LETTER Special Section on Nets-Oriented Software Specification and Design

On a Sufficient Condition for a Matrix to be the Synchronic Distance Matrix of a Marked Graph

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SUMMARY The synchronic distance is a fundamental concept in a Petri net. Marked graphs form a subclass of Petri nets. Given a matrix D, we are interested in the problem of finding a marked graph whose synchronic distance matrix is D. It is well-known that the synchronic distance matrix of a marked graph is a distance matrix. In this letter, we give a matrix D such that D is a distance matrix and there does not exist a marked graph whose synchronic distance matrix is D.
key words: *marked graph, synchronic distance matrix, distance matrix*

1. Introduction

Recently, it has been demanded to analysis dispersed or parallel processing systems with models of them, as they develop rapidly. Petri nets are useful models of discrete event systems, such as parallel computers, distributed systems, and so on. The synchronic distance is a measure of mutual dependence between two events defined in a condition/event system or Petri net.

Marked graphs form a subclass of Petri nets. The synchronic distance matrix of a marked graph is a symmetric matrix whose element represents the synchronic distance between corresponding two vertices. It is well-known that if a marked graph is live, then the synchronic distance matrix of the marked graph is a distance matrix. Conversely, for any distance matrix D, we do not know whether there is a marked graph whose synchronic distance is D or not. The efficient necessary and sufficient condition, in order to determine whether D is the synchronic distance matrix of a marked graph or not, is an open problem.

In this letter, we give a matrix D which is a distance matrix but there does not exist a marked graph whose synchronic distance matrix is D.

2. Preliminary

A *marked graph* $M = (V(M), E(M), m_M)$ is a directed graph which consists of the finite set $V(M)$ of vertices, the finite set $E(M)$ of directed edges and directed edge weight function called the *initial marking* $m_M : E(M) \to \boldsymbol{Z}^+$, where $\boldsymbol{Z}^+$ is a set of nonnegative integers. $m_M(e)$ represents the number of tokens assigned to a directed edge e.

A vertex v_i is said to be *firable* if each of its input directed edges has at least one token. And v_i *fires* by removing one token from its each input directed edges and adding one token to its each output directed edges. A marking resulting from a sequence of vertex firings is said to be *reachable* from m_M.

Let $R(m_M)$ be a set of all reachable markings from m_M. A vertex v_i is said to be *live* if v_i can be brought to firable by some firing sequence from any markings in $R(m_M)$. A marked graph M is said to be *live* if all vertices in $V(M)$ are live. The following theorem is well-known.

Theorem 1 [3]: A marked graph M is live if and only if M has no token-free directed circuits. □

Unless otherwise stated, a marked graph M is live in this letter.

For marked graphs, there is an important theorem, as follows.

Theorem 2 [1], [2]: The number of tokens on a directed cycle of a marked graph does not change as a result of vertex firings. □

Let d^*_{ij} be the synchronic distance between vertices v_i and v_j in a marked graph M. Then $d^*_{ij} = k$ means that there is a marking in $R(m_M)$ from which one vertex v_i or v_j may be fired maximally k times without the necessity of firing the other. The $n \times n$ matrix $D = (d^*_{ij})$, where $n = |V(M)|$ and d^*_{ij} is the (i,j)-element of D, is called the *synchronic distance matrix* of M. And D is simply called a *synchronic distance matrix* if there exists a marked graph whose synchronic distance matrix is D.

For synchronic distances, the following equation holds.

Theorem 3 [2]: Let M be a marked graph and v_i, v_j be vertices of M. Then

$$d^*_{ij} = \min(w(P_{ij})) + \min(w(P_{ji})),$$

where $w(P_{ij})$ represents the number of tokens on the path P_{ij} from v_i to v_j. □

Note that $d^*_{ij} > 0$ if $i \neq j$ from Theorem 2, 3.

A graph is called a *plane graph* if it is *embedded*

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in the plane [7] (see Appendix) and a cycle of length three is called *triangle.*

3. Realization of a Synchronic Distance Matrix to a Marked Graph

The following property is well-known for synchronic distances.

Property 1 [2], [4]: Let D be a synchronic distance matrix and d^*_{ij} be the (i,j)-element of D. Then

(i) $d^*_{ij} = 0 \Leftrightarrow i = j$.

(ii) $d^*_{ij} = d^*_{ji}$ for all i, j.

(iii) $d^*_{ij} \leqq d^*_{ik} + d^*_{kj}$ for all i, j, k.

(iv) d^*_{ij} is a nonnegative integer or ∞. □

In this letter, if a square matrix D satisfies (i)–(iv) in Property 1, then D is called a *distance matrix.*

An edge(v_i, v_j) of graph G is *redundant* if there exists a vertex $v_k (k \neq i, j)$ satisfied following inequality.

$$w(v_i, v_j) \geq w(v_i, v_k) + w(v_k, v_j),$$

where $w(v_i, v_j)$ represents the weight of (v_i, v_j).

N_D is the graph obtained from the complete graph, whose each edge is assigned to the corresponding element of a distance matrix $D = (d_{ij})$, by deleting all redundant edges and all edges whose weights are ∞.

The length of the shortest path between v_i and v_j in a graph is called the *distance* between v_i and v_j. Note that the distance between v_i and v_j in N_D is d_{ij}.

M_D is a directed graph (i.e., marked graph) which is obtained by replacing each edge(v_i, v_j) of N_D with a pair of oppositely directed edges, (v_i, v_j) and (v_j, v_i), and the sum of those two directed edges' weights is equal to the weight of the original edge (see Figs. 1(a)–(d)).

It has been shown the necessary and sufficient condition such that D is a synchronic distance matrix, as follows [5], [6].

Theorem 4: D is a synchronic distance matrix if and only if there exists a marked graph M_D whose synchronic distance matrix is equal to D. □

If we check the synchronic distance matrices of all M_D's, we can determine whether a matrix D is a synchronic distance matrix or not, from Theorem 4. Unfortunately, this is not an efficient algorithm.

However, we find that if N_D contains an induced subgraph, D is not a synchronic distance matrix.

Theorem 5: Let D be a distance matrix. If N_D contains an induced subgraph N' isomorphic a *wheel graph* W_n [7] satisfying the following condition,

(1) $n \geq 5$ and n is odd,

(2) the weight of each edge is one,

then, D is not a synchronic distance matrix.

(The wheel graph with $n = 5$, that satisfies the above condition, is shown in Fig. 2.)

Proof: Let N' be a plane graph, for a wheel graph is embeddable in the plane. Assume that there exists a

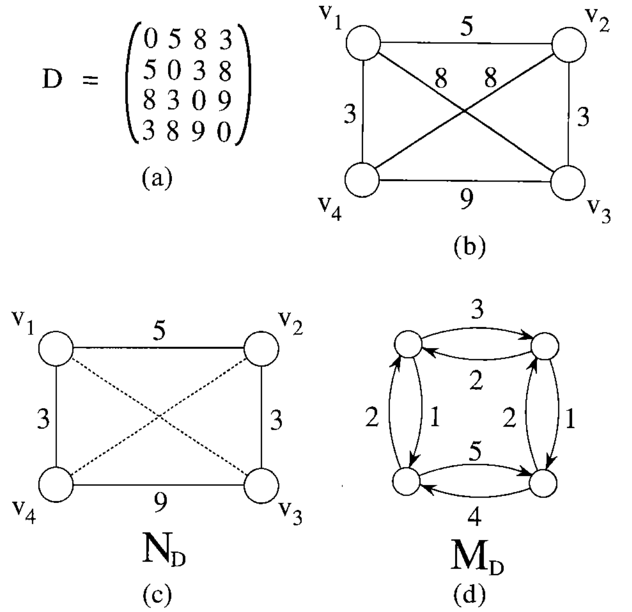

Fig. 1 A distance matrix D, the graph N_D and a directed graph M_D.
(a) A distance matrix D.
(b) The complete graph whose each edge is assigned to the corresponding element of D.
(c) The graph N_D obtained by deletion of all redundant edges. (The redundant edges are shown by broken lines.)
(d) A directed graph M_D obtained by replacing each edge in N_D with a pair of oppositely directed edges and the sum of those two directed edges' weights equals to the weight of the original edge.

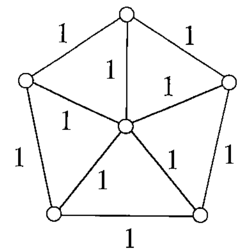

Fig. 2 The wheel graph W_5 with $n = 5$.

marked graph M_D whose synchronic distance matrix is D. Let M' be a subdigraph (i.e., sub-marked graph) of M_D corresponding N'. In M', one of two directed cycles, whose underlying graphs are the same triangle of N_D, has two tokens and the other has one. (If one of them has no tokens, it contradicts the fact that M_D is live.) Thus, we now assign each triangle of N' to the direction of the directed cycle with two tokens on M'.

It is clear that the number of triangles in N' is odd, for n is odd. And it is also clear that there are two adjacent triangles assigned the same direction each other. Let N'' be the subgraph of N' consisting of those two triangles (see Fig. 3(a)).

In N'', let u and v be two distinct nonadjacent vertices. It is obvious that the synchronic distance between

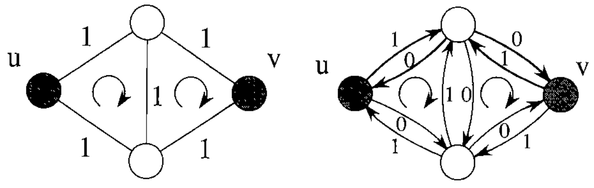

Fig. 3 The subgraph N'' and the subdigraph of M_D.
(a) The subgraph N'' which consists of two adjacent triangles assigned the same direction each other.
(b) The subdigraph of M_D whose underlying graph is N''.

u and v on M_D is one. On the other hand, since N' is an induced subgraph, (u, v) is not an edge of N_D. Since each edge weight in N_D is greater than zero and the synchronic distance between u and v is equal to the distance between u and v on N_D, the synchronic distance is two (see Figs. 3(a) and (b)). That is a contradiction. Hence such M_D does not exist.

From Theorem 4, D is not a synchronic distance matrix. □

From Property 1, the synchronic distance matrix of a marked graph is a distance matrix. However, from Theorem 5, a distance matrix is not always a synchronic distance matrix.

4. Conclusion

In this letter, we gave a matrix which was a distance matrix but was not a synchronic distance matrix. That is, the synchronic distance of a marked graph is a distance matrix, although the converse does not hold in general.

Thus as a future problem, for a matrix D, it is mentioned to find an efficient necessary and sufficient condition to exist a marked graph whose synchronic distance is equal to D.

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Appendix: Definition of Embeddable of a Graph G [7].

A (p, q) graph G is said to be *embeddable* on a surface S if it is possible to distinguish a collection of p distinct points of S that correspond to the vertices of G and a collection of q curves, pairwise disjoint except possibly for endpoints, on S that correspond to the edges of G such that if a curve A corresponds to the edge $e = uv$, then only the endpoints of A correspond to vertices of G, namely u and v.