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A Note on a Generalized Transmission Number and Its Sabidussi Difference

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SUMMARY Consider a connected network N. With each edge of N we associate a nonnegative real number called the length of the edge. Let N' be a network obtained from N by adding new edges and/or by decreasing the lengths of some edges. For a generalized transmission number $c_f(N, v)$ defined on vertex v in N, we call $\Delta_c(v)=c_f(N, v)-c_f(N', v)$ the Sabidussi difference of vertex v. In this note, we present a new theorem and its corollaries on the Sabidussi difference.

1. Introduction

Consider a connected network N with vertex set $V(N)$. With each edge of N we associate a nonnegative number called the length of the edge. We define the length of a path in N to be the sum of the lengths of all the edges of the path. We say that a path between two vertices in N is a shortest path between the two vertices in N if its length is equal to or less than the length of any other path between the two vertices in N. We define the distance between two vertices in N to be the length of a shortest path between the two vertices in N. As a criterion index for mathematically evaluating which vertex is one of the most central vertices in N, a real number called the transmission number has been used, where the transmission number of a vertex in N is the sum of all the distances from the vertex to all the others in N. A generalized transmission number[1] of vertex v in N is defined to be

$$c_f(N, v)=\sum_{u\in V(N)} f(d_N(v, u), \sigma(u)) \tag{1}$$

where $d_N(v, u)$ is the distance from vertex v to vertex u, $\sigma(u)$ is the weight of vertex u, and $f(x, y)$ is a real-valued function of two nonnegative real variables x and y with the property that it is increasing and convex downwards with respect to x and is increasing with respect to y. If $f(x, y)=xy$, $c_f(N, v)$ becomes the so-called transmission number of vertex v. For a network N' obtained from N by adding new edges and/or by decreasing the lengths of some edges, we say that a real number defined by

$$\Delta_c(v)=c_f(N, v)-c_f(N', v) \tag{2}$$

is the Sabidussi difference of vertex v, because it was introduced by Sabidussi[2] in 1966.

As an example, let us consider as N a system of roads between towns, in which each vertex corresponds to a town and each edge corresponds to a road. Then, $\sigma(u)$ in interpreted as the amount of commodities demanded in town u, $d_N(v, u)$ is as the cost required to transport a unit of commodities from town v to town u, $c_f(N, v)$ is as the total cost of transporting commodities from town v to all the other towns. $c_f(N, v)$ can be utilized in order to find the best location of a center distributing commodities from one town to all the other towns through roads. The addition of a new edge corresponds to the construction of a new road, and the decrease of the length of an edge corresponds to the improvement of a road. The Sabidussi difference $\Delta_c(v)$ is considered as the total benefit caused by constructing new roads and/or improving some roads when a center is located in town v.

In this note, a new theorem and its corollaries on the Sabidussi difference $\Delta_c(v)$ are given.

2. Theorem and Its Corollaries

Throughout this note, we assume that $f(x, y)$ is a bounded real-valued function of two nonnegative variables x and y, where $f(x, y)$ is said to be bounded if there exists a real number $M(s, t)$ such that $|f(x, y)| < M(s, t)$ for all (x, y) with $0 \leqq x \leqq s$ and $0 \leqq y \leqq t$ for any real numbers s and t. We say that $f(x, y)$ is increasing with respect to x if $f(x_1, y) \leqq f(x_2, y)$ for all x_1, x_2 and y with $x_1 \leqq x_2$. We say that $f(x, y)$ is convex downwards with respect to x if, for all x_1, x_2, x_3 and y with $x_1 < x_2 < x_3$, there holds $[f(x_2, y)-f(x_1, y)]/(x_2-x_1) \geqq [f(x_3, y)-f(x_1, y)]/(x_3-x_1)$. An increasing function $f(x, y)$ with respect to y can be defined analogously.

We denote an edge between vertices v and u of N by vu, and the length of edge vu by $\lambda(vu)$, and furthermore the weight of vertex u by $\sigma(u)$. Then, sometimes, we denote N by $N=N(\lambda, \sigma)$. For vertex v and a subset S of $V(N)$, we define the distance $d_N(v, S)$ between v and S by Min $\{d_N(v, x) | x \in S\}$. Since there is no loss of generality, we assume that N contains no parallel

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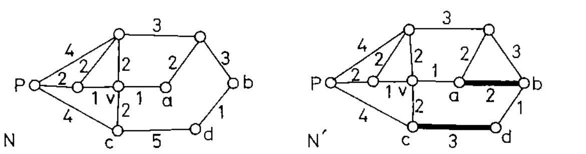

Fig. 1 Networks N and N' where numbers attached to edges denote the lengths of the edges.

edges and we also assume that new edges are added to N in such a way that the resulting network has no parallel edges. Under these preliminaries, the following theorem is proved as follows:

[Theorem 1] Let $N'=N'(\lambda', \sigma')$ be a network obtained from $N=N(\lambda, \sigma)$ by adding new edges a_ib_i with lengths $\lambda'(a_ib_i)$, $1\leqq i\leqq r'$, and by decreasing the lengths of edges a_ib_i of N from $\lambda(a_ib_i)$ to $\lambda'(a_ib_i)$, $r'+1\leqq i\leqq r$. Put $W=\{a_i, b_i|1\leqq i\leqq r\}\subset V(N)$, and let v and p be two distinct vertices of N such that $p\notin W$. Then, the Sabidussi difference $\Delta_c(v)$ satisfy

$$\Delta_c(v)\geqq\Delta_c(p) \tag{3}$$

if the following two conditions hold:

(a) $d_N(p, W)\geqq d_N(p, v)$ and $d_N(p, W)\geqq d_N(v, W)$.

(b) If $d_N'(p, q)< d_N(p, q)$ for some $q\ W$, then

$$d_N'(p, q)=d_N'(p, v)+d_N'(v, q).$$

Before getting down to the proof of this theorem, we give an example. Let N and N' be connected networks shown in Fig. 1, where N' is obtained from N by adding a new edge ab with length 2 and decreasing the length of edge cd from 5 to 3. Then, we have $W=\{a, b, c, d\}$, $d_N(p, W)=4\geqq d_N(p, v)=3$, $d_N(p, W)=4\geqq d_N(v, W)=1$, $d_N'(p, b)=d_N'(p, v)+d_N'(v, b)=6$ and $d_N'(p, d)=d_N'(p, v)+d_N'(v, d)=7$. Hence, in this example, we see that the conditions (a) and (b) are satisfied.

The following well-known lemma plays a key role in proving this theorem:

[Lemma 1] Let $f(x, y)$ be a bounded real-valued function of nonnegative variables x and y. If $f(x, y)$ is convex downwards with respect to x, then $f(a, y)+f(d, y)\leqq f(b, y)+f(c, y)$ for all a, b, c, d and y satisfying $a\leqq c\leqq d$, $a\leqq b\leqq d$ and $a+d=b+c$. Also, if $f(a, y)+f(b, y)\leqq 2f((a+b)/2, y)$ for all a, b and y, then $f(x, y)$ is convex downwards with respect to x.

(Proof of Theorem 1) For convenience' sake, we denote $d_N(x, y)$ and $d_N'(x, y)$ by $d(x, y)$ and $d'(x, y)$, respectively. It is clear that $d_N'(x, W)\leqq d_N(x, W)$ for every $x\in V(N)$. By the definition of the Sabidussi difference, we obtain

$$\begin{aligned}&\Delta_c(v)-\Delta_c(p)\\ &=\sum_{x\in V(N)}\{f(d(v, x), \sigma(x))-f(d'(v, x), \sigma(x))\\ &\quad -f(d(p, x), \sigma(x))\\ &\quad +f(d'(p, x), \sigma(x))\}.\end{aligned} \tag{4}$$

So it suffices to show that

$$\begin{aligned}\phi(x)=\{&f(d(v, x), \sigma(x))-f(d'(v, x), \sigma(x))\\ &-f(d(p, x), \sigma(x))\\ &+f(d'(p, x), \sigma(x))\}\\ \geqq 0&\end{aligned}$$

for every $x\in V(N)$. We consider two cases.

[Case 1] $d(x, v)\leqq d(x, W)$. If $d(x, v)>d'(x, v)$, then every shortest path in N' connecting v and x passes through some vertices in W, and then $d'(v, x)\geqq d(v, W)+d(W, x)\geqq d(v, W)\geqq d(v, x)$, a contradiction. Therefore, $d(v, x)=d'(v, x)$ as $d(v, x)\geqq d'(v, x)$. We next show that $d(p, x)=d'(p, x)$, which implies $\phi(x)=0$ and thus the proof of this case is complete. Suppose $d(p, x)>d'(p, x)$. Then every shortest path in N' connecting p and x passes through some vertices in W. Hence, by the assumption of this case and by the condition (1), we have $d'(p, x)\geqq d(p, W)+d(W, x)\geqq d(p, v)+d(v, x)\geqq d(p, x)$, a contradiction. Consequently, $d(p, x)=d'(p, x)$.

[Case 2] $d(x, v)>d(x, W)$. If $d(x, p)=d'(x, p)$, then $\phi(x)=\{f(d(v, x), \sigma(x))-f(d'(v, x), \sigma(x))\}\geqq 0$ since $d(v, x)\geqq d'(v, x)$ and $f(x, y)$ is increasing with respect to x. Hence we may assume $d(x, p)>d'(x, p)$. Then every shortest path in N' connecting p and x passes through some vertices in W. Thus $d'(p, x)=d'(p, q)+d(q, x)$ for some $q\in W$. Since $d(p, q)+d(q, x)\geqq d(p, x)>d'(p, x)=d'(p, q)+d(q, x)$, we have $d(p, q)>d'(p, q)$. By the conditions (2) and (1), we obtain $d'(p, q)=d'(p, v)+d'(v, q)=d(p, v)+d'(v, q)$. So it follows that $d'(p, x)=d(p, v)+d'(v, q)+d(q, x)\geqq d(p, v)+d'(v, x)\geqq d'(p, x)$. Therefore $d'(p, x)=d(p, v)+d'(v, x)$. It is immediate that $d'(v, x)=a$, $d(v, x)=b$, $d'(p, x)=c$ and $d(p, v)+d(v, x)=d$ satisfy the conditions on a, b, c and d in Lemma 1. On the other hand, we have

$$\begin{aligned}\phi(x)\geqq\{&f(d(v, x), \sigma(x))-f(d'(v, x), \sigma(x))\\ &-f(d(p, v)+d(v, x), \sigma(x))\\ &+f(d'(p, x), \sigma(x))\}\end{aligned}$$

since $d(p, x)\leqq d(p, v)+d(v, x)$, and since $f(x, y)$ is increasing with respect to x. Also, since $f(x, y)$ is convex downwards with respect to x, we conclude from Lemma 1 that $\phi(x)\geqq 0$. Hence, this completes the proof of the theorem. (Q. E. D.)

Roughly speaking along with the example stated already in this note, Theorem 1 means that the benefit of a vertex v is greater than that of a vertex p if v lies nearer to new edges or improved edges than p. It should be noted that the following corollaries are obtained as an immediate consequence of Theorem 1:

[Corollary 1-1] Consider the case where $\lambda(vu)=1$ for every edge vu of N and $\sigma(u)=1$ for every vertex of

N. Note that such a network N is often called a graph. Let N' be a network obtained from N by adding new edges a_ib_i with length one, $1 \leqq i \leqq r$, such that the set of these $r(r+1)/2$ new edges forms a complete graph of r vertices. Put $W=\{a_i, b_i | 1 \leqq i \leqq r\} < V(N)$. If, for two distinct vertices p and v of N such that $p \notin W$, there holds $d_N(p, W)= d_N(p, v)+d_N(v, W)$, then the Sabidussi difference $\Delta_c(v)$ satisfy

$$\Delta_c(v) \geqq \Delta_c(p). \tag{5}$$

(Proof) By Theorem 1, it suffices to show that if $d_N'(p, q) < d_N(p, q)$ for some $q \in W$, then $d_N'(p, q)= d_N'(p, v)+d_N'(v, q)$ for some vertex v. Suppose $d_N'(p, q) < d_N(p, q)$ for some $q \in W$. Then every shortest path in N' connecting p and q contains exactly one new edge, say bq, since the newly added edges form a complete graph. Thus we have $d_N'(p, q) \geqq d_N'(p, W) + d_N'(b, q) = d_N(p, W)+1$. Accordingly, if, for two distinct vertices p and v of N such that $p \notin W$, there holds $d_N(p, W)=d_N(p, v)+d_N(v, W)$, then it follows for some $u \in W$ that $d_N'(p, q) \geqq d_N(p, W)+1=d_N(p, v)+d_N(v, u)+d_N'(u, q) \geqq d_N'(p, v)+d_N'(v, q) \geqq d_N'(p, q)$. From this we see that there holds $d_N'(p, q)=d_N'(p, v)+d_N'(v, q)$. Hence, this completes the proof of this corollary. (Q. E. D.)

[Corollary 1-2] For any $N=N(\lambda, \sigma)$, let $N'=N'(\lambda', \sigma')$ be a network obtained from N by adding a new edge vw with $\lambda'(vw) < d_N(v, w)$. Then, the Sabidussi difference $\Delta_c(v)$ satisfy

$$\Delta_c(v) \geqq \Delta_c(p) \tag{6}$$

for every vertex p such that $d_N(p, v) < d_N(p, w)$ and $p \notin \{v, w\}$.

The proof of this corollary is omitted herein, because this corollary was already proved for graphs in Ref. (3), in the case where $f(x, y)$ can be expressed as $f(x, y)=g(x) \cdot y$, and subsequently was generalized for networks in Ref. (4).

It should be noted that a key of the proof of Theorem 1 is the property that $f(x, y)$ is increasing and convex downwards with respect to x, characterizing a generalized transmission number. The fundamental importance of the property of $f(x, y)$ will also be seen in the following two remarks:

[Remark 1[(2)-(4)]] For any $N=N(\lambda, \sigma)$, let $N'=N'(\lambda', \sigma')$ be a network obtained from N by adding a new edge vw with $\lambda'(vw) < d_N(v, w)$. Then, the property of $f(x, y)$ is proved to be a necessary condition for the Sabidussi difference $\Delta_c(v)$ to satisfy $\Delta_c(v) \geqq \Delta_c(p)$ for every vertex p such that $d_N(p, v) < d_N(p, w)$ and $p \notin \{v, w\}$.

[Remark 2] A vertex v of N with minimum value of c_f is called a f-center vertex of N. The set of f-center vertices of N is called the f-center of N. Let N be any connected tree in which each vertex has its own weight but the length of each edge is 1. Then, it can easily be proved that the property of $f(x, y)$ is a necessary and sufficient condition for the f-center of N to consist of one vertex or two adjacent vertices.

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