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On Eccentric Sets of Edges in Graphs

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SUMMARY We introduce the distance between two edges in a graph (nondirected graph) as the minimum number of edges in a tieset with the two edges. Using the distance between edges we define the eccentricity $\varepsilon_\tau(e_j)$ of an edge e_j. A finite non-empty set J of positive integers (no repetitions) is an eccentric set if there exists a graph G with edge set E such that $\varepsilon_\tau(e_j) \in J$ for all $e_j \in E$ and each positive integer in J is $\varepsilon_\tau(e_j)$ for some $e_j \in E$. In this paper, we give necessary and sufficient conditions for a set J to be eccentric.

1. Introduction

By a graph we mean a finite nondirected and connected graph which may include multiple edges. The eccentricity of a vertex which is defined using the distance between two vertices is well known as a basic concept of graphs[(1)], and some properties of eccentric sets in graphs has been examined[(2)]. Edges are also important elements as well as vertices in graphs. We propose the concept of the eccentricity $\varepsilon_\tau(e_j)$ of an edge e_j in a graph which is defined using the distance between two edges in a graph, where the distance between two edges is the minimum number of edges in a tieset with the two edges. We call a finite non-empty set J of positive integers (no repetitions) an eccentric set if there exists a graph G with edge set E such that $\varepsilon_\tau(e_j) \in J$ for all $e_j \in E$ and each positive integer in J is $\varepsilon_\tau(e_j)$ for some $e_j \in E$. We give necessary and sufficient conditions for a set J to be eccentric.

2. Preliminaries

In this section, we introduce a quantity corresponding to the distance between two edges. Let $G = (V, E)$ be a nondirected and connected graph, where V and E are the sets of vertices and edges in G respectively. A path in G is a subgraph and the set $P_i \subseteq E$ of edges in the subgraph is called a pathset. An elementary circuit (or tie) is a subgraph and the set of edges in the subgraph is called an elementary tieset, which is called simply a tieset. The number of edges in a path or the pathset is called its length and the number of edges in a tie or the tieset is called its length. That is, the length of a pathset p_i is $|p_i|$ and the length of a tieset τ_e is $|\tau_e|$, where $|A|$ denotes the cardinality of the set A. The subscript e of τ_e means elementary tieset. For simplicity, we use τ for τ_e, hereafter.

In general, the distance between two vertices of G is often defined as the length of the shortest path between the two vertices. In this letter, we define the distance between two edges of G using a tieset with the two edges. Let $\tau_k(e_i, e_j)$ be a tieset containing both e_i and e_j of G and let $R(e_i, e_j)$ be the set of these tiesets. That is,

$$R(e_i, e_j) = \{\tau_k(e_i, e_j)\} \qquad (1)$$

The length of $\tau_k(e_i, e_j)$ is $|\tau_k(e_i, e_j)|$. We define the distance $d(e_i, e_j)$ between two edges e_i and e_j as follows.

(a) $$d(e_i, e_j) = \min_{\tau_k(e_i, e_j) \in R(e_i, e_j)} |\tau_k(e_i, e_j)|, \qquad (2)$$

if $R(e_i, e_j) \neq \phi$ for $e_i, e_j \in E$

(b) $$d(e_i, e_j) = \infty, \text{ if } R(e_i, e_j) = \phi \text{ for } e_i, e_j \in E \qquad (3)$$

For an edge $e_i \in E$, let

(c) $$d(e_i, e_i) = 0 \qquad (4)$$

In the definition (a)-(c) of the distance between edges, (c) is optional. We may define the distance $d(e_i, e_i)$ by (a) and (b) instead of by (c), where $d(e_i, e_i) = \min_{\tau_k(e_i, e_i) \in R(e_i, e_i)} |\tau_k(e_i, e_i)|$ for $R(e_i, e_i) \neq \phi$ and $d(e_i, e_i) = \infty$ for $R(e_i, e_i) = \phi$. In this case, $d(e_i, e_i)$ has the minimum value among the distances $d(e_i, e_j)$

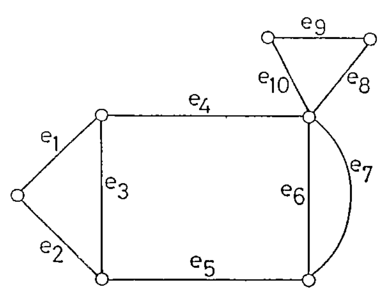

Fig. 1 A graph.

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for $e_j \in E$. In this paper, we use the above definition (a)-(c). We show an example. In a graph of Fig. 1,

$$\tau_1(e_1, e_2)=\{e_1, e_2, e_3\},\ \tau_2(e_1, e_2)=\{e_1, e_2, e_4, e_5, e_6\},$$

$$\tau_3(e_1, e_2)=\{e_1, e_2, e_4, e_5, e_7\}.$$

$$R(e_1, e_2)=\{\tau_1(e_1, e_2), \tau_2(e_1, e_2), \tau_3(e_1, e_2)\}$$

$$R(e_1, e_8)=\phi.$$

From the definition (a)-(c), $d(e_1, e_2)=3$ and $d(e_1, e_8)=\infty$.

[Lemma 1] If there are elementary circuits L_1 containing both e_i and e_j and L_2 containing both e_j and e_k in G, then there exists an elementary circuit containing both e_i and e_k in a subgraph $L_1 \cup L_2$. □

Using Lemma 1, we obtain Theorem 1.

[Theorem 1] $d(e_i, e_j)$ is a metric.

(Proof) In order to prove that $d(e_i, e_j)$ satisfies the distance axiom, it suffices to show that the following equations hold.

(I) $d(e_i, e_j)=0 \Longleftrightarrow e_i=e_j$

(II) $d(e_i, e_j)=d(e_j, e_i)$

(III) $d(e_i, e_k) \leqq d(e_i, e_j)+d(e_j, e_k)$.

From Eq. (4), $d(e_i, e_i)=0$ and from the definition of $d(e_i, e_j)$, $d(e_i, e_j) \neq 0$ for $e_i \neq e_j$, that is, (I) holds.

In G, a tieset containing e_i and e_j is also a tieset containing e_j and e_i and thus we have $R(e_i, e_j)=R(e_j, e_i)$. Hence, from Eqs. (2) and (3), (II) holds.

Next we prove (III). From Eq. (2), $d(e_i, e_j)$ is the minimum among lengths of tiesets containing both e_i and e_j. Let $\tau_1(e_i, e_j)$ be one of tiesets of the minimum length in $R(e_i, e_j)$. Similarly, let $\tau_2(e_j, e_k)$ be one of tiesets of the minimum length in $R(e_j, e_k)$. If both of $\tau_1(e_i, e_j)$ and $\tau_2(e_j, e_k)$ exist, then from Lemma 1 there is an elementary circuit containing both e_i and e_k in the edge-induced subgraph of $\tau_1(e_i, e_j) \cup \tau_2(e_j, e_k)$. Let $\tau_3(e_i, e_k)$ be the set of edges in this circuit. Then,

$$\tau_3(e_i, e_k) \subseteqq \tau_1(e_i, e_j) \cup \tau_2(e_j, e_k) \tag{5}$$

$$\therefore\ |\tau_3(e_i, e_k)| \leqq |\tau_1(e_i, e_j)|+|\tau_2(e_j, e_k)| \tag{6}$$

From Eq. (2) we have

$$d(e_i, e_k) \leqq |\tau_3(e_i, e_k)| \tag{7}$$

From the assumption

$$d(e_i, e_j)=|\tau_1(e_i, e_j)|$$

$$d(e_j, e_k)=|\tau_2(e_j, e_k)|$$

and therefore from Eqs. (6) and (7)

$$d(e_i, e_k) \leqq d(e_i, e_j)+d(e_j, e_k) \tag{8}$$

Next we consider the case where one of $\tau_1(e_i, e_j)$ and $\tau_2(e_j, e_k)$ does not exist. Without loss of generality we assume that $\tau_2(e_j, e_k)$ does not exist. Then G is a separable graph and $e_j(e_i)$ and e_k belong to different nonseparable components. Therefore from the Eq. (3) we have

$$d(e_j, e_k)=d(e_i, e_k)=\infty, \tag{9}$$

$$d(e_i, e_k) \leqq d(e_i, e_j)+d(e_j, e_k) \tag{10}$$

Further in the case where neither $\tau_1(e_i, e_j)$ nor $\tau_2(e_j, e_k)$ exist, we have

$$d(e_i, e_j)=d(e_j, e_k)=\infty. \tag{11}$$

It follows from $d(e_i, e_k) \leqq \infty$ that

$$d(e_i, e_k) \leqq d(e_i, e_j)+d(e_j, e_k) \tag{12}$$

(III) follows from the Eqs. (8), (10), and (12). □

3. Eccentric Sets with Respect to Edges

The concept of eccentricity with respect to vertices is defined using shortest path between vertices[(1)]. Here, we define the concept of eccentricity with respect to edges using the distance between edges. Let

$$\varepsilon_\tau(e_i)=\max_{e_j \in E} d(e_i, e_j) \tag{13}$$

we call $\varepsilon_\tau(e_i)$ the eccentricity of e_i. The radius $\mathrm{rad}_\tau(G)$ concerning edges of G is defined as $\min_{e_i \in E} \varepsilon_\tau(e_i)$ while the diameter $\mathrm{diam}_\tau(G)$ is $\max_{e_i \in E} \varepsilon_\tau(e_i)$. An edge e is a central edge if $\varepsilon_\tau(e)=\mathrm{rad}_\tau(G)$.

Let us show a simple example to explain the eccentricity of an edge and a central edge. Let us consider a road network where the length of each road is 1 and each road has a police station at the middle point of the road. A patrol car of a police station have to patrol his edge (road) and other edges besides. In order to avoid the repetition point of the patrol, we assume that a patrol car of a police station have to patrol some other edges from his station to his station passing once and only once through some vertices and some edges except for his edge. If a patrol car is ordered the patrol that have to make a round via a specific police station, the patrol car have to include the edge of the specific police station in the patrol. $d(e_i, e_j)$ represents the shortest patrol tour length from the police station of e_i going through that of e_j which is the specific station in the patrol. Note that the shortest patrol tour from the police station of e_i going through that of e_j may not include the shortest path between the two police stations. The above assumption of the patrol tour is only used for the patrol. Of course, in case of emergency at e_j, the patrol car of e_i can use the shortest path between the two police stations. The maximum value of the shortest patrol tour length of e_i in any specific police stations to make a

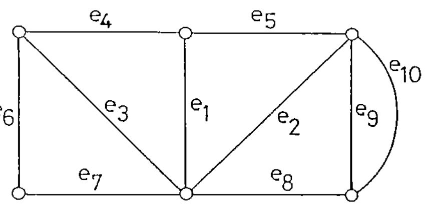

Fig. 2 A graph G.

round via is the eccentricity of e_i. The central edge e_k of G means that a patrol car of e_k can have minimum patrol tour length in any specific stations to make a round via.

A finite non-empty set J positive integer (no repetition) is called eccentric set if there exists a graph G with edge sets E such that $\varepsilon_\tau(e_i) \in J$ for all edges $e_i \in E$ and each positive integer in J is $\varepsilon_\tau(e_i)$ for some $e_i \in E$. We show an example. In a graph G in Fig. 2, $\varepsilon_\tau(e_1)=4$, $\varepsilon_\tau(e_2)=\varepsilon_\tau(e_3)=\varepsilon_\tau(e_4)=\varepsilon_\tau(e_5)=5$ and $\varepsilon_\tau(e_6)=\varepsilon_\tau(e_7)=\varepsilon_\tau(e_8)=\varepsilon_\tau(e_9)=\varepsilon_\tau(e_{10})=6$. Thus, $\mathrm{rad}_\tau(G)=4$, and $\mathrm{diam}_\tau(G)=6$. $J=\{4, 5, 6\}$ is an eccentric set.

The radius and diameter are related by the following inequalities.

[Theorem 2] For any graph G,

$$\mathrm{rad}_\tau(G) \leqq \mathrm{diam}_\tau(G) \leqq 2\mathrm{rad}_\tau(G)-2 \qquad (14)$$

(Proof) The inequality $\mathrm{rad}_\tau(G) \leqq \mathrm{diam}_\tau(G)$ follows from the definition of radius and diameter.

Next we show $\mathrm{diam}_\tau(G) \leqq 2\mathrm{rad}_\tau(G)-2$.

(1) The case where G is a separable graph.
When edges e_i and e_j belong to different non-separable components, there is no elementary circuit containing both e_i and e_j. Thus, $R(e_i, e_j)=\phi$. By the definition (3) of distance between edges,

$$d(e_i, e_j)=\infty, \qquad (15)$$

$$\therefore \quad \mathrm{rad}_\tau(G)=\mathrm{diam}_\tau(G)=\infty \qquad (16)$$

From Eq. (16) the inequality (14) follows.

(2) The case where G is a non-separable graph.

(i) The case $E=\{e_i, e_j\}(|E|=2)$.
Clearly, $\mathrm{rad}_\tau(G)=\mathrm{diam}_\tau(G)=2$. Then, we have Eq. (14).

(ii) The case where the cardinality of E is greater than 3 ($|E| \geqq 3$). Consider edges e_i, e_j and e_k such that $d(e_i, e_j)=\mathrm{diam}_\tau(G)$ and $\varepsilon_\tau(e_k)=\mathrm{rad}_\tau(G)$ hold. If $e_i=e_k$ or $e_j=e_k$, $\mathrm{diam}_\tau(G)=\mathrm{rad}_\tau(G) \geqq 2$ holds. Thus, the inequality (14) holds. Next we consider the case $(e_i \neq e_j)$ $e_i \neq e_k$ and $e_j \neq e_k$. Let L_1 be a circuit of the minimum length among elementary circuits containing both e_i and e_k, and let $\tau_1(e_i, e_k)$ be the set of edges in it. Also, let L_2 be a circuit of the minimum length among elementary circuits containing both e_j and e_k and let $\tau_2(e_j, e_k)$ be a set of edges in it. From Lemma 1, there is an elementary circuit containing both e_i and e_j on the subgraph of G consisting of L_1 and L_2. Let $\tau_3(e_i, e_j)$ be a set of edges in the elementary circuit. Then,

$$\tau_3(e_i, e_j) \subseteqq \tau_1(e_i, e_k) \cup \tau_2(e_j, e_k) \qquad (17)$$

We define a subset E_x of edges as

$$E_x=\tau_1(e_i, e_k)-(\tau_1(e_i, e_k) \cap \tau_3(e_i, e_j)) \qquad (18)$$

(a) The case $E_x=\phi$.
Since $\tau_3(e_i, e_j)$ is an elementary tieset, its subset does not contain any other elementary tieset. Thus we have $\tau_1(e_i, e_k)=\tau_3(e_i, e_j)$. Hence, e_j is contained in $\tau_1(e_i, e_k)$. Therefore from Eq. (17) we have

$$\begin{aligned} &|\tau_3(e_i, e_j)| \\ &\quad \leqq |\tau_1(e_i, e_k) \cup \tau_2(e_j, e_k)| \\ &\quad \leqq |\tau_1(e_i, e_k)|+|\tau_2(e_j, e_k)|-|\{e_k\}|-|\{e_j\}| \qquad (19) \end{aligned}$$

By the definition of distance between edges, $d(e_i, e_j) \leqq |\tau_3(e_i, e_j)|$. From the assumption,

$$\begin{aligned} &d(e_i, e_j)=\mathrm{diam}_\tau(G), \\ &|\tau_1(e_i, e_k)| \leqq \mathrm{rad}_\tau(G), \text{ and} \\ &|\tau_2(e_j, e_k)| \leqq \mathrm{rad}_\tau(G). \qquad (20) \end{aligned}$$

Thus, from the inequarity (19)

$$\mathrm{diam}_\tau(G) \leqq 2\,\mathrm{rad}_\tau(G)-2 \qquad (21)$$

(b) The case $E_x \neq \phi$.

Let e_a be an edge which belongs to E_x. From Eq. (17),

$$\tau_3(e_i, e_j) \subseteqq \tau_1(e_i, e_k) \cup \tau_2(e_j, e_k)-\{e_a\}. \qquad (22)$$

Thus we have

$$\begin{aligned} |\tau_3(e_i, e_j)| &\leqq |\tau_1(e_i, e_k) \cup \tau_2(e_j, e_k)|-|\{e_a\}| \\ &\leqq |\tau_1(e_i, e_k)|+|\tau_2(e_j, e_k)|-|\{e_k\}| \\ &\quad -|\{e_a\}| \qquad (23) \end{aligned}$$

Therefore, from Eq. (20)

$$\mathrm{diam}_\tau(G) \leqq 2\,\mathrm{rad}_\tau(G)-2. \qquad (24)$$

From Eqs. (21) and (24), the inequality (14) follows. □

Next, we characterize eccentric sets.

[Theorem 3] A non-empty set $J=\{b_1, b_2, \cdots, b_n\}$ of positive integers, listed in increasing order, is an eccentric set if and only if

$$b_n \leqq 2(b_1-1) \qquad (25)$$

(Proof) The necessity of the inequality (25) follows from Theorem 2. Next, we consider the sufficiency. We show a method to construct a graph G with a set J which satisfies (25). Let $P(m)$ be a path of order m where the order is the number of vertices. We first introduce a notation of combining two graphs to produce a new graph.

Let $G(H) \vee P(m)$ be a graph in which a path $P(m)$ connects adjacent vertices in the subgraph H of G (see Fig. 3). Note that $G(H) \vee P(m)$ is not always unique because the subgraph H may have more adjacent vertices than two. Let

$$m_1 = b_2 - b_1 + 2 \tag{26}$$

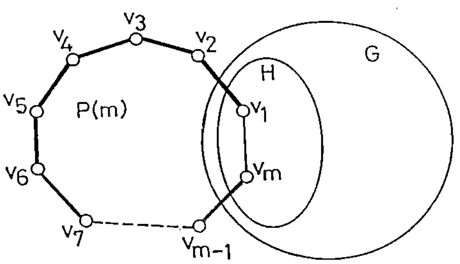

Fig. 3 The Combining Graph $G(H) \vee P(m)$ of G and $P(m)$.

$$m_2 = b_3 - b_2 + 2 \tag{27}$$

$$\vdots$$

$$m_{n-1} = b_n - b_{n-1} + 2 \tag{28}$$

At first, construct a graph $G_2 = G_1(C(b_1)) \vee p(m_1)$ where $G_1 = C(b_1)$ and $C(b_1)$ is a circuit of order b_1. Let the subgraph consisting of $P(m_1)$ in G_2 be P_1 and let the edge in G_1 whose end vertices are connected by $P(m_1)$ be e_1. Next, construct a graph $G_3 = G_2(P_1) \vee P(m_2)$ where the subgraph consisting of $P(m_2)$ is P_2 and the edge in G_2 whose end vertices are connected by $P(m_2)$ is e_2. Proceed the construction of $G_i (i = 1, 2, \cdots)$ to $G_n = G_{n-1}(P_{m-2}) \vee P(m_{n-1})$ where the subgraph consisting of $P(m_{n-1})$ in G_n is P_{n-1} and the edge in G_{n-1} whose end vertices are connected by $P(m_{n-1})$ is e_{n-1}. Let an edge of the subgraph P_{n-1} of G_n be e_n.

From the method of the construction of G_n, G_n has the eccentric set J, where $\varepsilon_T(e_i) = b_i$, $(1 \leqq i \leqq n)$, and $\varepsilon_T(e_i) \in J$ for any edge e_i of G_n, since J holds the

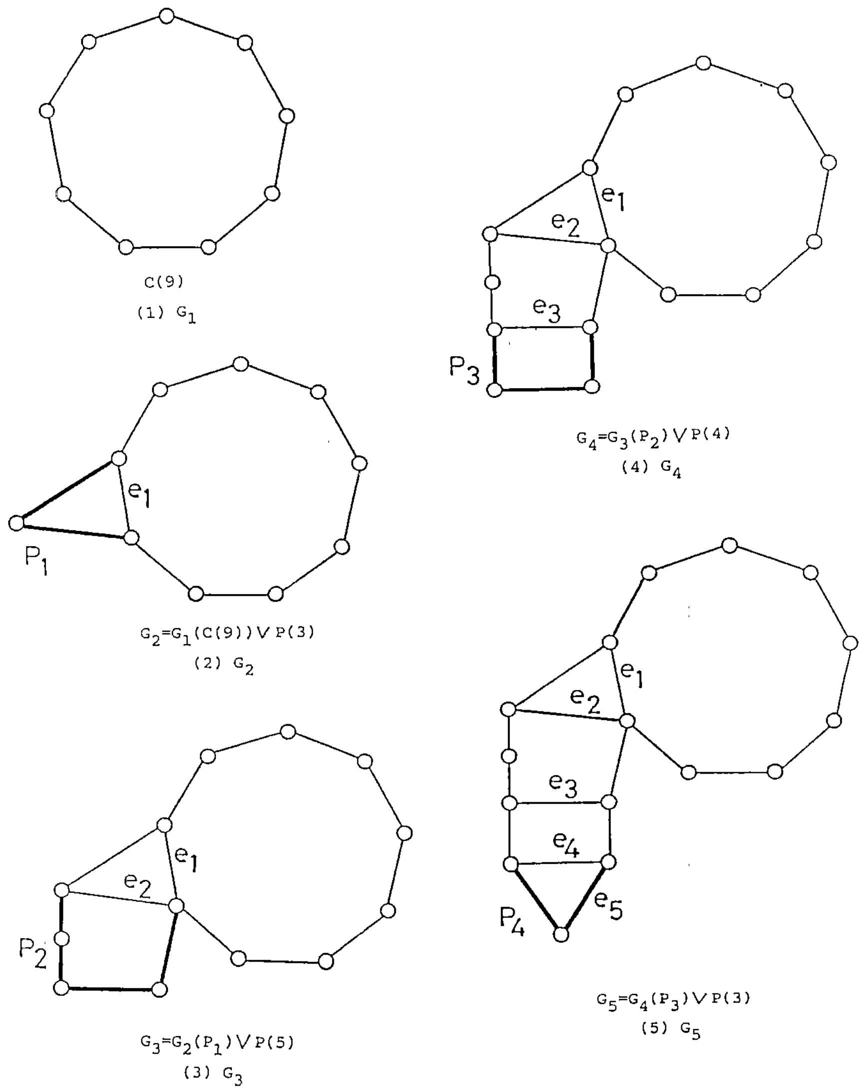

Fig. 4 Graphs G_1, G_2, G_3, G_4 and G_5.

inequality (25). □

We show examples.

[Examples]

(1) $J=\{4, 6, 7\}$

This set J is not an eccentric set because $b_1=4$ and $b_n=7$, $(n=3)$ do not satisfy Eq. (25).

(2) $J=\{9, 10, 13, 15, 16\}$

This set J is an eccentric set because $b_1=9$ and $b_n=16$, $(n=5)$ satisfy Eq. (25). Let us show the construction of a graph G with J. From Eqs. (26), (27) and (28),

$$m_1=3,\ m_2=5,\ m_3=4,\ m_4=3$$

Figure 4 shows the procedure of the construction of G from G_1 to G_5. $G=G_5$ is a graph with J where $\varepsilon_\tau(e_1)=9$, $\varepsilon_\tau(e_2)=10$, $\varepsilon_\tau(e_3)=13$, $\varepsilon_\tau(e_4)=15$ and $\varepsilon_\tau(e_5)=16$.

4. Conclusion

We have introduced the concept of the distance between edges using tiesets of a graph and we have defined the eccentricity of an edge and examined the properties of an eccentric set. Furthermore, we gave necessary and sufficient conditions for a set of positive integer to be eccentric. We note that the similar discussion of the eccentricity of an edge by the distance between edges using cutsets instead of tiesets in a graph is possible.

A sequence I : $a_1, a_2, \cdots, a_m$, is said to be an eccentric sequence if there exists a graph G whose edges can be labelled $e_1, e_2, \cdots, e_m$ so that $\varepsilon_\tau(e_i)=a_i$ for each i, $1\leqq i\leqq m$. The necessary and sufficient condition for a sequence to be an eccentric sequence have not been obtained and this is an open problem.

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