $V \in \mathcal{B}$ is closed and *balanced* (i.e., $\lambda V \subset V$ for all $\lambda \in [-1, 1]$).

(ii) For any $V \in \mathcal{B}$, there exists $V' \in \mathcal{B}$ such that $V' + V' \subset V$.

If we can take such a neighborhood base with an additional condition that every $V \in \mathcal{B}$ is *convex* (i.e., $\lambda V + (1 - \lambda)V \subset V$ for all $\lambda \in (0, 1)$), then the space is said to be *locally convex*. A set A is said to be *compact* if its every open cover has a finite subcover. If Z is *Hausdorff* (i.e., any two distinct points can be separated by disjoint neighborhoods), then every compact subset of Z is closed.

A function $f \colon Z \to \mathbb{R}$ is said to be

(i) *upper semicontinuous* at $z_0 \in Z$ if for any $\alpha > f(z_0)$, there exists a neighborhood V of z_0 such that $f(z) \leq \alpha$ for all $z \in V$.

(ii) *lower semicontinuous* at $z_0 \in Z$ if for any $\alpha < f(z_0)$, there exists a neighborhood V of z_0 such that $f(z) \geq \alpha$ for all $z \in V$.

(iii) upper semicontinuous (or lower semicontinuous) if it is so at every $z_0 \in Z$.

It holds that f is upper (resp., lower) semicontinuous if and only if $\{z \in Z \mid f(z) \geq \alpha\}$ (resp., $\{z \in Z \mid f(z) \leq \alpha\}$) is closed for every $\alpha \in \mathbb{R}$. As is well known, an upper (resp., lower) semicontinuous function defined on a compact set always has a maximum (resp., minimum).

A *set-valued mapping* is, as the name suggests, a mapping whose values are sets. Let X be a topological space. Then a set-valued mapping $F \colon X \to \mathcal{P}(Z)$ is said to be

(i) *upper continuous* at $x_0 \in X$ if for any open set O in Z with $F(x_0) \subset O$, there exists a neighborhood U of x_0 such that $F(x) \subset O$ for all $x \in U$.

(ii) *lower continuous* at $x_0 \in X$ if for any open set O in Z with $F(x_0) \cap O \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap O \neq \emptyset$ for all $x \in U$.

(iii) upper continuous (or lower continuous) if it is so at every $x_0 \in X$.

These kinds of continuities are unique to set-valued mappings. Intuitively speaking, the upper (resp., lower) continuity of F requires that the set $F(x)$ cannot suddenly expand (resp., shrink) as x changes.

In this section, we have mentioned several concepts related to the linearity and topology of the space in the respective settings. Throughout the rest of the thesis, we set Z as a real Hausdorff topological vector space preordered by a convex cone C with $0_Z \in C \neq Z$.

2.2 Set relations in set optimization

Set relations are extensions of the vector preorder $\leq_C$ to the power set $\mathcal{P}(Z)$ and are mainly used to compare the values of a set-valued objective function in set optimization (see [8, 18, 19, 23]). The original idea of six types of set relations can be found in Kuroiwa, Tanaka, and Ha's paper [24]. The relationship between two sets A and B in the sense that A is dominated by B from above or A dominates B from below is classified as follows:

(i) $A \subset \bigcap_{b \in B}(b - C)$, or equivalently $\bigcap_{a \in A}(a + C) \supset B$.

(ii) $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$.

(iii) $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$.

(iv) $A + C \supset B$.

(v) $A \subset B - C$.

(vi) $A \cap (B - C) \neq \emptyset$, or equivalently $(A + C) \cap B \neq \emptyset$.

This classification gives the following definition of set relations, where the superscript numbering is the same as originally employed in [12]. The letters L and U stand for "lower" and "upper," respectively.

Definition 2.1. Six types of set relations $\preceq_C^{(*)}$ $(* = 1, 2L, 2U, 3L, 3U, 4)$ are defined by

$$
\begin{aligned}
A \preceq_C^{(1)} B \; &:\Longleftrightarrow \; \forall a \in A \; \forall b \in B \colon a \leq_C b, \\
A \preceq_C^{(2L)} B \; &:\Longleftrightarrow \; \exists a \in A \; \forall b \in B \colon a \leq_C b, \\
A \preceq_C^{(2U)} B \; &:\Longleftrightarrow \; \exists b \in B \; \forall a \in A \colon a \leq_C b, \\
A \preceq_C^{(3L)} B \; &:\Longleftrightarrow \; \forall b \in B \; \exists a \in A \colon a \leq_C b, \\
A \preceq_C^{(3U)} B \; &:\Longleftrightarrow \; \forall a \in A \; \exists b \in B \colon a \leq_C b, \\
A \preceq_C^{(4)} B \; &:\Longleftrightarrow \; \exists a \in A \; \exists b \in B \colon a \leq_C b
\end{aligned}
$$

for $A, B \in \mathcal{P}(Z)$.

Here, we assume "$\forall z \in \emptyset \colon Q(z)$" is true and "$\exists z \in \emptyset \colon Q(z)$" is false for every propositional function Q. If $A = \emptyset$ and $B \neq \emptyset$ for example, then $A \preceq_C^{(*)} B$ holds for $* = 1, 2U, 3U$ and does not hold for $* = 2L, 3L, 4$.

The set relation $\preceq_C^{(*)}$ is transitive for $* = 1, 2L, 2U, 3L, 3U$ and a preorder only for $* = 3L, 3U$. For this reason, $\preceq_C^{(3L)}$ and $\preceq_C^{(3U)}$ especially play key roles in many existing studies. The propositions below describe four basic properties of the six types of set relations.

Proposition 2.1 ([24])**.** *Let $A, B \in \mathcal{P}(Z)$ be nonempty. Then the following implications hold:*

$$
\begin{aligned}
A \preceq_C^{(1)} B \implies A \preceq_C^{(2L)} B \implies A \preceq_C^{(3L)} B \implies A \preceq_C^{(4)} B, \\
A \preceq_C^{(1)} B \implies A \preceq_C^{(2U)} B \implies A \preceq_C^{(3U)} B \implies A \preceq_C^{(4)} B.
\end{aligned}
$$

Proof. The implications are derived from Definition 2.1 and the following properties of the universal and existential quantifiers $\forall, \exists$:

$$
\begin{aligned}
S \neq \emptyset, \;\; \forall z \in S \colon Q(z) \implies \exists z \in S \colon Q(z), \\
\exists z \in S \; \forall z' \in S' \colon Q'(z, z') \implies \forall z' \in S' \; \exists z \in S \colon Q'(z, z')
\end{aligned}
$$

for every sets S, S' and propositional functions Q on S and Q' on $S \times S'$. □

Proposition 2.2 ([25])**.** *Let $A, B \in \mathcal{P}(Z)$, $z \in Z$, and $\lambda > 0$. Then the following implication holds for each $* = 1, 2L, 2U, 3L, 3U, 4$:*

$$
A \preceq_C^{(*)} B \implies A + z \preceq_C^{(*)} B + z, \quad \lambda A \preceq_C^{(*)} \lambda B.
$$

Proof. The compatibility of $\leq_C$ with the linear structure of Z and Definition 2.1 immediately give the implication to be proved. $\square$

Proposition 2.3 ([11])**.** *Let $A, B \in \mathcal{P}(Z)$. Then the following equivalences hold:*

$$
\begin{aligned}
A \preceq_C^{(1)} B &\iff B \preceq_{-C}^{(1)} A, & A \preceq_C^{(2L)} B &\iff B \preceq_{-C}^{(2U)} A,\\
A \preceq_C^{(2U)} B &\iff B \preceq_{-C}^{(2L)} A, & A \preceq_C^{(3L)} B &\iff B \preceq_{-C}^{(3U)} A,\\
A \preceq_C^{(3U)} B &\iff B \preceq_{-C}^{(3L)} A, & A \preceq_C^{(4)} B &\iff B \preceq_{-C}^{(4)} A.
\end{aligned}
$$

Proof. For any $a, b \in Z$, $a \leq_C b$ is equivalent to $b \leq_{-C} a$ because $b - a \in C$ can be deformed into $a - b \in -C$. We therefore obtain the desired equivalences from Definition 2.1. $\square$

Proposition 2.4 ([10])**.** *Let $A, A', B, B' \in \mathcal{P}(Z)$. If*

(i) $A \supset A'$ *and* $B \supset B'$ *for* $* = 1$,

(ii) $A \subset A'$ *and* $B \supset B'$ *for* $* = 2L, 3L$,

(iii) $A \supset A'$ *and* $B \subset B'$ *for* $* = 2U, 3U$,

(iv) $A \subset A'$ *and* $B \subset B'$ *for* $* = 4$,

then

$$A \preceq_C^{(*)} B \implies A' \preceq_C^{(*)} B'.$$

Proof. For every sets S, S' with $S \supset S'$ and propositional function Q on S,

$$
\begin{aligned}
\forall z \in S \colon Q(z) &\implies \forall z \in S' \colon Q(z),\\
\exists z \in S' \colon Q(z) &\implies \exists z \in S \colon Q(z).
\end{aligned}
$$

The conclusion follows from Definition 2.1 and these facts. $\square$

Next, let us introduce a setting for set optimization. A *set optimization problem* is an optimization problem whose objective function is a set-valued mapping. Specifically, it is given in a general form as

$$
\text{(SOP)} \quad \begin{cases} \text{minimize} & F(x) \\ \text{subject to} & x \in X \end{cases}
$$

for a nonempty set X and a mapping $F \colon X \to \mathcal{P}(Z)$. We need to determine the meaning of "minimization." In the literature, there are the following three solution concepts for (SOP).

Definition 2.2 ([18] for (i)–(ii), [7] for (iii))**.** Let $\preceq$ be a binary relation on $\mathcal{P}(Z)$. An element $\bar{x} \in X$ is called

(i) an *optimal solution* of (SOP) with respect to $\preceq$ if

$$\forall x \in X\colon\ (F(x) \preceq F(\bar{x}) \implies F(\bar{x}) \preceq F(x))\,.$$

(ii) a *strongly optimal solution* of (SOP) with respect to $\preceq$ if

$$\forall x \in X \setminus \{\bar{x}\} : F(\bar{x}) \preceq F(x).$$

(iii) a *strictly optimal solution* of (SOP) with respect to $\preceq$ if

$$\nexists x \in X \setminus \{\bar{x}\} : F(x) \preceq F(\bar{x}).$$

It is easily verified that any strongly optimal or strictly optimal solution is also an optimal solution. In actual use, the relation $\preceq$ in this definition is substituted with each of the set relations.

At the end of this section, we mention a scalarization technique for sets. Scalarization is considered to be a powerful tool for handling set relations (see [6, 20, 25, 30]). In [25], two kinds of *scalarizing functions for sets* are defined for each $* = 1, 2L, 2U, 3L, 3U, 4$:

$$I^{(*)}_{k,B}(A) := \inf\left\{t \in \mathbb{R} \;\middle|\; A \preceq^{(*)}_C B + tk\right\},$$
$$S^{(*)}_{k,B}(A) := \sup\left\{t \in \mathbb{R} \;\middle|\; B + tk \preceq^{(*)}_C A\right\}$$

where $k \in \operatorname{int} C$ and $A, B \in \mathcal{P}(Z)$. By using these functions, we can measure the relative position between sets A and B with respect to each set relation. In [36, 37], calculation algorithms of the value $I^{(*)}_{k,B}(A)$ in certain polyhedral cases are discussed.

2.3 Fuzzy set and fuzzy relation

Fuzzy sets are a generalization of ordinary sets originated by Zadeh [38]. A fuzzy set $\tilde{A}$ in Z is uniquely determined by a function $\mu_{\tilde{A}}\colon Z \to [0,1]$. The value $\mu_{\tilde{A}}(z)$ represents the grade of membership of z in $\tilde{A}$, and hence $\mu_{\tilde{A}}$ is called the *membership function* of $\tilde{A}$. In ordinary set theory, the *characteristic function* of a set A in Z is given as

$$\chi_A\colon Z \to \{0,1\}\,,\quad \chi_A(z) := \begin{cases} 1 & (z \in A) \\ 0 & (z \notin A) \end{cases}.$$

This is a special case of the membership function of a fuzzy set. Ordinary sets are usually called *crisp sets* in fuzzy set theory.

The set of all fuzzy sets in Z is denoted by $\mathcal{F}(Z)$. Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. We say

(i) $\tilde{A}$ is *equal to* $\tilde{B}$, written as $\tilde{A} = \tilde{B}$, if $\mu_{\tilde{A}}(z) = \mu_{\tilde{B}}(z)$ for all $z \in Z$.

(ii) $\tilde{A}$ is *included in* $\tilde{B}$, written as $\tilde{A} \subset \tilde{B}$, if $\mu_{\tilde{A}}(z) \leq \mu_{\tilde{B}}(z)$ for all $z \in Z$.

The *complement* of $\tilde{A}$ is denoted by $\tilde{A}^{\mathrm{c}}$ and is defined by

$$\mu_{\tilde{A}^{\mathrm{c}}}(z) := 1 - \mu_{\tilde{A}}(z), \quad z \in Z.$$

For each $\alpha \in [0, 1]$, the *α-cut* (or *α-level set*) of $\tilde{A}$ is defined as

$$[\tilde{A}]_\alpha := \begin{cases} \{z \in Z \mid \mu_{\tilde{A}}(z) \geq \alpha\} & (\alpha \in (0, 1]) \\ \mathrm{cl}\,\{z \in Z \mid \mu_{\tilde{A}}(z) > 0\} & (\alpha = 0) \end{cases}.$$

Clearly, $\alpha \leq \beta$ for $\alpha, \beta \in [0, 1]$ implies $[\tilde{A}]_\alpha \supset [\tilde{A}]_\beta$. We refer, for convenience, to the set-valued mapping

$$[0, 1] \ni \alpha \mapsto [\tilde{A}]_\alpha \in \mathcal{P}(Z)$$

as the *cut mapping* of $\tilde{A}$. The *translation* $\tilde{A}+z$ for $z \in Z$ and the *scalar multiplication* $\lambda\tilde{A}$ for $\lambda \neq 0$ are defined by

$$\mu_{\tilde{A}+z}(z') := \mu_{\tilde{A}}(z' - z), \quad \mu_{\lambda\tilde{A}}(z') := \mu_{\tilde{A}}\left(\frac{1}{\lambda}z'\right), \quad z' \in Z.$$

Then $[\tilde{A} + z]_\alpha = [\tilde{A}]_\alpha + z$ and $[\lambda\tilde{A}]_\alpha = \lambda[\tilde{A}]_\alpha$ hold for every $\alpha \in [0, 1]$.

Moreover, a fuzzy set $\tilde{A}$ in Z is said to be

(i) *normal* if there exists $z \in Z$ such that $\mu_{\tilde{A}}(z) = 1$.

(ii) *closed* if its α-cut is closed for every $\alpha \in [0, 1]$.

(iii) *compact* if its α-cut is compact for every $\alpha \in [0, 1]$.

(iv) *convex* if its α-cut is convex for every $\alpha \in [0, 1]$.

(v) *strictly convex* if

$$\begin{aligned} \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\} \in (0, 1) &\implies \mu_{\tilde{A}}(\lambda z + (1 - \lambda)z') > \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\}, \\ \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\} = 1 &\implies \mu_{\tilde{A}}(\lambda z + (1 - \lambda)z') = 1 \end{aligned}$$

for every $z, z' \in Z$ with $z \neq z'$ and $\lambda \in (0, 1)$.

We easily notice that the normality of a fuzzy set is equivalent to the nonemptiness of all its α-cuts. It is known that a fuzzy set $\tilde{A}$ is convex if and only if its membership function is *quasiconcave*, i.e.,

$$\mu_{\tilde{A}}(\lambda z + (1-\lambda)z') \geq \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\}$$

for every $z, z' \in Z$ and $\lambda \in (0,1)$. Consequently, the strict convexity of a fuzzy set is a stronger condition than its convexity. Normal, convex, and compact fuzzy sets in $\mathbb{R}$ can be viewed as a generalization of real numbers and are called *fuzzy numbers*. We remark that the definition of a fuzzy number differs a little depending on the literature.

The following two propositions reveal sufficient conditions for the cut mapping of a fuzzy set satisfying each continuity of a set-valued mapping (cf. [10, Propositions 2.2 and 2.3]).

Proposition 2.5. *If $\tilde{A} \in \mathcal{F}(Z)$ is compact, then the cut mapping of $\tilde{A}$ is upper continuous.*

Proof. Fix any $\alpha \in [0,1]$ and open set O in Z with $[\tilde{A}]_\alpha \subset O$. If $[\tilde{A}]_0 \subset O$, then we have $[\tilde{A}]_\beta \subset [\tilde{A}]_0 \subset O$ for all $\beta \in [0,1]$.

Assume that $[\tilde{A}]_0 \not\subset O$. Now, the membership function $\mu_{\tilde{A}}$ is upper semicontinuous because the compactness of $\tilde{A}$ requires its closedness (by the Hausdorffness of the space), and $[\tilde{A}]_0 \setminus O$ is a nonempty compact set. We can hence take $\bar{z}$ maximizing $\mu_{\tilde{A}}$ on $[\tilde{A}]_0 \setminus O$. Since $\bar{z} \notin O \supset [\tilde{A}]_\alpha$, $U := (\mu_{\tilde{A}}(\bar{z}), 1]$ is a neighborhood of α. For any $\beta \in U$, it follows from the maximality of $\mu_{\tilde{A}}(\bar{z})$ that $[\tilde{A}]_\beta \setminus O = \emptyset$ and thus $[\tilde{A}]_\beta \subset O$. Therefore, the cut mapping of $\tilde{A}$ is upper continuous at α. $\square$

Proposition 2.6. *If $\tilde{A} \in \mathcal{F}(Z)$ is normal and strictly convex, then the cut mapping of $\tilde{A}$ is lower continuous.*

Proof. It is easily verified that the cut mapping of $\tilde{A}$ is lower continuous at 1 by the inclusion $[\tilde{A}]_1 \subset [\tilde{A}]_\beta$ for all $\beta \in [0,1]$. Let $\alpha \in [0,1)$ and O be any open set in Z with $[\tilde{A}]_\alpha \cap O \neq \emptyset$. Then it suffices to prove that there exists $\bar{z} \in O$ satisfying $\mu_{\tilde{A}}(\bar{z}) > \alpha$. In fact, for such a $\bar{z}$, putting $U := [0, \mu_{\tilde{A}}(\bar{z}))$ as a neighborhood of α we have $\bar{z} \in [\tilde{A}]_\beta \cap O$ for all $\beta \in U$. This directly means the lower continuity of the mapping at α.

When $\alpha = 0$, from $[\tilde{A}]_0 \cap O \neq \emptyset$ and the definition of the 0-cut we obtain the desirable $\bar{z}$. When $\alpha \in (0,1)$, take $z \in [\tilde{A}]_\alpha \cap O$. If $\mu_{\tilde{A}}(z) > \alpha$, let $\bar{z} := z$. Otherwise, $\mu_{\tilde{A}}(z) = \alpha$ holds. Choose $z' \in Z$ with $\mu_{\tilde{A}}(z') = 1$ (by the normality of $\tilde{A}$) and a sufficiently small $\lambda \in (0,1)$ with $\bar{z} := \lambda z' + (1-\lambda)z \in O$. Since $\tilde{A}$ is strictly convex, we deduce $\mu_{\tilde{A}}(\bar{z}) > \min\{\mu_{\tilde{A}}(z'), \mu_{\tilde{A}}(z)\} = \alpha$, which completes the proof. □

In Zadeh's paper [38], the concept of *fuzzy relations* is proposed as well as that of fuzzy sets. A (binary) fuzzy relation on Z is defined to be a fuzzy set in the Cartesian product $Z \times Z$. Given a fuzzy relation $\tilde{R}$ on Z and elements $z, z' \in Z$, the value $\mu_{\tilde{R}}(z, z') \in [0,1]$ represents the degree to which the relation between z and z' holds. Since an ordinary relation (called a *crisp relation* in fuzzy set theory) either holds or does not hold, fuzzy relations are a generalization of ordinary relations. In dealing with fuzzy relations, a parameter $\alpha \in [0,1]$ is often used. The α-cut of any fuzzy relation $\tilde{R}$ on Z naturally defines a crisp relation on Z, setting

$$z\tilde{R}_\alpha z' \;:\Longleftrightarrow\; (z, z') \in [\tilde{R}]_\alpha \;\Longleftrightarrow\; \mu_{\tilde{R}}(z, z') \geq \alpha$$

for $z, z' \in Z$. Note that in the next chapter, we will mainly consider some fuzzy relations on $\mathcal{F}(Z)$ instead of Z.

2.4 Cone-notions

This section deals with some notions prefixed by a convex cone K in Z with $0_Z \in K$. The convex cone K will later be substituted with C and $-C$.

We say that

(i) a set A in Z is *K-compact* if its every open cover of the form $\{O_i + K\}_{i \in I}$ where O_i, $i \in I$ are open sets has a finite subcover.

(ii) a fuzzy set $\tilde{A}$ in Z is *K-compact* if its α-cut is K-compact for every $\alpha \in [0,1]$.

We also say that a set-valued mapping $F\colon X \to \mathcal{P}(Z)$, where X is a topological space, is

(i) *K-upper continuous* at $x_0 \in X$ if for any open set O in Z with $F(x_0) \subset O$, there exists a neighborhood U of x_0 such that $F(x) \subset O + K$ for all $x \in U$.

(ii) *K-lower continuous* at $x_0 \in X$ if for any open set O in Z with $F(x_0) \cap O \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap (O - K) \neq \emptyset$ for all $x \in U$.

(iii) K-upper continuous (or K-lower continuous) if it is so at every $x_0 \in X$.

The above definitions of cone-compactness of a set and cone-continuities of a set-valued mapping are based on [27, Definition 3.1] and [5, Definition 2.5.16], respectively.

Roughly speaking, each K-notion leaves the directions of K out of consideration as compared with the corresponding unprefixed notion. The following proposition reflects this aspect.

Proposition 2.7. *Let K be a convex cone in Z with $0_Z \in K$.*

(i) Every compact set in Z is K-compact.

(ii) Every compact fuzzy set in Z is K-compact.

(iii) Every upper continuous (resp., lower continuous) set-valued mapping from a topological space to $\mathcal{P}(Z)$ is K-upper continuous (resp., K-lower continuous).

Proof. Comparing the definitions of K-notions and the corresponding unprefixed notions, we can easily check that these statements are valid. □

The classes of fuzzy sets whose cut mappings satisfy each cone-continuity will play a significant role in stating a theorem in the next chapter. We give an example to help our understanding of them.

Example 2.1 ([10])**.** Consider the fuzzy sets $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$ in $\mathbb{R}$ whose membership functions are illustrated in Figure 2.1 and the set of all nonnegative real numbers $\mathbb{R}_+$ as a convex cone in $\mathbb{R}$. Then one can see that

(i) the cut mappings of $\tilde{A}_1$ and $\tilde{A}_2$ are not $\mathbb{R}_+$-upper continuous at 0.5, while they are $\mathbb{R}_+$-lower continuous (and even lower continuous).

(ii) the cut mappings of $\tilde{A}_3$ and $\tilde{A}_4$ are not $\mathbb{R}_+$-lower continuous at 0.5, while they are $\mathbb{R}_+$-upper continuous (and even upper continuous).

In addition, the fuzzy sets $\tilde{A}_1$ and $\tilde{A}_2$ are not compact, $\tilde{A}_3$ is not strictly convex, and $\tilde{A}_4$ is not normal. These are consistent with Propositions 2.5–2.7.

2.5 Possibility and necessity measures

First introduced by Zadeh [40], possibility theory has been developed especially by the contribution of Dubois and Prade (e.g., [2, 4]). The key ideas in this theory are possibility and necessity.

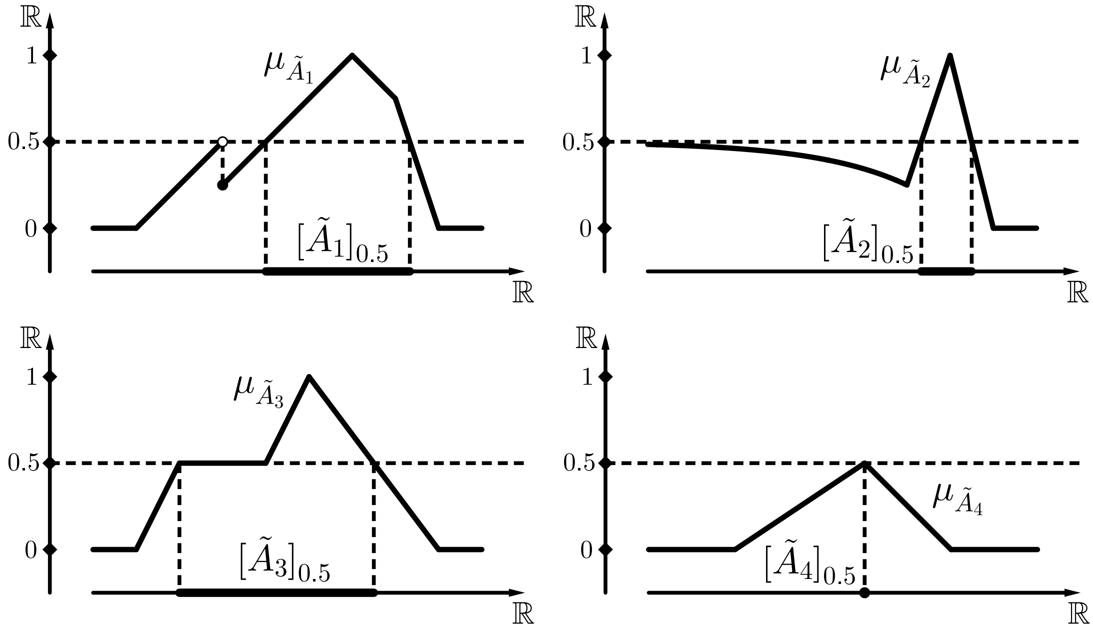

Figure 2.1: The membership functions of four fuzzy sets in $\mathbb{R}$ whose cut mappings are not $\mathbb{R}_+$-upper continuous or not $\mathbb{R}_+$-lower continuous ([10]).

For $A, B \in \mathcal{P}(Z)$, we put

$$\Pi_A(B) := \begin{cases} 1 & (A \cap B \neq \emptyset) \\ 0 & (A \cap B = \emptyset) \end{cases}, \quad N_A(B) := \begin{cases} 1 & (A \subset B) \\ 0 & (A \not\subset B) \end{cases}.$$

The quantity $\Pi_A(B)$ indicates whether $z \in B$ possibly holds or not when $z \in A$ holds, and hence $\Pi_A \colon \mathcal{P}(Z) \to \{0, 1\}$ is called a *possibility measure*. The quantity $N_A(B)$ indicates whether $z \in B$ necessarily holds or not when $z \in A$ holds, and hence $N_A \colon \mathcal{P}(Z) \to \{0, 1\}$ is called a *necessity measure*. We have another expression of these quantities

$$\Pi_A(B) = \sup_{z \in Z} \min \{\chi_A(z), \chi_B(z)\},$$
$$N_A(B) = \inf_{z \in Z} \max \{1 - \chi_A(z), \chi_B(z)\}.$$

By replacing the characteristic functions χ_A and χ_B with the membership functions of fuzzy sets, we extend the above two measures to the case of fuzzy sets.

Definition 2.3 ([3])**.** Let $\tilde{A} \in \mathcal{F}(Z)$. A possibility measure $\Pi_{\tilde{A}} \colon \mathcal{F}(Z) \to [0, 1]$ and a necessity measure $N_{\tilde{A}} \colon \mathcal{F}(Z) \to [0, 1]$ are defined by

$$\Pi_{\tilde{A}}(\tilde{B}) := \sup_{z \in Z} \min \{\mu_{\tilde{A}}(z), \mu_{\tilde{B}}(z)\}, \quad N_{\tilde{A}}(\tilde{B}) := \inf_{z \in Z} \max \{1 - \mu_{\tilde{A}}(z), \mu_{\tilde{B}}(z)\}$$

for $\tilde{B} \in \mathcal{F}(Z)$.

From this definition, we easily find the duality

$$\Pi_{\tilde{A}}(\tilde{B}) = 1 - N_{\tilde{A}}(\tilde{B}^{\mathrm{c}}), \quad N_{\tilde{A}}(\tilde{B}) = 1 - \Pi_{\tilde{A}}(\tilde{B}^{\mathrm{c}})$$

and the monotonicity

$$\tilde{B} \subset \tilde{B}' \implies \Pi_{\tilde{A}}(\tilde{B}) \leq \Pi_{\tilde{A}}(\tilde{B}'), \;\; N_{\tilde{A}}(\tilde{B}) \leq N_{\tilde{A}}(\tilde{B}')$$

for $\tilde{A}, \tilde{B}, \tilde{B}' \in \mathcal{F}(Z)$.

Proposition 2.8 ([3])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. If $\tilde{A}$ is normal, then*

$$N_{\tilde{A}}(\tilde{B}) \leq \Pi_{\tilde{A}}(\tilde{B}).$$

Proof. By the above duality, it is sufficient to prove

$$\Pi_{\tilde{A}}(\tilde{B}) + \Pi_{\tilde{A}}(\tilde{B}^{\mathrm{c}}) \geq 1.$$

We have

$$\begin{aligned}
\Pi_{\tilde{A}}(\tilde{B}) + \Pi_{\tilde{A}}(\tilde{B}^{\mathrm{c}}) &= \sup_{z \in Z} \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{B}}(z)\} + \sup_{z \in Z} \min\{\mu_{\tilde{A}}(z), 1 - \mu_{\tilde{B}}(z)\} \\
&\geq \sup_{z \in Z} (\min\{\mu_{\tilde{A}}(z), \mu_{\tilde{B}}(z)\} + \min\{\mu_{\tilde{A}}(z), 1 - \mu_{\tilde{B}}(z)\}) \\
&= \sup_{z \in Z} \min\{2\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z) + \mu_{\tilde{B}}(z), \mu_{\tilde{A}}(z) + 1 - \mu_{\tilde{B}}(z), 1\} \\
&\geq \sup_{z \in Z} \mu_{\tilde{A}}(z).
\end{aligned}$$

Since $\tilde{A}$ is normal, $\sup_{z \in Z} \mu_{\tilde{A}}(z) = 1$ holds. The proof is thus completed. □

Using the possibility and necessity measures, we can extend the order intervals $[z, +\infty)_C$ and $(-\infty, z]_C$ for $z \in Z$ to obtain four kinds of interval-like fuzzy sets. For $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$, we define

(i) a fuzzy set consisting of elements possibly greater than $\tilde{A}$

$$[\tilde{A}, +\infty)_C^{\Pi} \in \mathcal{F}(Z), \quad \mu_{[\tilde{A}, +\infty)_C^{\Pi}}(z) := \Pi_{\tilde{A}}((-\infty, z]_C) = \sup_{\substack{z' \in Z \\ z' \leq_C z}} \mu_{\tilde{A}}(z').$$

(ii) a fuzzy set consisting of elements necessarily greater than $\tilde{A}$

$$[\tilde{A}, +\infty)_C^{N} \in \mathcal{F}(Z), \quad \mu_{[\tilde{A}, +\infty)_C^{N}}(z) := N_{\tilde{A}}((-\infty, z]_C) = \inf_{\substack{z' \in Z \\ z' \not\leq_C z}} (1 - \mu_{\tilde{A}}(z')).$$

(iii) a fuzzy set consisting of elements possibly less than $\tilde{B}$

$$(-\infty, \tilde{B}]_C^{\Pi} \in \mathcal{F}(Z), \quad \mu_{(-\infty,\tilde{B}]_C^{\Pi}}(z) := \Pi_{\tilde{B}}([z, +\infty)_C) = \sup_{\substack{z' \in Z \\ z \leq_C z'}} \mu_{\tilde{B}}(z').$$

(iv) a fuzzy set consisting of elements necessarily less than $\tilde{B}$

$$(-\infty, \tilde{B}]_C^{N} \in \mathcal{F}(Z), \quad \mu_{(-\infty,\tilde{B}]_C^{N}}(z) := N_{\tilde{B}}([z, +\infty)_C) = \inf_{\substack{z' \in Z \\ z \not\leq_C z'}} \left(1 - \mu_{\tilde{B}}(z')\right).$$

Note that the last equalities in each item are valid because $\emptyset \neq C \neq Z$ ensures $\emptyset \neq (-\infty, z]_C \neq Z$ and $\emptyset \neq [z, +\infty)_C \neq Z$ for every $z \in Z$.

Chapter 3

Possibility-theoretical indices for comparing fuzzy sets

This chapter focuses on six types of order-like fuzzy relations on $\mathcal{F}(Z)$ associated with possibility theory. The idea of such relations was originally proposed by Dubois and Prade [3] to evaluate all of the relative positions of two fuzzy numbers. Inuiguchi, Ichihashi, and Kume [14] then defined the six types of fuzzy relations in a general setting. We here consider a vector preorder version of these relations. In [14, 16], some properties of these kinds of relations are investigated. Adding to them, we describe further results involving the relations newly shown in [10].

3.1 Definition and basic properties

The definition of the above-mentioned fuzzy relations given by Inuiguchi, Ichihashi, and Kume is the following.

Definition 3.1 ([14])**.** Let $\tilde{R}$ be a fuzzy relation on Z. Six types of fuzzy relations $\boxed{\mathrm{f}}\boxed{\mathrm{l}}\tilde{R}$, $\Diamond\mathrm{f}\,\boxed{\mathrm{l}}\tilde{R}$, $\Diamond\mathrm{l}\,\boxed{\mathrm{f}}\tilde{R}$, $\boxed{\mathrm{l}}\Diamond\mathrm{f}\,\tilde{R}$, $\boxed{\mathrm{f}}\Diamond\mathrm{l}\,\tilde{R}$, $\Diamond\mathrm{f}\,\Diamond\mathrm{l}\,\tilde{R}$ on $\mathcal{F}(Z)$ are defined by

$$
\begin{aligned}
\mu_{\boxed{\mathrm{f}}\boxed{\mathrm{l}}\tilde{R}}(\tilde{A}, \tilde{B}) &:= \inf_{a,b \in Z} \max\left\{1 - \mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b), \mu_{\tilde{R}}(a,b)\right\}, \\
\mu_{\Diamond\mathrm{f}\boxed{\mathrm{l}}\tilde{R}}(\tilde{A}, \tilde{B}) &:= \sup_{a \in Z} \inf_{b \in Z} \min\left\{\mu_{\tilde{A}}(a), \max\left\{1 - \mu_{\tilde{B}}(b), \mu_{\tilde{R}}(a,b)\right\}\right\}, \\
\mu_{\Diamond\mathrm{l}\boxed{\mathrm{f}}\tilde{R}}(\tilde{A}, \tilde{B}) &:= \sup_{b \in Z} \inf_{a \in Z} \min\left\{\max\left\{1 - \mu_{\tilde{A}}(a), \mu_{\tilde{R}}(a,b)\right\}, \mu_{\tilde{B}}(b)\right\}, \\
\mu_{\boxed{\mathrm{l}}\Diamond\mathrm{f}\tilde{R}}(\tilde{A}, \tilde{B}) &:= \inf_{b \in Z} \sup_{a \in Z} \max\left\{\min\left\{\mu_{\tilde{A}}(a), \mu_{\tilde{R}}(a,b)\right\}, 1 - \mu_{\tilde{B}}(b)\right\}, \\
\mu_{\boxed{\mathrm{f}}\Diamond\mathrm{l}\tilde{R}}(\tilde{A}, \tilde{B}) &:= \inf_{a \in Z} \sup_{b \in Z} \max\left\{1 - \mu_{\tilde{A}}(a), \min\left\{\mu_{\tilde{B}}(b), \mu_{\tilde{R}}(a,b)\right\}\right\},
\end{aligned}
$$

$$\mu_{\diamond\!\!\mathrm{f}\,\diamond\!\!\mathrm{l}\,\tilde{R}}(\tilde{A}, \tilde{B}) := \sup_{a,b \in Z} \min \{\mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b), \mu_{\tilde{R}}(a, b)\}$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$.

By replacing the fuzzy relation $\tilde{R}$ with the vector preorder $\leq_C$, we obtain the next definition.

Definition 3.2 ([10])**.** Six types of fuzzy relations $\precsim_C^{(*)}$ $(* = 1, 2L, 2U, 3L, 3U, 4)$ on $\mathcal{F}(Z)$ are defined by

$$\begin{aligned}
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) &:= \inf_{\substack{a,b \in Z \\ a \not\leq_C b}} \max \{1 - \mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\
\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) &:= \sup_{a \in Z} \inf_{\substack{b \in Z \\ a \not\leq_C b}} \min \{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\
\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) &:= \sup_{b \in Z} \inf_{\substack{a \in Z \\ a \not\leq_C b}} \min \{1 - \mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}, \\
\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) &:= \inf_{b \in Z} \sup_{\substack{a \in Z \\ a \leq_C b}} \max \{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\
\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) &:= \inf_{a \in Z} \sup_{\substack{b \in Z \\ a \leq_C b}} \max \{1 - \mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}, \\
\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) &:= \sup_{\substack{a,b \in Z \\ a \leq_C b}} \min \{\mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}
\end{aligned}$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$.

Note here that by $\emptyset \neq C \neq Z$, all of the infimums and supremums above have definite meanings. These fuzzy relations are "possibility-theoretical" as shown in the following proposition. For convenience, we refer to these fuzzy relations as *PN fuzzy relations*, which come from the words "possibility" and "necessity."

Proposition 3.1 ([10], cf. [14])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. Then the following equalities hold:*

$$\begin{aligned}
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) &= N_{\tilde{A}}((-\infty, \tilde{B}]_C^N) = N_{\tilde{B}}([\tilde{A}, +\infty)_C^N), \\
\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) &= \Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^N), \\
\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) &= \Pi_{\tilde{B}}([\tilde{A}, +\infty)_C^N), \\
\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) &= N_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi), \\
\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) &= N_{\tilde{A}}((-\infty, \tilde{B}]_C^\Pi), \\
\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) &= \Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^\Pi) = \Pi_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi).
\end{aligned}$$

Proof. We first prove a general fact that

$$\max\left\{\alpha, \inf_{z\in S} f(z)\right\} = \inf_{z\in S} \max\{\alpha, f(z)\}$$

holds for every nonempty set S, function $f\colon S \to \mathbb{R}$, and $\alpha \in \mathbb{R}$. Let $\beta := \max\{\alpha, \inf_{z\in S} f(z)\}$ and $\gamma := \inf_{z\in S}\max\{\alpha, f(z)\}$. We have $\beta \le \max\{\alpha, f(z)\}$ for all $z \in S$ and hence $\beta \le \gamma$. If $\beta < \gamma$, then there exists $\bar{z} \in S$ such that $f(\bar{z}) < \gamma$. It follows that $\max\{\alpha, f(\bar{z})\} < \gamma$, but this is impossible. Therefore, $\beta = \gamma$.

Using the above fact, we deduce

$$\begin{aligned}
N_{\tilde{A}}((-\infty, \tilde{B}]_C^N) &= \inf_{z\in Z} \max\left\{1-\mu_{\tilde{A}}(z), \mu_{(-\infty,\tilde{B}]_C^N}(z)\right\} \\
&= \inf_{z\in Z} \max\left\{1-\mu_{\tilde{A}}(z),\ \inf_{\substack{z'\in Z \\ z \not\precsim_C z'}} (1-\mu_{\tilde{B}}(z'))\right\} \\
&= \inf_{z\in Z} \inf_{\substack{z'\in Z \\ z \not\precsim_C z'}} \max\{1-\mu_{\tilde{A}}(z), 1-\mu_{\tilde{B}}(z')\} \\
&= \mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}).
\end{aligned}$$

The other desired equalities are similarly proved. □

Consequently, the values $\mu_{\precsim_C^{(*)}}(\tilde{A}, \tilde{B})$ $(* = 1, 2L, 2U, 3L, 3U, 4)$ can be interpreted as follows:

(i) $\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B})$ is the necessity that $\tilde{A}$ is necessarily less than $\tilde{B}$ (or equivalently, the necessity that $\tilde{B}$ is necessarily greater than $\tilde{A}$).

(ii) $\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B})$ is the possibility that $\tilde{A}$ is necessarily less than $\tilde{B}$.

(iii) $\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B})$ is the possibility that $\tilde{B}$ is necessarily greater than $\tilde{A}$.

(iv) $\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B})$ is the necessity that $\tilde{B}$ is possibly greater than $\tilde{A}$.

(v) $\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B})$ is the necessity that $\tilde{A}$ is possibly less than $\tilde{B}$.

(vi) $\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B})$ is the possibility that $\tilde{A}$ is possibly less than $\tilde{B}$ (or equivalently, the possibility that $\tilde{B}$ is possibly greater than $\tilde{A}$).

The propositions below show three basic properties of the PN fuzzy relations.

Proposition 3.2 ([10], cf. [14])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$ be normal. Then the following inequalities hold:*

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}),$$
$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}).$$

Proof. We check the first line of the inequalities and omit the second line. Since $\tilde{A}$ and $\tilde{B}$ are normal, by Propositions 2.8 and 3.1

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) = N_{\tilde{A}}((-\infty, \tilde{B}]_C^N) \le \Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^N) = \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}),$$
$$\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) = N_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi) \le \Pi_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi) = \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}).$$

Suppose that $\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) > \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B})$ and let $\alpha := \frac{1}{2}\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) + \frac{1}{2}\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B})$. Then for some $\bar{a}, \bar{b} \in Z$, we have

$$\inf_{\substack{b \in Z \\ \bar{a} \not\le_C b}} \min\{\mu_{\tilde{A}}(\bar{a}), 1 - \mu_{\tilde{B}}(b)\} > \alpha > \sup_{\substack{a \in Z \\ a \le_C \bar{b}}} \max\left\{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(\bar{b})\right\}.$$

This implies

$$\mu_{\tilde{A}}(\bar{a}) > \sup_{\substack{a \in Z \\ a \le_C \bar{b}}} \mu_{\tilde{A}}(a), \quad \inf_{\substack{b \in Z \\ \bar{a} \not\le_C b}} (1 - \mu_{\tilde{B}}(b)) > 1 - \mu_{\tilde{B}}(\bar{b}).$$

These inequalities deny both $\bar{a} \le_C \bar{b}$ and $\bar{a} \not\le_C \bar{b}$, which is a contradiction. Therefore, $\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \le \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B})$. $\square$

Proposition 3.3 ([10])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$, $z \in Z$, and $\lambda > 0$. Then the following equalities hold for each $* = 1, 2L, 2U, 3L, 3U, 4$:*

$$\mu_{\precsim_C^{(*)}}(\tilde{A} + z, \tilde{B} + z) = \mu_{\precsim_C^{(*)}}(\tilde{A}, \tilde{B}), \quad \mu_{\precsim_C^{(*)}}(\lambda\tilde{A}, \lambda\tilde{B}) = \mu_{\precsim_C^{(*)}}(\tilde{A}, \tilde{B}).$$

Proof. By the compatibility of $\le_C$ with the linear structure of Z,

$$a' + z \le_C b' + z \iff a' \le_C b', \quad \lambda a' \le_C \lambda b' \iff a' \le_C b'$$

for every $a', b' \in Z$. Thus,

$$\begin{aligned}\mu_{\precsim_C^{(4)}}(\tilde{A} + z, \tilde{B} + z) &= \sup_{\substack{a,b \in Z \\ a \le_C b}} \min\{\mu_{\tilde{A}}(a - z), \mu_{\tilde{B}}(b - z)\} \\ &= \sup_{\substack{a',b' \in Z \\ a'+z \le_C b'+z}} \min\{\mu_{\tilde{A}}(a'), \mu_{\tilde{B}}(b')\} = \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}),\end{aligned}$$

$$\mu_{\precsim_C^{(4)}}(\lambda\tilde{A}, \lambda\tilde{B}) = \sup_{\substack{a,b\in Z \\ a\leq_C b}} \min\left\{\mu_{\tilde{A}}\left(\frac{1}{\lambda}a\right), \mu_{\tilde{B}}\left(\frac{1}{\lambda}b\right)\right\}$$
$$= \sup_{\substack{a',b'\in Z \\ \lambda a'\leq_C \lambda b'}} \min\{\mu_{\tilde{A}}(a'), \mu_{\tilde{B}}(b')\} = \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}).$$

The proofs for the other types are the same. □

Proposition 3.4 (cf. [14])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. Then the following equalities hold:*

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(1)}}(\tilde{B}, \tilde{A}), \qquad \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(2U)}}(\tilde{B}, \tilde{A}),$$
$$\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(2L)}}(\tilde{B}, \tilde{A}), \qquad \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(3U)}}(\tilde{B}, \tilde{A}),$$
$$\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(3L)}}(\tilde{B}, \tilde{A}), \qquad \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) = \mu_{\precsim_{-C}^{(4)}}(\tilde{B}, \tilde{A}).$$

Proof. For every $a, b \in Z$,

$$b \leq_{-C} a \iff a \leq_C b, \quad b \not\leq_{-C} a \iff a \not\leq_C b.$$

Thus,

$$\mu_{\precsim_{-C}^{(2U)}}(\tilde{B}, \tilde{A}) = \sup_{a\in Z} \inf_{\substack{b\in Z \\ b\not\leq_{-C} a}} \min\{1 - \mu_{\tilde{B}}(b), \mu_{\tilde{A}}(a)\}$$
$$= \sup_{a\in Z} \inf_{\substack{b\in Z \\ a\not\leq_C b}} \min\{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\} = \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}),$$
$$\mu_{\precsim_{-C}^{(3U)}}(\tilde{B}, \tilde{A}) = \inf_{b\in Z} \sup_{\substack{a\in Z \\ b\leq_{-C} a}} \max\{1 - \mu_{\tilde{B}}(b), \mu_{\tilde{A}}(a)\}$$
$$= \inf_{b\in Z} \sup_{\substack{a\in Z \\ a\leq_C b}} \max\{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\} = \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}).$$

The others are proved in the same way. □

3.2 Relationship to set relations

In this section, we show that the six types of PN fuzzy relations are completely related to the six types of set relations. That is the reason why we intentionally employ the same superscript numbering for both groups of relations.

Theorem 3.1 ([10])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$. Then the following equalities hold:*

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}\right\},$$

$$\mu_{\precsim_C^{(2L)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_\alpha \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha}\right\},$$
$$\mu_{\precsim_C^{(2U)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_{\alpha}\right\},$$
$$\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}\right\},$$
$$\mu_{\precsim_C^{(3U)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_{\alpha}\right\},$$
$$\mu_{\precsim_C^{(4)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_{\alpha}\right\}$$

where $\sup\emptyset := 0$ *is assumed.*

Proof. All of these equalities are proved in similar manners. We check only the fourth equality and omit the others.

Let

$$\bar{\alpha} := \sup\left\{\alpha \in [0,1] \;\middle|\; [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}\right\}, \quad \beta := \frac{1}{2}\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) + \frac{1}{2}\bar{\alpha}.$$

Suppose first that $\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) < \bar{\alpha}$. Then there exists $\gamma \in [0,1]$ satisfying $\beta < \gamma$ and $[\tilde{A}]_\gamma \preceq_C^{(3L)} [\tilde{B}]_{1-\gamma}$. By Proposition 2.4, it follows from $[\tilde{A}]_\gamma \subset [\tilde{A}]_\beta$ and $[\tilde{B}]_{1-\gamma} \supset [\tilde{B}]_{1-\beta}$ that $[\tilde{A}]_\beta \preceq_C^{(3L)} [\tilde{B}]_{1-\beta}$. In addition, $\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) < \beta$ implies that some $\bar{b} \in Z$ satisfies $\max\left\{\mu_{\tilde{A}}(a), 1-\mu_{\tilde{B}}(\bar{b})\right\} < \beta$ for any $a \in Z$ with $a \leq_C \bar{b}$. We hence deduce $\bar{b} \in [\tilde{B}]_{1-\beta}$ and $a \not\leq_C \bar{b}$ for all $a \in [\tilde{A}]_\beta$, which contradict $[\tilde{A}]_\beta \preceq_C^{(3L)} [\tilde{B}]_{1-\beta}$.

Suppose next that $\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) > \bar{\alpha}$. Then we have $[\tilde{A}]_\beta \not\preceq_C^{(3L)} [\tilde{B}]_{1-\beta}$ and $\max\{\mu_{\tilde{A}}(a_b), 1-\mu_{\tilde{B}}(b)\} > \beta$ for any $b \in Z$ and some $a_b \in Z$ with $a_b \leq_C b$. For any $b \in [\tilde{B}]_{1-\beta}$, it follows from $1-\mu_{\tilde{B}}(b) \leq \beta$ that $a_b \in [\tilde{A}]_\beta$. This is a contradiction to $[\tilde{A}]_\beta \not\preceq_C^{(3L)} [\tilde{B}]_{1-\beta}$. Therefore, $\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) = \bar{\alpha}$. ◻

By using this theorem, we can give another proof of each of Propositions 3.2–3.4 on the basis of the basic properties of the set relations shown in Section 2.2.

Another proof of Proposition 3.2. We check the first line of the inequalities and omit the second line. Fix any $\alpha \in [0,1]$. By Proposition 2.4, we have

$$[\tilde{A}]_{1-\alpha} \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha},$$
$$[\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_\alpha$$

when $0 \leq \alpha \leq 0.5$ and

$$[\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(1)} [\tilde{B}]_{1-\alpha},$$
$$[\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_\alpha$$

when $0.5 \leq \alpha \leq 1$. It follows from these implications and Proposition 2.1 that

$$\begin{aligned} &[\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha} \\ &\implies [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha} \implies [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_\alpha. \end{aligned}$$

Thus, we obtain the desired inequalities by Theorem 3.1. Note that the normality of $\tilde{A}$ and $\tilde{B}$ is necessary for the nonemptiness of all α-cuts of them. □

Another proof of Proposition 3.3. Use Proposition 2.2 and Theorem 3.1. □

Another proof of Proposition 3.4. Use Proposition 2.3 and Theorem 3.1. □

The following theorem requires some assumptions unlike Theorem 3.1 and provides a more practical relationship between the PN fuzzy relations and the set relations. It should be remarked that the assumption of the space being locally convex, which is imposed in [10, Theorem 3.2], is in fact not needed. The same holds true for the two theorems presented later.

Theorem 3.2 ([10])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$ and $\alpha \in (0, 1]$. Assume that C is closed.*

(i) If the cut mapping of $\tilde{A}$ is $(-C)$-lower continuous and the cut mapping of $\tilde{B}$ is C-lower continuous, then

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}.$$

(ii) If $\tilde{A}$ is C-compact, the cut mapping of $\tilde{A}$ is C-upper continuous, and the cut mapping of $\tilde{B}$ is C-lower continuous, then

$$\begin{aligned} \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \geq \alpha &\iff [\tilde{A}]_\alpha \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha}, \\ \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \geq \alpha &\iff [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}. \end{aligned}$$

(iii) If the cut mapping of $\tilde{A}$ is $(-C)$-lower continuous, $\tilde{B}$ is $(-C)$-compact, and the cut mapping of $\tilde{B}$ is $(-C)$-upper continuous, then

$$\begin{aligned} \mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) \geq \alpha &\iff [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_\alpha, \\ \mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) \geq \alpha &\iff [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_\alpha. \end{aligned}$$

(iv) If $\tilde{A}$ is C-compact, the cut mapping of $\tilde{A}$ is C-upper continuous, $\tilde{B}$ is $(-C)$-compact, and the cut mapping of $\tilde{B}$ is $(-C)$-upper continuous, then

$$\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_\alpha.$$

Proof. It is clear from Theorem 3.1 that the right-hand side of each equivalence implies the left-hand side. The converse implications are each proved as follows.

(i) Let $\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \geq \alpha$ and suppose to the contrary that $[\tilde{A}]_{1-\alpha} \not\preceq_C^{(1)} [\tilde{B}]_{1-\alpha}$. Then we have $\bar{b} - \bar{a} \notin C$ for some $\bar{a} \in [\tilde{A}]_{1-\alpha}$ and $\bar{b} \in [\tilde{B}]_{1-\alpha}$. It follows from the closedness of C that there exists an open neighborhood V of 0_Z such that $\big(\bar{b} - \bar{a} + V\big) \cap C = \emptyset$. Since C is a convex cone, it holds that

$$\big(\bar{b} - \bar{a} + V - C\big) \cap C = \emptyset.$$

In fact, if some $z \in V$ and $k \in C$ satisfy $\bar{b} - \bar{a} + z - k \in C$, then $\bar{b} - \bar{a} + z \in C + k \subset C + C \subset C$. This is a contradiction to $\big(\bar{b} - \bar{a} + V\big) \cap C = \emptyset$.

Now, we can take a neighborhood V' of 0_Z with $V' + V' \subset V$. If V' is not open, then let $V' := \operatorname{int} V'$. Since $\bar{a} - V'$ is open, $[\tilde{A}]_{1-\alpha} \cap (\bar{a} - V') \neq \emptyset$, and the cut mapping of $\tilde{A}$ is $(-C)$-lower continuous at $1 - \alpha$, there exists a neighborhood U_1 of $1 - \alpha$ such that $[\tilde{A}]_\beta \cap (\bar{a} - V' + C) \neq \emptyset$ for all $\beta \in U_1$. Similarly, by the C-lower continuity of the cut mapping of $\tilde{B}$, some neighborhood U_2 of $1 - \alpha$ satisfies $[\tilde{B}]_\beta \cap \big(\bar{b} + V' - C\big) \neq \emptyset$ for all $\beta \in U_2$. Choose $\beta \in [0, \alpha)$ with $1 - \beta \in U_1 \cap U_2$. From

$$[\tilde{A}]_{1-\beta} \cap (\bar{a} - V' + C) \neq \emptyset, \quad [\tilde{B}]_{1-\beta} \cap \big(\bar{b} + V' - C\big) \neq \emptyset,$$

for some $\hat{a} \in [\tilde{A}]_{1-\beta}$ and $\hat{b} \in [\tilde{B}]_{1-\beta}$ we obtain

$$\hat{b} - \hat{a} \in \big(\bar{b} + V' - C\big) - (\bar{a} - V' + C) \subset \bar{b} - \bar{a} + V - C.$$

This requires $\hat{b} - \hat{a} \notin C$ and hence $[\tilde{A}]_{1-\beta} \not\preceq_C^{(1)} [\tilde{B}]_{1-\beta}$.

However, $\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \geq \alpha > \beta$ together with Theorem 3.1 and Proposition 2.4 implies the opposite condition $[\tilde{A}]_{1-\beta} \preceq_C^{(1)} [\tilde{B}]_{1-\beta}$. Therefore, $[\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}$.

(ii) We check only the implication of type $2L$ because a similar argumentation is available for that of type $3L$.

Suppose that $\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \geq \alpha$ and $[\tilde{A}]_\alpha \not\preceq_C^{(2L)} [\tilde{B}]_{1-\alpha}$ are simultaneously fulfilled. The latter means $b_a - a \notin C$ for any $a \in [\tilde{A}]_\alpha$ and some $b_a \in [\tilde{B}]_{1-\alpha}$ depending on a. For each $a \in [\tilde{A}]_\alpha$, let V_a be an open neighborhood of 0_Z satisfying

$$(b_a - a + V_a - C) \cap C = \emptyset$$

and take another open neighborhood V'_a of 0_Z with $V'_a + V'_a \subset V_a$. Then the family $\{a - V'_a + C\}_{a \in [\tilde{A}]_\alpha}$ is an open cover of the C-compact set $[\tilde{A}]_\alpha$. Hence

$$[\tilde{A}]_\alpha \subset \bigcup_{i=1,\ldots,m} \big(a_i - V'_{a_i} + C\big)$$

for some $a_1, \ldots, a_m \in [\tilde{A}]_\alpha$. Put $V' := \bigcap_{i=1,\ldots,m} V'_{a_i}$ and take an open neighborhood V'' of 0_Z with $V'' + V'' \subset V'$. Since the cut mapping of $\tilde{A}$ is C-upper continuous at α, there exists a neighborhood U of α such that

$$[\tilde{A}]_\beta \subset [\tilde{A}]_\alpha - V'' + C$$

for all $\beta \in U$. Since the cut mapping of $\tilde{B}$ is C-lower continuous at $1-\alpha$, for each $i = 1, \ldots, m$, there exists a neighborhood U_i of α such that

$$[\tilde{B}]_{1-\beta} \cap (b_{a_i} + V'' - C) \neq \emptyset$$

for all $\beta \in U_i$.

Choose $\beta \in U \cap \left(\bigcap_{i=1,\ldots,m} U_i\right) \cap [0, \alpha)$ and let a' be any element of $[\tilde{A}]_\beta$. Then we can take $a \in [\tilde{A}]_\alpha$ with $a' \in a - V'' + C$, i with $a \in a_i - V'_{a_i} + C$, and $b'_{a'} \in [\tilde{B}]_{1-\beta}$ with $b'_{a'} \in b_{a_i} + V'' - C$. We deduce

$$\begin{aligned} b'_{a'} - a' &\in (b_{a_i} + V'' - C) - (a - V'' + C) \subset b_{a_i} - \left(a_i - V'_{a_i} + C\right) + V' - C \\ &\subset b_{a_i} - a_i + V'_{a_i} + V'_{a_i} - C \subset b_{a_i} - a_i + V_{a_i} - C \end{aligned}$$

and hence $b'_{a'} - a' \notin C$. This implies $[\tilde{A}]_\beta \not\preceq_C^{(2L)} [\tilde{B}]_{1-\beta}$ although the inequality $\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) > \beta$ implies $[\tilde{A}]_\beta \preceq_C^{(2L)} [\tilde{B}]_{1-\beta}$. This is a contradiction.

(iii) By reversing $\tilde{A}$ and $\tilde{B}$ and replacing C with $-C$ in (ii), we have

$$\begin{aligned} \mu_{\precsim_{-C}^{(2L)}}(\tilde{B}, \tilde{A}) \geq \alpha &\iff [\tilde{B}]_\alpha \preceq_{-C}^{(2L)} [\tilde{A}]_{1-\alpha}, \\ \mu_{\precsim_{-C}^{(3L)}}(\tilde{B}, \tilde{A}) \geq \alpha &\iff [\tilde{B}]_\alpha \preceq_{-C}^{(3L)} [\tilde{A}]_{1-\alpha}. \end{aligned}$$

Thus, the conclusion follows from Propositions 2.3 and 3.4.

(iv) Let $\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) \geq \alpha$. Supposing that $[\tilde{A}]_\alpha \not\preceq_C^{(4)} [\tilde{B}]_\alpha$, we obtain

$$(b - a + V_{ab} - C) \cap C = \emptyset$$

for any $a \in [\tilde{A}]_\alpha$, $b \in [\tilde{B}]_\alpha$, and some open neighborhood V_{ab} of 0_Z. Let, for each $a \in [\tilde{A}]_\alpha$ and $b \in [\tilde{B}]_\alpha$, V'_{ab} be an open neighborhood of 0_Z with $V'_{ab} + V'_{ab} \subset V_{ab}$. Fix any $b \in [\tilde{B}]_\alpha$. Since $[\tilde{A}]_\alpha$ is C-compact and $\{a - V'_{ab} + C\}_{a \in [\tilde{A}]_\alpha}$ is its open cover,

$$[\tilde{A}]_\alpha \subset \bigcup_{i=1,\ldots,m_b} \left(a_{ib} - V'_{a_{ib}b} + C\right)$$

for some $a_{1b}, \ldots, a_{m_b b} \in [\tilde{A}]_\alpha$. Put $V'_b := \bigcap_{i=1,\ldots,m_b} V'_{a_{ib}b}$ and take an open neighborhood V''_b of 0_Z with $V''_b + V''_b \subset V'_b$. Since $[\tilde{B}]_\alpha$ is $(-C)$-compact and $\{b + V''_b - C\}_{b \in [\tilde{B}]_\alpha}$ is its open cover,

$$[\tilde{B}]_\alpha \subset \bigcup_{j=1,\ldots,n} \left(b_j + V''_{b_j} - C\right)$$

for some $b_1, \dots, b_n \in [\tilde{B}]_\alpha$. Put $V'' := \bigcap_{j=1,\dots,n} V''_{b_j}$ and take an open neighborhood V''' of 0_Z with $V''' + V''' \subset V''$. The C- and $(-C)$-upper continuities of the cut mappings respectively of $\tilde{A}$ and $\tilde{B}$ at α allow us to take some $\beta \in [0, \alpha)$ satisfying

$$[\tilde{A}]_\beta \subset [\tilde{A}]_\alpha - V''' + C, \quad [\tilde{B}]_\beta \subset [\tilde{B}]_\alpha + V''' - C.$$

Fix any $a' \in [\tilde{A}]_\beta$ and $b' \in [\tilde{B}]_\beta$. It then follows that, for some $a \in [\tilde{A}]_\alpha$, $b \in [\tilde{B}]_\alpha$, i, and j,

$$\begin{aligned} b' - a' &\in (b + V''' - C) - (a - V''' + C) \subset \left(b_j + V''_{b_j} - C\right) - a + V'' - C \\ &\subset b_j - a + V''_{b_j} + V''_{b_j} - C \subset b_j - \left(a_{ib_j} - V'_{a_{ib_j} b_j} + C\right) + V'_{b_j} - C \\ &\subset b_j - a_{ib_j} + V'_{a_{ib_j} b_j} + V'_{a_{ib_j} b_j} - C \subset b_j - a_{ib_j} + V_{a_{ib_j} b_j} - C. \end{aligned}$$

Therefore, we have $b' - a' \notin C$ and conclude $[\tilde{A}]_\beta \not\preceq_C^{(4)} [\tilde{B}]_\beta$, which contradicts $\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) > \beta$. $\square$

Furthermore, the above theorem can be rewritten into the following one with a little excessive but simple assumptions.

Theorem 3.3 ([10])**.** *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$ and $\alpha \in (0, 1]$. Assume that C is closed.*

(i) If $\tilde{A}$ and $\tilde{B}$ are normal and strictly convex, then

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \ge \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}.$$

(ii) If $\tilde{A}$ is compact and $\tilde{B}$ is normal and strictly convex, then

$$\begin{aligned} \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \ge \alpha &\iff [\tilde{A}]_\alpha \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha}, \\ \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \ge \alpha &\iff [\tilde{A}]_\alpha \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}. \end{aligned}$$

(iii) If $\tilde{A}$ is normal and strictly convex and $\tilde{B}$ is compact, then

$$\begin{aligned} \mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) \ge \alpha &\iff [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_\alpha, \\ \mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) \ge \alpha &\iff [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_\alpha. \end{aligned}$$

(iv) If $\tilde{A}$ and $\tilde{B}$ are compact, then

$$\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) \ge \alpha \iff [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_\alpha.$$

Proof. From Propositions 2.5–2.7, the following statements hold:

(i) Every compact fuzzy set in Z is C- and $(-C)$-compact.

(ii) The cut mapping of every compact fuzzy set in Z is C- and $(-C)$-upper continuous.

(iii) The cut mapping of every normal and strictly convex fuzzy set in Z is C- and $(-C)$-lower continuous.

By using these statements, we rewrite the assumptions of each statement in Theorem 3.2 and thus conclude that this theorem is true. □

We remark that (iv) of this theorem is a natural consequence of [16, Propositions 2.7 and 2.13].

3.3 Application to fuzzy optimization

In this last section, we provide a possible application of Theorem 3.3 to the area of fuzzy optimization.

Let X be a nonempty set. We consider a *fuzzy optimization problem* in a general form

$$\text{(FOP)} \quad \begin{cases} \text{minimize} & \tilde{F}(x) \\ \text{subject to} & x \in X \end{cases}$$

for $\tilde{F}\colon X \to \mathcal{F}(Z)$. The objective function in this problem is a *fuzzy set-valued mapping*. We have many potential options for the solution concepts for (FOP) because they depend on what we use to compare the values of $\tilde{F}$ (e.g., a crisp relation, a fuzzy relation, and a ranking function) and how we define "minimization." If we use a fuzzy relation on $\mathcal{F}(Z)$ and a parameter α, the following three solution concepts are naturally considered.

Definition 3.3 ([10] for (i)–(ii), [35] for (iii))**.** Let $\precsim$ be a fuzzy relation on $\mathcal{F}(Z)$ and $\alpha \in (0,1]$. An element $\bar{x} \in X$ is called

(i) an *α-optimal solution* of (FOP) with respect to $\precsim$ if

$$\forall x \in X\colon \left(\mu_{\precsim}(\tilde{F}(x), \tilde{F}(\bar{x})) \geq \alpha \implies \mu_{\precsim}(\tilde{F}(\bar{x}), \tilde{F}(x)) \geq \alpha\right).$$

(ii) an *α-strongly optimal solution* of (FOP) with respect to $\precsim$ if

$$\forall x \in X \setminus \{\bar{x}\} : \mu_{\precsim}(\tilde{F}(\bar{x}), \tilde{F}(x)) \geq \alpha.$$

(iii) an *α-strictly optimal solution* of (FOP) with respect to $\precsim$ if

$$\nexists x \in X \setminus \{\bar{x}\} : \mu_{\precsim}(\tilde{F}(x), \tilde{F}(\bar{x})) \geq \alpha.$$

By the definition, any α-strongly optimal or α-strictly optimal solution is also an α-optimal solution. We here focus on the case in which the fuzzy relation $\precsim$ is each of the PN fuzzy relations. Moreover, we introduce two kinds of set optimization problems

$$(\mathrm{SOP})'_\beta \quad \begin{cases} \text{minimize} & [\tilde{F}(x)]_\beta \\ \text{subject to} & x \in X \end{cases},$$

$$(\mathrm{SOP})''_\beta \quad \begin{cases} \text{minimize} & [\tilde{F}(x)]_\beta \times [\tilde{F}(x)]_{1-\beta} \\ \text{subject to} & x \in X \end{cases}$$

for each $\beta \in [0,1]$ and an extension of the set relations $\preceq_C^{(*)}$ $(* = 2L, 2U, 3L, 3U)$ defined by

$$A_1 \times A_2 \mathrel{\hat{\preceq}_C^{(*)}} B_1 \times B_2 \;:\Longleftrightarrow\; A_1 \preceq_C^{(*)} B_2$$

for $A_1, A_2, B_1, B_2 \in \mathcal{P}(Z)$.

The following theorem shows that solving the fuzzy optimization problem (FOP) with respect to each PN fuzzy relation is equivalent to solving a certain set optimization problem with respect to the corresponding set relation or its extension.

Theorem 3.4 ([10])**.** *Let $\alpha \in (0,1]$ and assume that C is closed.*

(i) Assume that $\tilde{F}(x)$ is normal and strictly convex for every $x \in X$. Then $\bar{x} \in X$ is an α-optimal (resp., α-strongly optimal, α-strictly optimal) solution of (FOP) *with respect to $\precsim_C^{(1)}$ if and only if it is an optimal (resp., strongly optimal, strictly optimal) solution of* $(\mathrm{SOP})'_{1-\alpha}$ *with respect to $\preceq_C^{(1)}$.*

(ii) Assume that $\tilde{F}(x)$ is compact, normal, and strictly convex for every $x \in X$. Then, for each $* = 2L, 3L$, $\bar{x} \in X$ is an α-optimal (resp., α-strongly optimal, α-strictly optimal) solution of (FOP) *with respect to $\precsim_C^{(*)}$ if and only if it is an optimal (resp., strongly optimal, strictly optimal) solution of* $(\mathrm{SOP})''_\alpha$ *with respect to $\hat{\preceq}_C^{(*)}$.*

(iii) Assume that $\tilde{F}(x)$ *is compact, normal, and strictly convex for every* $x \in X$*. Then, for each* $* = 2U, 3U$*,* $\bar{x} \in X$ *is an* α*-optimal (resp.,* α*-strongly optimal,* α*-strictly optimal) solution of* (FOP) *with respect to* $\precsim_C^{(*)}$ *if and only if it is an optimal (resp., strongly optimal, strictly optimal) solution of* $(\mathrm{SOP})''_{1-\alpha}$ *with respect to* $\hat{\preceq}_C^{(*)}$*.*

(iv) Assume that $\tilde{F}(x)$ *is compact for every* $x \in X$*. Then* $\bar{x} \in X$ *is an* α*-optimal (resp.,* α*-strongly optimal,* α*-strictly optimal) solution of* (FOP) *with respect to* $\precsim_C^{(4)}$ *if and only if it is an optimal (resp., strongly optimal, strictly optimal) solution of* $(\mathrm{SOP})'_{\alpha}$ *with respect to* $\preceq_C^{(4)}$*.*

Proof. We prove (i) and (ii) and omit the others.

(i) From Theorem 3.3, we deduce for every $x, x' \in X$ that

$$\mu_{\precsim_C^{(1)}}(\tilde{F}(x), \tilde{F}(x')) \geq \alpha \iff [\tilde{F}(x)]_{1-\alpha} \preceq_C^{(1)} [\tilde{F}(x')]_{1-\alpha}.$$

The sets $[\tilde{F}(x)]_{1-\alpha}$, $x \in X$ are the values of the objective function of $(\mathrm{SOP})'_{1-\alpha}$. Therefore, by comparing Definition 3.3 with Definition 2.2, we find that this statement holds.

(ii) For each $* = 2L, 3L$, it holds by Theorem 3.3 that

$$\mu_{\precsim_C^{(*)}}(\tilde{F}(x), \tilde{F}(x')) \geq \alpha \iff [\tilde{F}(x)]_{\alpha} \preceq_C^{(*)} [\tilde{F}(x')]_{1-\alpha}$$

for every $x, x' \in X$. The right-hand side of this is equivalent to

$$[\tilde{F}(x)]_{\alpha} \times [\tilde{F}(x)]_{1-\alpha} \hat{\preceq}_C^{(*)} [\tilde{F}(x')]_{\alpha} \times [\tilde{F}(x')]_{1-\alpha}.$$

Since the sets $[\tilde{F}(x)]_{\alpha} \times [\tilde{F}(x)]_{1-\alpha}$, $x \in X$ are the values of the objective function of $(\mathrm{SOP})''_{\alpha}$, the proof is completed. □

Chapter 4

Conclusion

In this thesis, we have presented a further study on the six types of possibility-theoretical fuzzy relations between fuzzy sets (referred to as PN fuzzy relations) in a preordered vector space. We described their relationship to the six types of set relations in Theorems 3.1–3.3 with different assumptions. This implies that many existing studies involving the set relations, such as scalarization techniques for sets, have a potential to be utilized for the PN fuzzy relations. Moreover, we applied Theorem 3.3 to fuzzy optimization, showing in Theorem 3.4 that solving a general fuzzy optimization problem with respect to each PN fuzzy relation is equivalent to solving a certain set optimization problem with respect to the corresponding set relation or its extension. Theorems 3.1–3.3 can be applied to other fuzzy decision-making areas, such as fuzzy game theory, when the PN fuzzy relations are employed.

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