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Realization Problems of a Tree with a Transmission Number Sequence

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SUMMARY Problems of realizing a vertex-weighted tree with a given weighted transmission number sequence are discussed in this paper. First we consider properties of the weighted transmission number sequence of a vertex-weighted tree. Let S be a sequence whose terms are pairs of a non-negative integer and a positive integer. The problem determining whether S is the weighted transmission number sequence of a vertex-weighted tree or not, is called w-TNS. We prove that w-TNS is *NP*-complete, and we show an algorithm using backtracking. This algorithm always gives a correct solution. And, if each transmission number of S is different to the others, then the time complexity of this is only $O(|S|^2)$. Next we consider the d_2-transmission number sequence so that the distance function is defined by a special convex function.

key words: *transmission number, NP-complete problem*

1. Introduction

As the problem of realizing a graph from an integer sequence, degree sequences [1] and eccentric sequences [2] et al. are well-known. Although the transmission number of a graph is as important as the eccentric number in the Location Problem, few properties about a (weighted) transmission number sequence are known.

Let $G(V, E)$ be the vertex-weighted graph which is an undirected connected graph with a positive integer $w(v)$ (called the weight of v) associated with each vertex v of G. Let $d(v, u)$ be the distance which is the length of the shortest path between vertices u and v of G. The transmission number [3] of a vertex v of G is defined as follows:

$$t(G; v) = \sum_{u \in V} d(v, u)w(u)$$

(usually dropping the first term G when the graph is clear from context). A vertex v of G is called a median if $t(v) = \min_{u \in V} t(u)$. Let S be a non-negative integer sequence:

$$S = (s_1, s_2, \cdots, s_p).$$

S is a transmission number sequence if there exists a vertex-weighted graph $G(V, E)$ whose vertices can be labeled $v_1, v_2, \cdots, v_p$ so that $t(v_i) = s_i$ for all i. If G is vertex-unweighted ($w(v) = 1$ for all $v \in V$), some results are given in Ref. [4]. In this paper, although we do not consider such transmission number sequences, we will define a weighted transmission number sequence and extend results given in Ref. [4] to the case of it.

Let S be the sequence whose terms are pairs of a non-negative integer and a positive integer:

$$S = ((s_1, w_1), (s_2, w_2), \cdots, (s_p, w_p))$$

with $s_i \geq 0$ and $w_i \geq 1$ for all i. (call each w_i weight of S, and call S an NPI sequence.) S is a weighted transmission number sequence if there exists a vertex-weighted graph $G(V, E)$ whose vertices can be labeled $v_1, v_2, \cdots, v_p$ so that $t(v_i) = s_i$ and $w(v_i) = w_i$ for all i. Figure 1 shows an example of a weighted transmission number sequence. We discuss realizing of a *vertex-weighted tree* with a weighted transmission number sequence hereafter.

We show some properties of transmission numbers and weighted transmission number sequences of vertex-weighted trees in Sect. 2. We call the problem determining whether an NPI sequence is the weighted transmission number sequence of a vertex-weight tree or not, w-TNS. (The restricted problem of this so that $w_i = 1$ for all i, is called TNS in Ref. [4]. There is a mistake in it. The correction about this is in the end of this paper.) We prove that w-TNS is *NP*-complete in Sect. 3. An algorithm is given by using a backtrack method in Sect. 4. w-TNS is not solvable in polynomial time unless $P = NP$, but w-TNS can be solved in polynomial

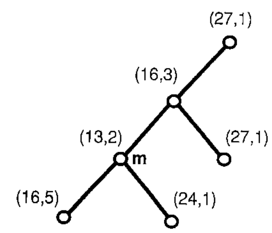

Fig. 1 The weighted transmission number sequence of this vertex-weighted tree is $(27, 1)$,$(27, 1)$,$(24, 1)$,$(16, 3)$,$(16, 5)$,$(13, 2)$. The vertex m is the median.

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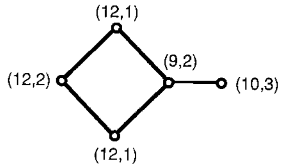

Fig. 2 The weighted d_2-transmission number sequence of this graph is $(12,1)$,$(12,1)$,$(12,2)$,$(10,3)$,$(9,2)$.

time if $s_1 > s_2 > \cdots > s_p$.

We discuss the following case in Sect. 5. We define $d_2(u,v)$ which is the distance between vertices u and v of G as follows:

$$d_2(u,v) = \begin{cases} d(u,v) & \text{if } d(u,v) < 2 \\ 2 & \text{if } d(u,v) \geqq 2, \end{cases}$$

and the d_2-transmission number of a vertex v of G as follows:

$$t_2(G;v) = \sum_{u \in V} d_2(v,u)w(u)$$

(usually dropping the first term G when the graph is clear from context). Figure 2 shows an example of a d_2-transmission number sequence. We consider the case where the weight of each vertex is equal to the others. In this case, a vertex-weighted tree can be also constructed in polynomial time.

2. Properties of Transmission Number

In this section, we show some properties of the weighted transmission number sequence of a vertex-weighted tree, and we show a nondeterministic algorithm for w-TNS.

Let $T(V,E)$ be a vertex-weighted tree, and let u and v be adjacent vertices of T. Let $T - \{u,v\}$ be the disconnected graph which is obtained by removing an edge $\{u,v\}$ from T. We define T_u and T_v as the two subtrees (components) of $T - \{u,v\}$ so that $u \in V(T_u)$ and $v \in V(T_v)$. We define $M(T)$ as follows

$$M(T) = \sum_{v \in V} w(v).$$

The following properties about the transmission number of a vertex-weighted tree are known [5].

Property 1: Let $T(V,E)$ be a vertex-weighted tree. For all u and v of T which are adjacent to each other;

$$M(T_u) - M(T_v) = t(v) - t(u).$$

Proof: By the definition of the transmission number,

$$\begin{aligned} t(v) &= \sum_{v' \in V} d(v,v')w(v') \\ &= \sum_{v' \in V(T_v)} d(v,v')w(v') \\ &\quad + \sum_{u' \in V(T_u)} d(v,u')w(u'). \end{aligned}$$

Since the (shortest) v-u' path passes on the edge $\{v,u\}$,

$$\begin{aligned} t(v) &= \sum_{v' \in V(T_v)} d(v,v')w(v') \\ &\quad + \sum_{u' \in V(T_u)} (d(u,u')+1)w(u') \\ &= \sum_{v' \in V(T_v)} d(v,v')w(v') \\ &\quad + \sum_{u' \in V(T_u)} d(u,u')w(u') \\ &\quad + \sum_{u' \in V(T_u)} w(u') \\ &= \sum_{v' \in V(T_v)} d(v,v')w(v') \\ &\quad + \sum_{u' \in V(T_u)} d(u,u')w(u') + M(T_u). \end{aligned}$$

Similarly,

$$\begin{aligned} t(u) &= \sum_{u' \in V(T_u)} d(u,u')w(u') \\ &\quad + \sum_{v' \in V(T_v)} d(v,v')w(v') + M(T_v). \end{aligned}$$

Therefore,

$$t(v) - t(u) = M(T_u) - M(T_v). \qquad \square$$

Using Property 1, we obtain the following property.

Property 2: Let $T(V,E)$ be a vertex-weighted tree, and let v be a vertex of T. Then v is a median of T if and only if for each vertex u that is adjacent to v;

$$M(T_u) \leqq M(T_v).$$

(The vertex u is a median of T if and only if the equal sign is valid.) □

Immediately, Property 1 and Property 2 imply the following Property 3.

Property 3: The number of medians of a vertex-weighted tree is one or two. If the tree has two medians, they are adjacent to each other. □

We will show a necessary and sufficient condition for an NPI sequence to be a weighted transmission number sequence. Immediately, the following lemmas are obtained by using Property 1 and Property 2.

Lemma 1: Let $T(V,E)$ be a vertex-weighted tree, and let $v_1v_2\cdots v_n$ be a v_1-v_n path of T. If v_n is a median,

$$t(v_1) > t(v_2) > \cdots > t(v_{n-1}) \geqq t(v_n).$$

$v_{n-1} = v_n$ if and only if both v_{n-1} and v_n are medians.)

Lemma 2: Let $T(V, E)$ be a vertex-weighted tree so that $|V| \geqq 2$, and let v be a vertex of $T(V, E)$ so that $t(v) = \max_{u \in V} t(u)$. Then the degree of v is 1.

We prove the following theorem using these lemmas.

Theorem 1: An NPI sequence : $S = ((s_1, w_1), (s_2, w_2), \cdots, (s_p, w_p))$ with $s_1 \geqq s_2 \geqq \cdots \geqq s_p$, is the weighted transmission number sequence of a vertex-weighted tree if and only if there exists j $(2 \leqq j \leqq p)$ such that

$$s_j = s_1 + w_1 - \sum_{2 \leq i \leq p} w_i$$

and the sequence:

$$S' = ((s_2 - w_1, w_2), \cdots, (s_{j-1} - w_1, w_{j-1}), (s_j - w_1, w_j + w_1), (s_{j+1} - w_1, w_{j+1}), \cdots, (s_p - w_1, w_p))$$

is the weighted transmission number sequence of a vertex-weighted tree.

Proof: We suppose that S is the weighted transmission number sequence of a vertex-weighted tree. Then there exists a vertex-weighted tree T whose vertices are labeled $v_1, v_2, \cdots, v_p$ so that $t(v_i) = s_i$ and $w(v_i) = w_i$ for all i. The degree of the vertex v_1 (corresponding to (s_1, w_1)) is 1 by Lemma 2. Let v_j be the vertex which is adjacent to v_1. Then $s_j = s_1 + w_1 - \sum_{2 \leq i \leq p} w_i$ by Property 1. Let T' be the vertex-weighted tree which is obtained by removing v_1 from T and setting $w_j + w_1$ on the weight of v_j. We shall show that $s_i - w_1$ is the transmission number of each v_i of T'.

$$\begin{aligned} t(T'; v_i) &= \sum_{2 \leq k < j,\, j < k \leq p} w(v_k) d(v_i, v_k) \\ &\quad + (w(v_j) + w(v_1)) d(v_i, v_j) \\ &= \sum_{2 \leq k \leq p} w(v_k) d(v_i, v_k) \\ &\quad + w(v_1)(d(v_i, v_j) + 1) - w(v_1) \\ &= t(T; v_i) - w(v_1) \\ &= s_i - w_1. \end{aligned}$$

Therefor U' is the weighted transmission number sequence of T'.

Conversely, we assume that U' is a weighted transmission number sequence. Then there exists a vertex-weighted tree T' whose vertices are labeled $v_2, v_3, \cdots, v_p$ so that for all i, $t(v_i) = s_i - w_1$ and

$$w(v_i) = \begin{cases} w_i + w_1 & \text{if } i = j \\ w_i & \text{otherwise.} \end{cases}$$

Let T be the vertex-weighted tree which is obtained by adding a vertex v_1 and the edge $\{v_1, v_j\}$ to T', and setting $w(v_i) = w_i$ for all v_i. We shall show that each s_i is the transmission number v_i of T. For all v_i $(2 \leqq i \leqq p)$;

$$\begin{aligned} t(T; v_i) &= \sum_{1 \leq k \leq p} w(v_k) d(v_i, v_k) \\ &= \sum_{2 \leq k \leq p} w(v_k) d(v_i, v_k) \\ &\quad + w(v_1)(d(v_i, v_j) + 1) \\ &= \sum_{2 \leq k < j,\, j < k \leq p} w(v_k) d(v_i, v_k) \\ &\quad + (w(v_j) + w(v_1)) d(v_i, v_j) + w(v_1) \\ &= t(T'; v_i) + w(v_1) \\ &= s_i. \end{aligned}$$

And for v_1;

$$\begin{aligned} t(T; v_1) &= \sum_{2 \leq k \leq p} w(v_k) d(v_1, v_k) \\ &= \sum_{2 \leq k \leq p} w(v_k)(d(v_1, v_k) - 1) \\ &\quad + \sum_{2 \leq k \leq p} w(v_k) \\ &= \sum_{1 \leq k \leq p} w(v_k) d(v_j, v_k) \\ &\quad - w_1 + \sum_{2 \leq k \leq p} w(v_k) \\ &= t(T; v_j) - w_1 + \sum_{2 \leq k \leq p} w(v_k) \\ &= s_j - w_1 + \sum_{2 \leq k \leq p} w(v_k). \end{aligned}$$

Since $s_j = s_1 + w_1 - \sum_{2 \leq i \leq p} w_i$, $t(T; v_1) = s_1$. Therefore U is the weighted transmission number sequence of T. We have proved Theorem 1. □

Next we easily have the following nondeterministic algorithm for w-TNS using Theorem 1. The following NPI sequence is inputted;

$$S = ((s_1, w_1), (s_2, w_2), \cdots, (s_p, w_p))$$

with $s_1 \geqq s_2 \geqq \cdots \geqq s_p$.

Algorithm 1: Set $W = w_1 + w_2 + \cdots + w_p$. A variable **i** is used as a counter in Algorithm 1.

Step 1. (Initialize) Set $\mathbf{i} \leftarrow 1$.

Step 2. If $S = ((0, W))$ then stop (accept).

Step 3. If there exists j $(\mathbf{i} < j \leqq p)$ which satisfies the following condition:
Condition A: $s_\mathbf{i} - (W - 2w_\mathbf{i}) = s_j$,
then go to Step 4 else stop (not accept).

Step 4. An integer j $(\mathbf{i} < j \leqq p)$ which meets Condition A is guessed oracularly.

Step 5. Set $s_k \leftarrow s_k - w_j$ for all k $(i < k \leqq p)$.

Step 6. Set $w_j \leftarrow w_i + w_j$, and eliminate (s_i, w_i) from S.

Step 7. $i \leftarrow i + 1$ and go to Step 2.

This algorithm exactly solves *w*-TNS, and also constructs a vertex-weighted tree from a given weighted transmission number sequence. In Sect. 3, we prove that *w*-TNS is *NP*-complete using Theorem 1. And, we show a deterministic algorithm in Sect. 3.

3. Proof of *NP*-Completeness

In this section, we prove that *w*-TNS is *NP*-complete using Algorithm 1. We explain PARTITION before this. The proof which PARTITION is *NP*-complete, is given in Ref. [6].

PARTITION

INSTANCE : A positive integer sequence $i_1, i_2, \cdots, i_p$.

QUESTION : Is there J which is a subset of $I = \{1, 2, \cdots, p\}$ so that $\sum_{k\in J} i_k = \sum_{k'\in I-J} i_{k'}$?

***w*-TNS**

INSTANCE : An NPI sequence :

$$S = ((s_1, w_1), (s_2, w_2), \cdots, (s_p, w_p))$$

with $s_1 \geqq s_2 \geqq \cdots \geqq s_p$.

QUESTION : Is there a vertex-weighted tree whose weighted transmission number sequence is equal to S ?

We will prove that *w*-TNS is *NP*-complete by transforming PARTITION to *w*-TNS.

Theorem 2: *w*-TNS is *NP*-complete.

Proof: First it is easy to see *w*-TNS $\in$ *NP*, since (nondeterministic) Algorithm 1 need to guess j on step 4 oracularly and to check that S is the weighted transmission number sequence of a vertex-weighted tree in polynomial time.

We will transform PARTITION to *w*-TNS. Let $\mathcal{I} = (i_1, i_2, \cdots, i_p)$ be an instance of PARTITION, and set $C = i_1 + i_2 + \cdots + i_p + 2$. As $\mathcal{I}$ can be sorted in polynomial time, it is proper to suppose that $i_1 \leqq i_2 \leqq \cdots \leqq i_p$ without loss of generality. Moreover if $i_k > (C/2) - 1$ for some k, then $\mathcal{I}$ does not have a partition evidently. Hence if instances of PARTITION are restricted so that $i_k \leqq (C/2) - 1$ for all k, then such a subproblem is yet *NP*-complete. So we assume $i_k \leqq (C/2) - 1$ for all k. Set

$$s_k = \begin{cases} \frac{5}{2}C - 2(i_k + 1) & \text{if } 1 \leqq k \leqq p \\ \frac{3}{2}C - 2 & \text{if } p < k \leqq p + 2, \end{cases}$$

and set

$$w_k = \begin{cases} i_k & \text{if } 1 \leqq k \leqq p \\ 1 & \text{if } p < k \leqq p + 2. \end{cases}$$

Then $s_1 \geqq s_2 \geqq \cdots \geqq s_{p+2}$. Let S be an instance of *w*-TNS as follows;

$$S = ((s_1, w_1), (s_2, w_2), \cdots, (s_{p+2}, w_{p+2}))$$

It is clear that PARTITION is transformed to *w*-TNS in polynomial time.

We assume that $\mathcal{I}$ has a partition. Set $I = \{1, 2, \cdots, p\}$. Then there exists a subset J of I so that

$$\sum_{k\in J} i_k = \sum_{k'\in I-J} i_{k'}.$$

Let V be a vertex set whose vertices are labeled $v_1, v_2, \cdots, v_{p+2}$ so that $w(v_k) = w_k$ for all k $(1 \leqq k \leqq p+2)$, and let E be the edge set as follows;

$$\begin{aligned} E = &\{\{v_k, v_{p+1}\} | k \in J\} \\ &\cup\{\{v_k, v_{p+2}\} | k \in I - J\} \\ &\cup\{\{v_{p+1}, v_{p+2}\}\}. \end{aligned}$$

Then $T(V, E)$ is a tree. Since the definition of s_k and Property 1, $t(v_k) = s_k$ for each vertex v_k of $T(V, E)$. Therefore if $\mathcal{I}$ has a partition, S is the weighted transmission number sequence of a vertex-weighted tree. Figure 3 shows an example of a tree corresponding to PARTITION.

Conversely, we assume that S is the weighted transmission number sequence of a vertex-weighted tree. Then there exists a vertex-weighted tree $T(V, E)$ whose vertices are labeled $v_1, v_2, \cdots, v_{p+2}$ so that $t(v_k) = s_k$ and $w(v_k) = w_k$ for all k $(1 \leqq k \leqq p+2)$. Then v_{p+1} and v_{p+2} are medians, and are adjacent to each other by Property 3. Using an inductive method, we will show that for each v_k $(1 \leqq k \leqq p)$, v_k is adjacent only to either v_{p+1} or v_{p+2} and the degree of v_k is 1. If $k = 1$, it easy to see that these hold by Lemma 2 and the definition of s_1. We suppose that these hold for all j $(1 \leqq j < k \leqq p)$. Then for all j, each v_j is not adjacent to v_k, and $t(v_j) \geqq t(v_k)$. We assume that there exists $v_{k'}$ $(k < k' \leqq p)$ which is adjacent to v_k. Then we observe the following two cases:

Case 1: $v_{k'}$ is not included in the path between v_k and v_{p+1}. Then $t(v_k)\ (= s_k) < t(v_{k'})\ (= s_{k'})$ by Lemma 1. This is contradictory to $s_k \geqq s_{k'}$.

Case 2: In the other case, the degree of v_k is 1. Then Property 1 does not hold.

Hence v_k is not adjacent to all $v_{k'}$ $(k < k' \leqq p)$. Therefore the degree of v_k is 1 and v_k must be adjacent only

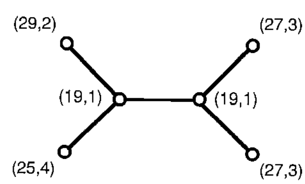

Fig. 3 This is a corresponding tree to a partition of (4, 3, 3, 2).

to either v_{p+1} or v_{p+2}. Thus for all k $(1 \leqq k \leqq p)$, v_k is adjacent only to either v_{p+1} or v_{p+2}. Set

$$J = \{k \leq p \mid v_k \text{ is adjacent to } v_{p+1}\},$$
$$J' = \{k \leq p \mid v_k \text{ is adjacent to } v_{p+2}\}.$$

We suppose $\sum_{k \in J} w_k \neq \sum_{k' \in J'} w_{k'}$. Then $M(T_{v_{p+1}}) \neq M(T_{v_{p+2}})$. By Property 1, we have $t(v_{p+1})$ $(= s_{p+1}) \neq t(v_{p+2})$ $(= s_{p+2})$. This contradicts $s_{p+1} = s_{p+2}$. Therefore $\sum_{k \in J} w_k = \sum_{k' \in J'} w_{k'}$. Since $J \cup J' = I$ and $J \cap J' = \emptyset$, J is a partition of $\mathcal{I}$. Hence $\mathcal{I}$ has a partition if and only if S is the weighted transmission number sequence of a vertex-weighted tree. We have proved w-TNS is *NP*-complete. □

4. A Deterministic Algorithm for w-TNS

We show a deterministic algorithm for w-TNS in this section. Algorithm 2 (deterministic) is easily given by transforming Algorithm 1 (nondeterministic) using a backtrack method. Algorithm 2 is improved so that the stage corresponding to Step 5 in Algorithm 1 is omitted using a variable h. (Notice that $S = (h, W)$ replaces the condition of the stage corresponding to Step 2.)

Algorithm 2: Set

$$S = ((s_1, w_1), (s_2, w_2), \cdots, (s_p, w_p))$$

with $s_1 \geqq s_2 \geqq \cdots \geqq s_p \geqq 0$ and $w_i \geqq 1$ for all i, and set $W = w_1 + w_2 + \cdots + w_p$.

```
INPUT:S
OUTPUT:boolean
function w-TNS(S);
    function link(i, h);
        val oldw, j:integer;
        begin
            if i = p then
                if S = (h, W)
                    then return(TRUE);
                    else return(FALSE);
            j ← i + 1;
            while j ≦ p do
                begin
                    if s_j = s_i + 2w_i − W then
                        begin
                            oldw ← w_j;
                            w_j ← w_i + w_j;
                            if link(i + 1, h + w_i)
                                then return(TRUE);
                            w_j ← oldw;
                        end;
                    j ← j + 1;
                end;
            return(FALSE)
        end;
begin
    return(link(1, 0));
end.
```
□

Algorithm 2 always gives a correct solution. If $s_i \neq s_j$ for all i, j $(i \neq j)$, then the time complexity of this is only $O(|S|^2)$. Algorithm 2 is so fast as to decrease the number of vertices whose transmission numbers are equal to the others. Figure 4 shows an example of a behavior of Algorithm 2.

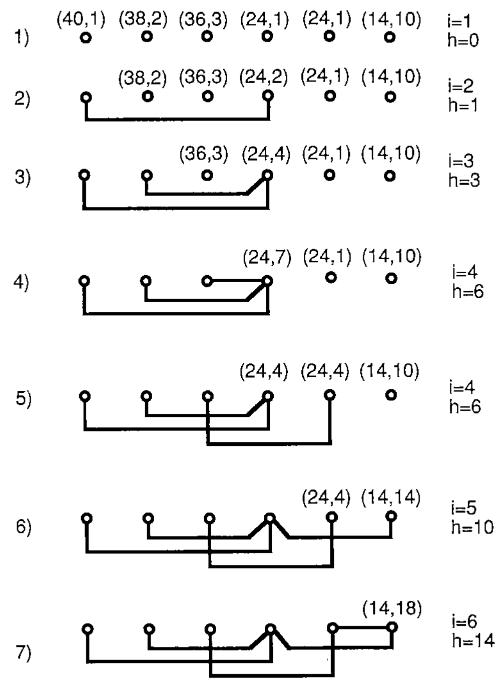

Fig. 4 An example of a behavior of Algorithm 2 whose input is (40, 1),(38, 2),(36, 3),(24, 1),(24, 1),(14, 10). A backtracking is done in 4).

5. A Convex Distance Function

We consider the case where the distance function is the convex function $d_2(\cdot, \cdot)$ hereafter. We will show a necessary and sufficient condition of a weighted d_2-transmission number sequence in the special case where each weight of this sequence is equal to the others. Before this, we show the following three properties.

Property 4: Let $G(V, E)$ be a vertex-weighted connected graph, and let v be a vertex of G. We define U as a subset of $V - \{v\}$ so that

$$U = \{u \in V - \{v\} \mid u \text{ is adjacent to } v\}.$$

Then

$$t_2(v) + \sum_{u \in U} w(u) = 2(M(G) - w(v)).$$ □

Immediately, we obtain the following two properties by Property 4.

Property 5: If $w(u) = w$ for each vertex u of a vertex-weighted graph G, for each v of G whose the degree is $\deg(v)$:

$$t_2(v) + w \cdot \deg(v) = 2w \cdot (|V| - 1).$$ □

Property 6: If $S = ((s_1, w), (s_2, w), \cdots, (s_p, w))$ is a weighted d_2-transmission number sequence of a vertex-weighted graph G, then

$$\sum_{i=1}^{p} s_i \leqq 2w \cdot (p-1)^2.$$

(G is a tree if and only if the equal sign is valid.) □

We obtain an upper bound of a weighted d_2-transmission number sequence in such a case. We prove the following lemma and theorem using Property 5. Let $G(V,E)$ be a connected graph. We define $\deg(G;v)$ as the degree of each v of G (usually dropping the first term G when the graph is clear from context).

By Property 5, $t_2(G;v)/w + \deg(G;v)$ is constant. Hence this problem comes to the discussion in the case of a degree sequence. But we must consider a connected graph in the case of a weighted d_2-transmission number sequence. (A graph constructed from a degree sequence is not always connected.)

Lemma 3: Let $G(V,E)$ be a connected graph, and let v be a vertex with the minimum degree in V. There exists a connected graph whose degree sequence equals the degree sequence of $G - \{v\}$

Proof: Set $G'(V',E') = G - \{v\}$, and set $\delta = \deg(G;v)$. Let $c(G)$ and $k(G)$ be the number of fundamental circuits and components of G respectively. Then it is known that the following equation holds [7]:

$$k(G') - c(G') = |V'| - |E'|.$$

By the definition of G',

$$k(G') - c(G') = (|V|-1) - (|E|-\delta).$$

If v is not a cut-vertex, then G' is connected. Therefore we consider the case that v is a cut-vertex of G. Then G' is disconnected. We will reconstruct G' to be connected without changing the degree of each vertex of V'. If $\delta = 1$, v is never a cut-vertex. Hence we assume $\delta \geqq 2$. Then $|V| \geqq 3$. It is well-known that the following inequality holds:

$$|E| \geqq \frac{\delta|V|}{2}.$$

Hence

$$\begin{aligned} k(G') - c(G') &\leqq (|V|-1) - (\frac{\delta|V|}{2} - \delta) \\ &\leqq (|V|-2) - \frac{\delta(|V|-2)}{2} + 1 \\ &\leqq \frac{(2-\delta)(|V|-2)}{2} + 1 \leqq 1. \end{aligned}$$

Since $k(G') \geqq 2$ (G' is disconnected),

$$c(G') \geqq k(G') - 1 \geqq 1$$

Hence there exists a component which has a circuit. Let $\{u_1, u_2\}$ be an edge on the circuit of the component, and let $\{u_3, u_4\}$ be an edge of another component. Let G'' be a graph which is obtained by removing $\{u_1, u_2\}$ and $\{u_3, u_4\}$, and adding $\{u_1, u_3\}$ and $\{u_2, u_4\}$ to G'. Then $k(G'') = k(G') - 1$ and $c(G'') = c(G') - 1$, and the degree of each vertex of V' do not change (Fig. 5). Since $k(G'') - c(G'') = k(G') - c(G') \leqq 1$, the similar way is repeated (until $k(\cdot) = 1$) to get a connected graph (of all vertices whose degrees equal to degrees of all vertices in G'). □

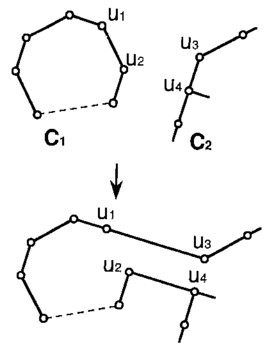

Fig. 5 Edges $\{u_1, u_2\}$ and $\{u_3, u_4\}$ are removed, and edges $\{u_1, u_3\}$ and $\{u_2, u_4\}$ are added to make components C_1 and C_2 connected each other.

We prove the following theorem using this lemma.

Theorem 3: For $p \geq 1$, $S = ((s_0, w), (s_1, w), (s_2, w), \cdots, (s_p, w))$ with $s_0 \geqq s_1 \geqq \cdots \geqq s_p \geqq 0$ and $w \geqq 1$, is a weighted d_2-transmission number sequence if and only if $S' = ((s'_1, w), (s'_2, w) \cdots, (s'_p, w))$ is a weighted d_2-transmission number sequence so that

$$s'_i = \begin{cases} s_i - 2w & \text{if } 1 \leqq i \leqq s_0/w - p \\ s_i - w & \text{if } s_0/w - p < i \leqq p. \end{cases}$$

Proof: Set $D = (2|V| - s_0/w - 2, 2|V| - s_1/w - 2, \cdots, 2|V| - s_p/w - 2)$. By Property 5, S is the weighted d_2-transmission number sequence of a connected graph G if and only if D is the (graphical) degree sequence of G. Generally, a degree sequence:$D = (d_0, d_1, \cdots, d_p)(d_0 \leqq d_1 \leqq \cdots \leqq d_p)$ is graphical if and only if a degree sequence:$D' = (d'_1, d'_2, \cdots, d'_p)$ is graphical so that

$$d'_i = \begin{cases} d_i & \text{if } 1 \leqq i \leqq p - d_0 \\ d_i - 1 & \text{if } p - d_0 < i \leqq p. \end{cases}$$

(This is proved similarly as the proof by *Havel-Hakimi* [1].) Since G is connected and d_0 is minimum in this graph, there exists a connected graph $G'(V', E')$ with the degree sequence D' by Lemma 3. For all i ($1 \leqq i \leqq p$), set $s''_i = 2w \cdot (|V'| - 1) - wd'_i$. By Property 5, D' is a (graphical) degree sequence of G' if and only if $(s''_1, w), (s''_2, w), \cdots, (s''_p, w)$ is a weighted d_2-transmission number sequence of G'. Then if $1 \leqq i \leqq p - d_0 (= s_0/w - p)$,

$$\begin{aligned} s''_i &= 2w \cdot (|V'| - 1) - wd'_i \\ &= 2w \cdot ((|V| - 1) - 1) - wd_i \\ &= 2w \cdot (|V| - 1) - wd_i - 2w \\ &= s_i - 2w = s'_i. \end{aligned}$$

And, if $p - d_0 < i \leqq p$,

$$\begin{aligned} s_i'' &= 2w \cdot (|V'| - 1) - wd_i' \\ &= 2w \cdot ((|V| - 1) - 1) - w(d_i - 1) \\ &= 2w \cdot (|V| - 1) - wd_i - w \\ &= s_i - w = s_i'. \end{aligned}$$

Therefore S is the weighted d_2-transmission number sequence of G if and only if S' is the weighted d_2-transmission number sequence of G'. We have proved this theorem. □

By applying Theorem 3, we can determine whether S is a weighted d_2-transmission number sequence and construct a graph with S in $O(|S|^2)$ time if each weight of S is equal to the others.

6. Conclusion

In this paper, we discussed realizing a vertex-weighted tree from a given sequence whose terms consist of a pair of non-negative integer and positive integer. We proved that w-TNS was *NP*-complete, and we showed the deterministic algorithm for w-TNS which used a backtracking.

Next, we considered d_2-transmission number sequence. We showed the necessary and sufficient condition for an NPI sequence whose weight is equal to the others to be d_2-transmission number sequence.

The following problems are unsolved:

- Average-behavior complexity of Algorithm 2 ?
- Is w-TNS *NP*-complete when each weight of an NPI sequence is 1 ?

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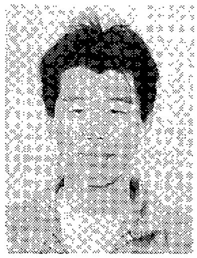
Correction

Please excuse us for mistaking in Ref. [4]. The proof of *NP*-completeness in Sect. 4 is uncorrect. Therefore Theorem 2 is false, and whether the subproblem of w-TNS whose weight is 1 (called TNS in Ref. [4]) is *NP*-complete or not, is unsolved. But w-TNS (which is discussed in this paper) is *NP*-complete.



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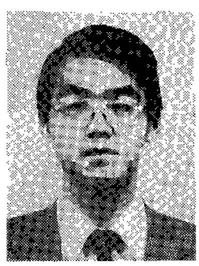

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