

PAPER

The Problem of where to Locate p -Sinks in a Flow Network: Complexity Approach

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SUMMARY The p -collection problem is where to locate p sinks in a flow network such that the value of a maximum flow is maximum. In this paper we show complexity results of the p -collection problem. We prove that the decision problem corresponding to the p -collection problem is strongly NP-complete. Although location problems (the p -center problem and the p -median problem) in networks and flow networks with tree structure is solvable in polynomial time, we prove that the decision problem of the p -collection problem in networks with tree structure, is weakly NP-complete. And we show a polynomial time algorithm for the subproblem of the p -collection problem such that the degree sum of vertices with degree ≥ 3 in a network, is bound to some constant $K \geq 0$.

key words: location problem, flow network, NP-complete, optimization problem

1. Introduction

Recently the authors discussed the problem of where to locate p sinks in a flow network such that the value of a maximum flow is maximum [1], and called this problem the p -collection problem. This problem is an important location problem in a flow network because we can apply this problem to locating p resources (e.g. data-bases, file-servers, etc.) in a computer network such that these p resources can be used by as many terminals (clients) as possible.

A network $N = (D, c, d)$ is a digraph $D = (V, A)$ with a positive integer $c(a)$ (called the capacity of a) associated with each of its $|A|$ arcs, and a nonnegative integer $d(v)$ (called the weight of v) associated with each of its $|V| (= n)$ vertices. Let s be the specified vertex with $s \notin V$ called a *source*, and let A_s be the arc set $\{(s, v) : v \in V \text{ and } d(v) > 0\}$. Let $V^* = V \cup \{s\}$, and let $A^* = A \cup A_s$. Let D^* be the digraph with the vertex set V^* and the arc set A^* . The *adjoint network* $N^* = (D^*, s, c^*)$ of N is the digraph D^* with the source s and the capacity $c^*(a)$ of each arc $a \in A^*$ such that

$$c^*(a) = \begin{cases} c(a) & \text{if } a \in A \\ d(v) & \text{if } a = (s, v) \in A_s. \end{cases}$$

Figure 1 shows examples of a network and its adjoint network.

Let X be an arbitrary subset of V . The flow network N_X^* is the adjoint network N^* with $|X|$ sinks fixed on all vertices of X . The *collection number* $h_N(X)$ of X is the value $val(f)$ of a maximum flow f in N_X^* . Let $H_N(p) = \max\{h_N(X) : |X| = p \text{ and } X \subseteq V\}$. (We usually drop the subscript N in $h_N(X)$ and $H_N(p)$ when the network N is clear from context.) Then we call the value of $H(p)$ the *maximum p -collection number* of N . A *maximum p -collection set* of N is a subset X' of V with $|X'| = p$ such that $h(X') = H(p)$, and the *p -collection problem* is the optimization problem of searching a maximum p -collection set in N .

Similarly we can discuss the p -collection problem in an undirected network M defined as the triple of the graph G with a vertex set V and an edge set E , a weight function d on V , and a capacity function c on E . This undirected version comes to the p -collection problem in a directed network without loss of generality by replacing each edge of the undirected network with two arcs whose orientations are opposite to each other, and whose capacities are equal to the capacity of the replaced edge. The adjoint network M^* of M is defined as the adjoint network of the (directed) network obtained by such a replacement in M .

The authors showed an $O(n)$ time algorithm for the 1-collection problem in a network whose underlying graph is a tree (called a tree network), and a pseudo-polynomial time algorithm for the p -collection problem in a tree network [1]. In this paper we discuss the complexity of the p -collection problem. In Sect. 2 we prove that the decision problem corresponding to the p -collection problem is strongly NP-complete. It is known that location problems (the p -center prob-

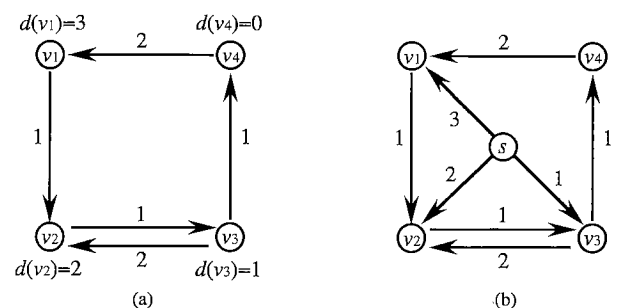


Fig. 1 (a) A network. (b) Its adjoint network.

Manuscript received December 7, 1995.

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lem and the p -median problem) in networks and flow networks restricted within tree networks is solvable in polynomial time [2]–[4]. In Sect. 3 we prove that the decision problem of the p -collection problem is weakly NP-complete even if the decision problem is restricted within classes of very simple networks as undirected star networks whose underlying graphs are stars (where a star is a bipartite graph $K_{(1,n-1)}$). The DSR (Degree Sum Restricted) p -collection problem is the subproblem of the p -collection problem such that the degree sum of vertices with degree ≥ 3 in a network is bound to some constant $K \geq 0$. In Sect. 4 we present a polynomial time algorithm for the DSR p -collection problem.

2. General Network

In this section we prove strong NP-completeness of the p -collection problem in general networks.

First we explain the min-cut max-flow theorem [8]. Generally the min-cut max-flow theorem refers to a flow network with a single source and a single sink, but we can easily extend it to a multi-sink version. Let N_X^* be a flow network with a source s and a set X of sinks, and let V^* be the vertex set of N_X^* . Let Y be a subset of V^* with $s \in Y$ involving no vertices of X , and let $\langle Y, V^* - Y \rangle$ denote the set of arcs incident from some vertex of Y and to some vertex of $V^* - Y$ each in N_X^* (called the cut $\langle Y, V^* - Y \rangle$). Let $\text{cap}(Y)$ denote the sum of capacities of arcs in $\langle Y, V^* - Y \rangle$ (called the capacity of the cut $\langle Y, V^* - Y \rangle$). A cut with capacity $\min_Y \{\text{cap}(Y)\}$ is called a minimum cut in N_X^* . The following lemma is an extended min-cut max-flow theorem. (We can easily verify the lemma by identifying all sinks of X .)

Lemma 1: A cut C is a minimum cut in a flow network with a source and plural sinks if and only if the capacity of C is equal to the value of a maximum flow in the flow network.

Hence $h_N(X)$ is equal to the capacity of a minimum cut. And the following lemma holds.

Lemma 2: Let f be a maximum flow in a flow network with a source and plural sinks. If a is an arc of a minimum cut in the flow network, the value of $f(a)$ is equal to the capacity of a (i.e., the arc a is saturated).

If we prove strong NP-completeness of the p -collection problem in undirected networks, then the p -collection problem (in directed networks) is strongly NP-complete. Thus we deal with undirected networks. We define the decision problem corresponding to the p -collection problem in the undirected network (denoted by CUN) as follows.

Problem 1: CUN

INSTANCE : An undirected network $M = (G, c, d)$ with a graph $G = (V, E)$, a positive integer $p \leq |V|$ and a nonnegative integer $r \leq \sum_{v \in V} d(v)$.

QUESTION : $H(p) \geq r$, i.e., is there $V' \subseteq V$ with $|V'| \leq p$ such that $h(V') \geq r$?

In our proof we will transform the following VERTEX COVER problem (denoted by VC) to CUN.

Problem 2: VC

INSTANCE : A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION : Is there a vertex cover $V' \subseteq V$ with $|V'| \leq k$, i.e., a subset $V' \subseteq V$ with $|V'| \leq k$ such that for each edge $\{u, v\} \in E$ at least one of u and v belongs to V' ?

It is known that VC is NP-complete if a graph is restricted within planar graphs with maximum degree 3 [5]. We prove the following theorem using this result.

Theorem 1: CUN is strongly NP-complete even for planar graphs with maximum degree 3.

Proof: CUN belongs to the class NP since this problem can be solved in polynomial time by guessing a subset V' of V with $|V'| = p$ and by checking $h(V') \geq r$.

We transform an instance of VC to one of CUN. Let $G = (V, E)$ be a planar graph with maximum degree 3, and k be a positive integer. We assign the degree $\deg(v)$ of v to the weight $d(v)$ for each vertex v and one to the capacity $c(e)$ of each edge e . Then the triple of $M = (G, c, d)$, k and $\sum_{v \in V} \deg(v) = r$ is an instance of CUN, and this transformation has been performed in polynomial time. An example of transformation is illustrated in Fig. 2.

It is clear that $h(V') = r$ if a subset V' of V is a vertex cover of G .

Conversely we assume that V' is a subset of V with $h(V') = r$. To show that V' is a vertex cover of G , suppose that there exists an edge $\{u, v\}$ uncovered by V' (i.e., $u, v \notin V'$). Let M^* denote the adjoint network of M , let V^* denote the vertex set of M^* , and let A_s denote the set of arcs incident from s in M^* . In this paper, for a subset X of V^* , we say that an arc (x, y) of M^* is incident from X if $x \in X$ and $y \notin X$. Let B denote the set of arcs incident from the set $\{u, v\}$. (By the definition of “incident from vertex-set”, both arcs

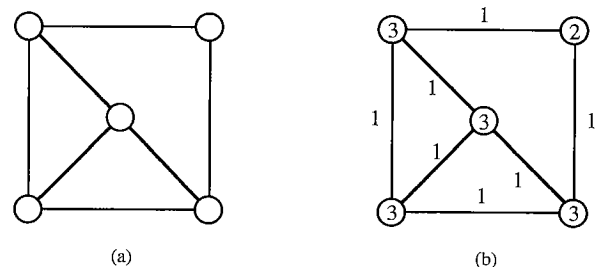


Fig. 2 (a) An instance of VC. (b) Its transformed instance.

(u, v) and (v, u) do not belong to B .) Then

$$\sum_{a \in B} c(a) = \deg(u) + \deg(v) - 2$$

$$< \deg(u) + \deg(v),$$

where $\deg(x)$ denotes the degree of a vertex x in G . However notice that the arc set A_s is the cut $\langle \{s\}, V^* - \{s\} \rangle$ with capacity r in $M_{V'}^*$. Let f be a maximum flow in $M_{V'}^*$ (where the flow network $M_{V'}^*$ is the adjoint network M^* with the sink set V'). Since $\text{val}(f) = h(V') = r$, the arc set A_s is a minimum cut by Lemma 1. From Lemma 2 we obtain $f(s, u) = \deg(u)$ and $f(s, v) = \deg(v)$. Therefore

$$\deg(u) + \deg(v) = f(s, u) + f(s, v)$$

$$\leq \sum_{a \in B} f(a) \leq \sum_{a \in B} c(a).$$

This result is unreasonable, and so V' must be a vertex cover of G . Therefore a subset V' of V with $|V'| \leq k$ is a vertex cover if and only if $h(V') = r$. Thus CUN is NP-complete. However the underlying graph of the transformed network is a planar graph with maximum degree 3, and the maximum number occurring in the transformed instance is three. Hence CUN is strongly NP-complete for planar graphs with maximum degree 3. \square

In the next section we shall restrict CUN within tree networks because the general CUN problem is too hard.

3. Tree Network

In this section we discuss the complexity of the p -collection problem in tree networks. We will obtain an interesting result. It is the fact that, CUN is weekly NP-complete although location problems (e.g. the p -center problem and the p -median problem) in a network and a flow network with tree structure can be solved in polynomial time [2]–[4]. It is known that the following lemma holds [1].

Lemma 3: There exists a pseudo-polynomial time algorithm for the p -collection problem in tree networks.

To begin with, we define CUN in star networks (denoted by CUS) as follows.

Problem 3: CUS

INSTANCE : An undirected network $M = (S, c, d)$ with a star S , a positive integer p and a nonnegative integer r .

QUESTION : $H(p) \geq r$?

In our proof we will transform to CUS the following EQUICARDINAL PARTITION problem (written as EP).

Problem 4: EP

INSTANCE : A positive integer sequence (t_1, \dots, t_{2m}) where m is a positive integer.

QUESTION : Is there a subset I' of $I = \{1, \dots, 2m\}$ with $|I'| = m$ such that $\sum_{i \in I'} t_i = \sum_{i \in I - I'} t_i$?

EP is a subproblem of the PARTITION problem and known to be NP-complete [6]. We prove the following theorem using this result.

Theorem 2: CUS is weakly NP-complete.

Proof: CUS belongs to the class NP since CUS is a subproblem of CUN. We transform an instance of EP to one of CUS. Let (t_1, \dots, t_{2m}) be a positive integer sequence in an instance of EP. Let $Q = \sum_{i \in I} t_i$. Let S be the star with the vertex set $V = \{u, v_1, \dots, v_{2m}\}$ and the edge set $E = \{\{u, v_1\}, \dots, \{u, v_{2m}\}\}$. For each $i \in I$, we assign Q to weight $d(v_i)$ of each vertex v_i and assign t_i to capacity $c(u, v_i)$ of the edge $\{u, v_i\}$, and we associate zero with the weight $d(u)$ of the vertex u . The triple of the undirected network $M = (S, c, d)$, the positive integer m and the nonnegative integer $(2m + 1)Q/2 = R$ is an instance of CUS, and this transformation can be accomplished in polynomial time. An example of a transformed instance is illustrated in Fig. 3.

We assume that I' is a subset of I with $|I'| = m$ such that $\sum_{i \in I'} t_i = \sum_{i \in I - I'} t_i$. Let $V' = \{v_i | i \in I'\}$. Let f be a flow in $M_{V'}^*$ such that

$$f(x, y) = \begin{cases} Q & \text{if } x = s \text{ and } y = v_i \ (i \in I') \\ t_i & \text{if } x = s \text{ and } y = v_i \ (i \notin I') \\ t_i & \text{if } x = v_i \ (i \notin I') \text{ and } y = u \\ t_i & \text{if } x = u \text{ and } y = v_i \ (i \in I') \\ 0 & \text{otherwise.} \end{cases}$$

Since $\text{val}(f) = R$, we have $h(V') \geq \text{val}(f) = R$.

Conversely we suppose that V' is a subset of V with $|V'| = m$ such that $h(V') \geq R$. If $u \in V'$, the inequality $\text{cap}(V^* - V') < mQ$ must hold in $M_{V'}^*$, and by Lemma 1 we obtain $h(V') \leq \text{cap}(V^* - V') < mQ < R$. Thus we have $u \notin V'$. Let I' be the subset of I such that $I' = \{i \in I | v_i \in V'\}$. We hope to show that $\sum_{i \in I'} t_i = \sum_{i \in I - I'} t_i$. To verify that this equation holds, assume the opposite. We examine the case where $\sum_{i \in I'} t_i < Q/2 < \sum_{i \in I - I'} t_i$. By Lemma 1 and $\text{cap}(V^* - V') = \sum_{i \in I'} t_i + mQ$, we obtain

$$h(V') \leq \text{cap}(V^* - V')$$

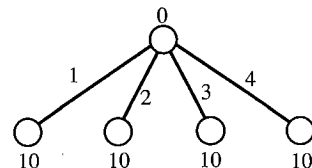


Fig. 3 The instance transformed from the instance $\{1, 2, 3, 4\}$ of EP.

$$\begin{aligned} &\leq \sum_{i \in I'} t_i + mQ \\ &< Q/2 + mQ = R. \end{aligned}$$

In the case where $\sum_{i \in I'} t_i > Q/2 > \sum_{i \in I-I'} t_i$, we have $h(V') < R$ since $\text{cap}(V^* - (V' \cup \{u\})) = \sum_{i \in I-I'} t_i + mQ$. These two cases contradict $h(V') \geq R$, and so $\sum_{i \in I'} t_i = \sum_{i \in I-I'} t_i$.

Hence there exists a subset I' of I with $|I'| = m$ such that $\sum_{i \in I'} t_i = \sum_{i \in I-I'} t_i$ if and only if there exists a subset of V with $|V'| = m$ such that $h(V') \geq R$. By Lemma 3, CUS is weakly NP-complete. \square

Now we study the complexity of the restricted CUN problem such that the maximum degree of an undirected tree network is three. Let M' be the tree network with the vertex set $\{u_1, \dots, u_{2m}, v_1, \dots, v_{2m}\}$ and with the edge set $\{\{u_i, u_{i+1}\} | 1 \leq i < 2m\} \cup \{\{u_j, v_j\} | 1 \leq j \leq 2m\}$ such that $d(u_i) = 0$ ($1 \leq i \leq 2m$), $d(v_i) = Q$ ($1 \leq i \leq 2m$), $c(u_i, u_{i+1}) = Q$ ($1 \leq i < 2m$), and $c(u_i, v_i) = t_i$ ($1 \leq i \leq 2m$). Considering M' as a network transformed from an instance of EP, we obtain the following theorem.

Theorem 3: If an undirected tree network is of maximum degree 3, CUN is weakly NP-complete.

All of the obtained theorems hold in the directed case since the directed case includes the undirected case.

4. Subproblem in P

In this section we show a polynomial time algorithm for the DSR p -collection problem. First we deal with a connected network.

Let $N = (D, c, d)$ be a connected DSR network where D is the digraph with a vertex set V and an arc set A , let $G = (V, E)$ be the underlying graph of D , and let K be a constant binding the degree sum of vertices of V with degree ≥ 3 .

We call a subset W of V a *wall* in G if W is empty, or if there exists an edge e of E such that for any vertex $w \in W$, there exists a w - e path (between the vertex w and the edge e) including no vertices of $W - \{w\}$. Let W be a nonempty wall in G , and let e be an edge such that for any vertex $w \in W$, there exists a w - e path including no vertices of $W - \{w\}$. Let E' be the maximum subset of E such that for any $e' \in E'$, there exists an e - e' path having no vertices of W . Then we call the pair (W, E') a *garden* in G . The wall W is contained in the set of vertices of V incident to edges of E' since each vertex of W is adjacent to some edge of E' . Figure 4 shows an example of a wall and a garden.

We introduce some expressions. Let (W, E') be an arbitrary garden in G . Let V/E' denote the set of vertices of V incident to edges of E' , let G/E' denote the subgraph of G with the vertex set V/E' and the edge set E' , let A/E' denote the arc set $\{(x, y) \in A : \{x, y\} \in E'\}$, and let N/E' denote the subnetwork in N restricted

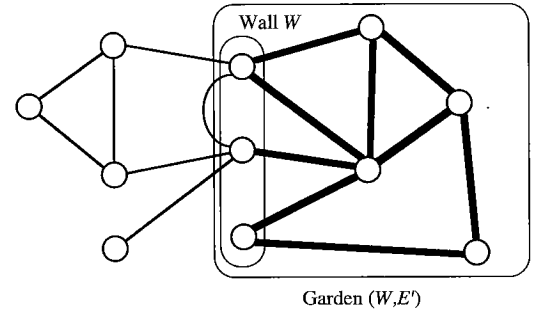


Fig. 4 An example of a wall W and a garden (W, E') . The edges of E' are illustrated with thick lines.

on the digraph with the vertex set V/E' and the arc set A/E' .

Let's consider enumerating all gardens, proving the following lemma.

Lemma 4: Let G be a graph with n vertices and m edges such that the degree sum of vertices with degree ≥ 3 is equal to K . Then the size of each wall in G is at most $K + 2$, the number of walls in G is $O(n^{K+2})$, and the number of gardens in G is $O(mn^{K+2})$.

Proof: To begin with, we estimate the size of a wall W in G . There exists a garden (W, E') with the wall W . By the definition of walls, there exists an edge e of E' such that for any $w \in W$, there exists some w - e path P_w involving no vertices of $W - \{w\}$. Let $G' = (\bigcup_{w \in W} V(P_w), \bigcup_{w \in W} E(P_w))$ where $V(P_w)$ and $E(P_w)$ denote the vertex set and the edge set of P_w respectively. Then each w of W has degree 1 in G' . Let n' denote the number of vertices of G' , and let m' denote the number of edges of G' . It is well known that the degree sum of all vertices of G' is equal to $2m'$, and (if G' is connected,) the inequality $n' \leq m' + 1$ holds [9]. In G' , let V_1 denote the set of vertices with degree 1, and let V_2 denote the set of vertices with degree 2, and let V_3 denote the set of vertices with degree ≥ 3 . Then

$$\begin{aligned} \sum_{v \in V_3} \deg(v) &= 2m' - \sum_{v \in V_1} \deg(v) - \sum_{v \in V_2} \deg(v) \\ &\geq 2n' - 2 - |V_1| - 2|V_2| \\ &\geq 2(n' - |V_2| - |V_1|) - 2 + |V_1| \\ &\geq |V_1| - 2 \geq |W| - 2. \end{aligned}$$

Since $\sum_{v \in V_3} \deg(v) \leq K$, we get $|W| \leq K + 2$.

Let r denote the number of walls in G . The number of walls with size q is equal or less than $\binom{n}{q}$. Hence

$$\begin{aligned} r &\leq \sum_{q=1}^{\min\{n, K+2\}} \binom{n}{q}. \text{ Since } \sum_{q=1}^{\min\{n, K+2\}} \binom{n}{q} \leq \\ &\sum_{q=1}^{\min\{n, K+2\}} n^q \leq \sum_{q=1}^{K+2} n^q, \text{ we obtain } r = O(n^{K+2}). \end{aligned}$$

The number of gardens in G is $O(mn^{K+2})$ since it is equal or less than mr . \square

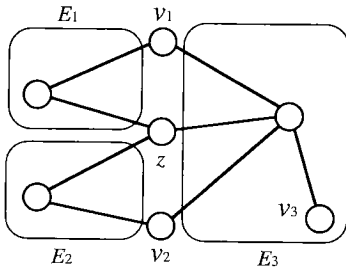


Fig. 5 A garden $\mathcal{G} = (W, E')$ with $W = \{v_1, v_2, v_3\}$, and $\mathcal{E}(\mathcal{G}, z) = \{E_1, E_2, E_3\}$.

Suppose that X is a subset of V with size $\leq K+2$. Using depth-first search, in $O(m)$ time, one can know whether X is a wall, and simultaneously enumerate gardens with the wall X when X is a wall. By Lemma 4 it takes $O(mn^{K+2})$ time to enumerate all walls in G .

Let $\mathcal{G} = (W, E')$ be a garden in G , and consider the subgraph G/E' . If z is a vertex of $V/E' - W$, let $\mathcal{E}(\mathcal{G}, z)$ denote the family of maximal subsets of E' such that in G/E' , there is an e - e' path containing no vertices of $W \cup \{z\}$ for any edges e and e' of each subset, and such that $\mathcal{E}(\mathcal{G}, z)$ is a partition of E' . Figure 5 illustrates an example of such a partition.

To examine properties of such a partition, we prove the following lemma.

Lemma 5: Let G be the graph with a vertex set V and an edge set E , let $\mathcal{G} = (W, E')$ be a garden in G , let z be a vertex of $V/E' - W$, and let $W' = W \cup \{z\}$. Then the following conditions hold.

(I) If E_1 and E_2 are different edge-sets of $\mathcal{E}(\mathcal{G}, z)$, then $V/E_1 - W'$ and $V/E_2 - W'$ are disjoint.

(II) If E_1 and E_2 are different edge-sets of $\mathcal{E}(\mathcal{G}, z)$, then $V/E_1 \cap V/E_2 \subseteq W'$.

(III) The following equations hold.

$$\begin{aligned} V/E' &= \bigcup_{E'' \in \mathcal{E}(\mathcal{G}, z)} V/E'', \\ W' &= \bigcup_{E'' \in \mathcal{E}(\mathcal{G}, z)} W' \cap V/E''. \end{aligned}$$

(IV) For any $E'' \in \mathcal{E}(\mathcal{G}, z)$, the pair $(W' \cap V/E'', E'')$ is a garden in G .

Proof: (I) Assume that there exist different edge-sets E_1 and E_2 of $\mathcal{E}(\mathcal{G}, z)$ such that $(V/E_1 - W') \cap (V/E_2 - W') \neq \emptyset$. Let v be a vertex of $(V/E_1 - W') \cap (V/E_2 - W')$. Then we have $v \in V/E_1$, $v \in V/E_2$, and $v \notin W'$. Hence there exist edges e_1 of E_1 and e_2 of E_2 incident to v . Since $v \notin W'$, there exists e_1 - e_2 path involving no vertex of W' . This consequence contradicts maximality of E_1 and E_2 . Hence we have proved (I).

(II) If $V/E_1 \cap V/E_2 = \emptyset$, clearly $V/E_1 \cap V/E_2 \subseteq W'$.

Hence we consider the case where $V/E_1 \cap V/E_2 \neq \emptyset$. Let v be an arbitrary vertex of $V/E_1 \cap V/E_2$. To establish that $v \in W'$, assume the opposite: $v \notin W'$. Then there must exist edges $e_1 \in E_1$ and $e_2 \in E_2$ incident to v . By $v \notin W'$ there exists an e_1 - e_2 path having no vertices of W' . This result contradicts maximality of E_1 and E_2 .

Hence $v \in W'$. Therefore we obtain $V/E_1 \cap V/E_2 \subseteq W'$.

(III) Let v be an arbitrary vertex of V/E' . Then there exists an edge e of E' incident to v . Since $\mathcal{E}(\mathcal{G}, z)$ is a partition of E' , there exists an edge set E'' of $\mathcal{E}(\mathcal{G}, z)$ with $e \in E''$. Hence we obtain $v \in V/E'' \subseteq \bigcup \{V/F : F \in \mathcal{E}(\mathcal{G}, z)\}$.

Conversely let v be an arbitrary vertex of $\bigcup \{V/F : F \in \mathcal{E}(\mathcal{G}, z)\}$. Then there exists an edge set E'' of $\mathcal{E}(\mathcal{G}, z)$ with $v \in V/E''$, and there exists an edge e of E'' incident to v . Since $e \in E'' \subseteq E'$, we have $v \in V/E'$. Thus we obtain $V/E' = \bigcup \{V/F : F \in \mathcal{E}(\mathcal{G}, z)\}$.

Since $W \subseteq V/E'$ and $z \in V/E'$, we obtain $W' \subseteq V/E'$. Hence we get

$$\begin{aligned} &\bigcup \{W' \cap V/F : F \in \mathcal{E}(\mathcal{G}, z)\} \\ &= W' \cap \bigcup \{V/F : F \in \mathcal{E}(\mathcal{G}, z)\} \\ &= W' \cap V/E' = W' \end{aligned}$$

(IV) Let $W'' = W' \cap V/E''$. To begin with, we show that W'' is a wall in G . If $|E''| = 1$, the pair (W'', E'') is a garden in G . Hence we examine the case where $|E''| \geq 2$. There exists a path between arbitrary two edges of E'' having no vertices of W' in G/E' by the definition of E'' . Since $W'' \subseteq V/E''$, there exists an edge of E'' incident to w for any w of W'' . Hence if e is an edge of E'' , then for any vertex w of W'' , there exists a w - e path in G' including no vertices of $W' - \{w\}$, that is, there exists a w - e path in G containing no vertices of $W'' - \{w\} \subseteq W' - \{w\}$. Therefore W'' is a wall in G .

We should show that (W'', E'') is a garden in G . Let e be an edge of E'' , and let e' be an arbitrary edge of $E - E''$. We suppose that in G , there exists some e - e' path P having no vertices of W'' . Since $e \in E''$ and $e' \in E - E''$, P must have vertices of W' . Let w be the vertex of W' nearest to e in P . As $w \notin W''$, the edge set E'' does not involve any edge incident to w . Let e'' be the edge nearer to e which is incident to w , and which lies in P . Then $e'' \notin E''$, and no vertices of W' lie in the subpath of P between e and e'' . This is a contradiction to maximality of E'' . Thus for any $e' \in E - E''$, there is not any e - e' path in G having no vertices of W'' . Hence the pair (W'', E'') is a garden in G . \square

Now we prove the following lemma to study the collection number of the subnetwork constructed of a garden in G .

Lemma 6: Let $\mathcal{G} = (W, E')$ be a garden in G , let z be a vertex of $V/E' - W$, and let $W' = W \cup \{z\}$. For any subset X of V/E' with $W' \subseteq X$,

$$\begin{aligned} h_{N/E'}(X) &= \sum_{w \in W'} d(w) \\ &= \sum_{i=1}^{|\mathcal{E}(\mathcal{G}, z)|} \left(h_{N/E_i}(X \cap V/E_i) - \sum_{w \in W_i} d(w) \right) \end{aligned}$$

where $E_1, \dots, E_{|\mathcal{E}(\mathcal{G}, z)|}$ are all members of $\mathcal{E}(\mathcal{G}, z)$ and $W_i = W' \cap V/E_i$ for each $1 \leq i \leq |\mathcal{E}(\mathcal{G}, z)|$.

Proof: In this proof, for simplicity, we write m as $|\mathcal{E}(\mathcal{G}, z)|$, and we write \sum_i as $\sum_{i=1}^{|\mathcal{E}(\mathcal{G}, z)|}$.

Let $(N/E')_X^*$ denote the adjoint network of N/E' with a source s and the sink set X , and let V', A' denote the vertex set and the arc set of $(N/E')_X^*$ respectively. For any $1 \leq i \leq m$, let $(N/E_i)_X^*$ denote the adjoint network of N/E_i with the sink set $X \cap V/E_i$, and let V_i, A_i denote the vertex set and the arc set of the adjoint network of $(N/E_i)_X^*$ respectively. Then $V_i = V/E_i \cup \{s\}$, and $A_i = A/E_i \cup \{(s, v) : v \in V/E_i \text{ and } d(v) > 0\}$ for any $1 \leq i \leq m$. Hence, for any $1 \leq i \leq m$, if $(u, v) \in A_i$, then $u, v \in V_i$.

We define functions g, g_1, \dots, g_m on A' as

$$g(u, v) = \begin{cases} d(v) & \text{if } u = s \text{ and } v \in W' \\ 0 & \text{otherwise,} \end{cases}$$

and for any $1 \leq i \leq m$

$$g_i(u, v) = \begin{cases} d(v) & \text{if } u = s \text{ and } v \in W_i \\ 0 & \text{otherwise.} \end{cases}$$

Functions g, g_1, \dots, g_m are flows in $(N/E')_X^*$. Let f be a maximum flow in $(N/E')_X^*$. We define functions f_1, \dots, f_m on A' as

$$f_i(u, v) = \begin{cases} f(u, v) & \text{if } (u, v) \in A_i \\ 0 & \text{otherwise.} \end{cases}$$

We verify whether f_i is a flow in $(N/E')_X^*$ for any $1 \leq i \leq m$. Each function f_i satisfies the capacity constraint, and is flow-conserved on all vertices of $(V' - V_i) - (X \cup \{s\})$ since $f_i(a) = 0$ for any arc a of A' incident to-or-from vertices of $V' - V_i$ (i.e., $a \in A' - A_i$). It remains to show that in $(N/E')_X^*$, each f_i is flow-conserved on any vertex v of $V_i - (X \cup \{s\})$. Such a vertex v must belong to V/E_i . Since $W' \subseteq X$, we have $v \notin W'$. If v is incident to arcs of some A_j ($i \neq j$), then $v \in V/E_j$, and so $v \in V/E_i \cap V/E_j \subseteq W'$ by Lemma 5 (II). Therefore v must not be incident to any arcs of A_j for any $j \neq i$. Thus for any $1 \leq i \leq m$, the function f_i is flow-conserved on all vertices of $V_i - (X \cup \{s\})$, and so f_i turns out to be a flow in $(N/E')_X^*$.

Now we examine values of these functions on each arc (u, v) of A' separating three cases.

Case 1) $u \neq s$: Then the edge $\{u, v\}$ of G/E' must belong to E_k for some $1 \leq k \leq m$. Hence $(u, v) \in A/E_k$, and $(u, v) \notin A/E_i$ for any $i \neq k$. By $u \neq s$ we have $(u, v) \notin A_i$ for any $i \neq k$. Thus we have $f_k(u, v) = f(u, v)$, and $f_i(u, v) = 0$ for any $i \neq k$. However, by $u \neq s$, we have $g(u, v) = 0$, and $g_i(u, v) = 0$ for any $1 \leq i \leq m$. Hence we get $f(u, v) - g(u, v) = \sum_i (f_i(u, v) - g_i(u, v))$.

Case 2) $u = s$ and $v \notin W'$: Then $v \in V/E_k - W'$ for some $1 \leq k \leq m$. For any $i \neq k$, vertex sets $V/E_i - W'$

and $V/E_k - W'$ are disjoint by Lemma 5 (I), and so $v \notin V/E_i - W'$. Since $v \notin W'$, we have $v \notin V/E_i$ for any $i \neq k$. Hence $(u, v) \in A_k$, and $(u, v) \notin A_i$ for any $i \neq k$. Thus we have $f_k(u, v) = f(u, v)$, and $f_i(u, v) = 0$ for any $i \neq k$. However, by $v \notin W'$, we have $g(u, v) = 0$, and $g_i(u, v) = 0$ for any $1 \leq i \leq m$. Hence we get $f(u, v) - g(u, v) = \sum_i (f_i(u, v) - g_i(u, v))$.

Case 3) $u = s$ and $v \in W'$: If i is an index with $v \in W_i$, then $(u, v) \in A_i$ since $v \in W_i \subseteq V/E_i$. If i is an index with $v \notin W_i = W' \cap V/E_i$ conversely, then $v \notin V/E_i$, and so $(u, v) \notin A_i$. Consequently $\{i : v \in W_i\} = \{i : (u, v) \in A_i\} (= I)$. Thus we have $f_i(u, v) = f(u, v)$ for any $i \in I$, and $f_i(u, v) = 0$ otherwise. Therefore $\sum_i f_i(u, v) = \sum_{i \in I} f_i(u, v) = |I|f(u, v)$. We obtain $f(u, v) = d(v)$ (i.e., (u, v) is saturated in $(N/E')_X^*$ with the flow f) since (u, v) joins the source s and the sink v . Hence $\sum_i f_i(u, v) = |I|d(v)$. Since $\sum_i g_i(u, v) = |I|d(v)$, we have $\sum_i f_i(u, v) = \sum_i g_i(u, v)$. Since $f(u, v) = d(v) = g(u, v)$, we get $f(u, v) - g(u, v) = \sum_i (f_i(u, v) - g_i(u, v))$.

These results in the three cases induce $f(u, v) - g(u, v) = \sum_i (f_i(u, v) - g_i(u, v))$ for any $(u, v) \in A'$. Since $f, g, f_1, \dots, f_m, g_1, \dots, g_m$ are flows in $(N/E')_X^*$, we have

$$val(f) - val(g) = \sum_i (val(f_i) - val(g_i)).$$

We know that $val(f) = h_{N/E'}(X)$, that $val(g) = \sum_{w \in W'} d(w)$, and that $val(g_i) = \sum_{w \in W_i} d(w)$ for any $1 \leq i \leq m$. We concentrate our attention on the value of each f_i . For any $1 \leq i \leq m$, let f'_i denote the restriction of f_i on A_i . Then each f'_i is a flow in $(N/E_i)_X^*$. To establish that each f'_i is a maximum flow in $(N/E_i)_X^*$, assume the opposite. Then there exists an f'_i -augmenting directed-path P in $(N/E_i)_X^*$. The directed-path P does not include any arcs incident from s to vertices of W_i since all of these arcs are saturated in $(N/E_i)_X^*$ with the flow f'_i . Hence P is an f -augmenting directed-path in $(N/E')_X^*$. Thus f is not a maximum flow in $(N/E')_X^*$. This consequence is irrational. Thus f'_i is a maximum flow in $(N/E_i)_X^*$. Therefore $val(f_i) = h_{N/E_i}(X \cap V/E_i)$ for any $1 \leq i \leq m$. Hence we have proved Lemma 6. \square

Let (W, E') be a garden in G with $|W| \leq p$. We define that for any nonnegative integer k with $|W| \leq k \leq \min\{p, |V/E'|\}$,

$$H_{N/E'}^W(k) = \max_X h_{N/E'}(X),$$

where X satisfies that $W \subseteq X \subseteq V/E'$ with $|X| = k$. Then the following proposition holds. (Hereafter, for convenience, we refer operators \sum_i, \bigcup_i and \max_k to

$$\sum_{i=1}^{|\mathcal{E}(\mathcal{G}, z)|}, \bigcup_{1 \leq i \leq |\mathcal{E}(\mathcal{G}, z)|} \text{ and } \max_{k_1, \dots, k_{|\mathcal{E}(\mathcal{G}, z)|}} \text{ respectively.)}$$

Proposition 1: If $\mathcal{G} = (W, E')$ is a garden in G with $|W| < p$, then for any $|W| < k \leq \min\{p, |V/E'|\}$,

$$\begin{aligned} & H_{N/E'}^W(k) \\ &= \max_{z \in V/E' - W} \left\{ \max_{\vec{k}} \sum_i \left(H_{N/E_i}^{W_i}(k_i) - \sum_{w \in W_i} d(w) \right) \right. \\ & \quad \left. + \sum_{w \in W \cup \{z\}} d(w) \right\} \end{aligned}$$

with the following condition :

(a) Each variable k_i ($1 \leq i \leq |\mathcal{E}(\mathcal{G}, z)|$) runs from $|W_i|$ to $\min\{k, |V/E_i|\}$ satisfying

$$\sum_{j=1}^{|\mathcal{E}(\mathcal{G}, z)|} (k_j - |W_j|) + |W \cup \{z\}| = k,$$

where $E_1, \dots, E_{|\mathcal{E}(\mathcal{G}, z)|}$ denote all members of $\mathcal{E}(\mathcal{G}, z)$, and W_i denotes $(W \cup \{z\}) \cap V/E_i$ for any $1 \leq i \leq |\mathcal{E}(\mathcal{G}, z)|$.

Proof: Let $m = |\mathcal{E}(\mathcal{G}, z)|$, and let $W' = W \cup \{z\}$. Let Y be a subset of V/E' with $|Y| = k$ such that $W' \subseteq Y$. Then $\max_X h_{N/E'}(X) = \max_z \max_Y h_{N/E'}(Y)$ since the domain on which X runs is equal to one on which Y runs. Fixing z , we should show the following equation.

$$\begin{aligned} & \max_Y h_{N/E'}(Y) - \sum_{w \in W'} d(w) \\ &= \max_{\vec{k}} \sum_i \left(H_{N/E_i}^{W_i}(k_i) - \sum_{w \in W_i} d(w) \right) \end{aligned} \quad (1)$$

First we verify that the left side is equal or less than the right side in Eq.(1). Let Y' be a subset of V/E' with $|Y'| = k$ such that $W' \subseteq Y'$, and such that $h_{N/E'}(Y') = \max_Y h_{N/E'}(Y)$. By $|Y'| = k$ we have $|Y' \cap V/E_i| \leq \min\{k, |V/E_i|\}$ for any $1 \leq i \leq m$. Since $W_i = W' \cap V/E_i \subseteq Y' \cap V/E_i$, we obtain $|W_i| \leq |Y' \cap V/E_i|$ for any $1 \leq i \leq m$. However we can obtain the following equation.

$$\sum_i (|Y' \cap V/E_i| - |W_i|) + |W'| = k \quad (2)$$

(Because calculating this equation tires us, we note it in Appendix.) Hence, for each $1 \leq i \leq m$, there exists a value of variable k_i equal to $|Y' \cap V/E_i|$ satisfying the condition (a). Hence, using Lemma 6, we can verify that

$$\begin{aligned} & h_{N/E'}(Y) - \sum_{w \in W'} d(w) \\ &= \sum_i \left(h_{N/E_i}(Y' \cap V/E_i) - \sum_{w \in W_i} d(w) \right) \\ &\leq \sum_i \left(H_{N/E_i}^{W_i}(|Y' \cap V/E_i|) - \sum_{w \in W_i} d(w) \right) \end{aligned}$$

$$\leq \max_{\vec{k}} \sum_i \left(H_{N/E_i}^{W_i}(k_i) - \sum_{w \in W_i} d(w) \right).$$

Conversely we verify that the left side is equal or greater than the right side in Eq.(1). Let k'_1, \dots, k'_m be values of variables k_1, \dots, k_m respectively such that $\sum_i H_{N/E_i}^{W'_i}(k'_i) = \max_{\vec{k}} \sum_i H_{N/E_i}^{W_i}(k_i)$. Then, for any $1 \leq i \leq m$, there exists a subset Y'_i of V/E_i with $W_i \subseteq Y'_i$ such that $|Y'_i| = k_i$, and $h_{N/E_i}(Y'_i) = H_{N/E_i}^{W_i}(k'_i)$. Let $Y' = \bigcup_i Y'_i$. Now we show that $Y'_i = Y' \cap V/E_i$ for any $1 \leq i \leq m$. Since $Y'_i \subseteq Y'$ and $Y'_i \subseteq V/E_i$, we know that $Y'_i \subseteq Y' \cap V/E_i$. To show that $Y'_i \supseteq Y' \cap V/E_i$, we deal with an arbitrary vertex y of $Y' \cap V/E_i$. In the case where $y \in W'$, the vertex y must belong to $V/E_i \cap W'$, and so $y \in Y'_i$. We examine the case where $y \notin W'$. To establish that $y \in Y'_i$ in this case, we suppose the opposite : $y \notin Y'_i$. Then y must belong to V/E_i and also belong to Y'_j for some $j \neq i$. Hence we have $y \in V/E_i \cap V/E_j$, and so $y \in W'$ by Lemma 5 (II). Since we are examining the case where $y \notin W'$ now, this result is absurd. Hence we have $y \in Y'_i$ also in the case where $w \notin W'$. The consequences in two cases reads to $y \in Y' \cap V/E_i$. Hence we get $Y'_i \supseteq Y' \cap V/E_i$. Consequently we get $Y'_i = Y' \cap V/E_i$ for any $1 \leq i \leq m$. However we can obtain the following equation.

$$|Y'| = k \quad (3)$$

(Because calculating this equation tires us, we note it in Appendix.), and we have $W' = \bigcup_i W_i \subseteq \bigcup_i Y'_i = Y'$. Hence, using Lemma 6, we can verify that

$$\begin{aligned} & \max_{\vec{k}} \sum_i \left(H_{N/E_i}^{W_i}(k_i) - \sum_{w \in W_i} d(w) \right) \\ &= \sum_i \left(h_{N/E_i}(Y'_i) - \sum_{w \in W_i} d(w) \right) \\ &= h_{N/E'}(Y') - \sum_{w \in W'} d(w) \\ &\leq \max_Y h_{N/E'}(Y) - \sum_{w \in W'} d(w) \end{aligned}$$

Therefore we obtain Eq. (1). \square

If N/E' has no sinks without vertices of W (i.e. $|W| = k$), the value of $H_{N/E'}^W(k)$ can be evaluated in $O(|V| \cdot |E| \log |V|^2/|E|)$ time using the maximum flow algorithm [7]. Since a maximum p -collection set of N includes the wall \emptyset , the equation $H_N(p) = H_N^\emptyset(p)$ holds. Thus we should calculate the value of the right side of this equation to obtain the maximum collection number of N . The DSR p -collection problem on a network N is solvable in polynomial time, taking apart subnetworks induced by gardens in G . We show the following algorithm for the DSR p -collection problem using the dynamic-programming approach.

Table 1 Complexity results of subproblems.Restriction A : The maximum degree is bound to some constant $K \geq 3$.Restriction B : The degree sum of vertices with degree ≥ 3 is bound to some constant $K \geq 0$.

| | No restriction | Restriction A | Restriction B |
|---------------|----------------|---------------|---------------|
| Networks | Strongly NPC | Strongly NPC | P |
| Tree networks | Weakly NPC | Weakly NPC | P |
| Star networks | Weakly NPC | P | P |

Algorithm 1:**Step 1:** Enumerate all gardens each with a wall of size equal or less than $\min\{p, K+2\}$.**Step 2:** For any garden (W, E') in G , using the maximum flow algorithm, calculate the value of $H_{N/E'}^W(|W|)$.**Step 3:** $i \leftarrow 2$.**Step 4:** $j \leftarrow \min\{i, p\} - 1$.**Step 5:** For any garden (W, E') in G with $|W| = j$ such that $|V/E'| = i$, using the equation in Proposition 1, calculate the value of $H_{N/E'}^W(k)$ for each k ($|W| < k \leq \min\{i, p\}$).**Step 6:** $j \leftarrow j - 1$.**Step 7:** If $j \geq 0$, go to Step 5.**Step 8:** $i \leftarrow i + 1$.**Step 9:** If $i \leq |V|$, go to Step 4.**Step 10:** Return $H_N^\emptyset(p)$.

It is easy to improve Algorithm 1 to calculate a maximum p -collection set. So we omit its computation. We examine the complexity of Algorithm 1. Let $n = |V|$, let $m = |E|$, and let r be the number of gardens each with a wall of size equal or less than $\min\{p, K+2\}$. We know that Algorithm 1 can perform Step 1 in $O(mn^{K+2})$ time, and Step 2 in $O(rnm \log n^2/m)$ time. The frequency of calculating the value of $H_{N/E'}^W(k)$ in Step 5 is $O(pr)$. The time to evaluate the value of each $H_{N/E'}^W(k)$ in Step 5, is $O(nmk)$. The complexity is $O(pr \cdot nmk)$. By Lemma 4, we know $r = O(mn^{K+2})$. Since $k \leq p$, the complexity of Algorithm 1 is $O(p^2 m^2 n^{K+3})$.

Now we examine the case of the disconnected DSR network N . Let c_1, \dots, c_ω be connected components of N where ω is the number of these components of N , and let $C_k = (\bigcup_{1 \leq i \leq k} V(c_i), \bigcup_{1 \leq i \leq k} E(c_i))$ for any $1 \leq k \leq \omega$ where $V(c_i), E(c_i)$ denote the vertex set and the edge set of c_i for any $1 \leq i \leq \omega$. We can calculate the maximum p -collection number of N by the following algorithm with dynamic-programming type.

Algorithm 2:**Step 1:** Using Algorithm 1, calculate $H_{c_i}(k)$ for $1 \leq i \leq \omega$ and $0 \leq k \leq \min\{p, |V(c_i)|\}$.**Step 2:** For any $1 \leq k \leq \min\{p, |V(c_1)|\}$

$$H_{C_1}(k) \leftarrow H_{c_1}(k).$$

Step 3: $i \leftarrow 2$.**Step 4:** If $i > \omega$, then go to Step 7.**Step 5:** For $0 \leq k \leq \min\{p, |V(C_i)|\}$,

$$H_{C_i}(k) \leftarrow \max_{k_1, k_2} \{H_{C_{i-1}}(k_1) + H_{c_i}(k_2)\},$$

where k_1 runs from zero to $\min\{p, |V(C_{i-1})|\}$, and k_2 runs from zero to $\min\{p, |V(c_i)|\}$ satisfying $k_1 + k_2 = k$.

Step 6: $i \leftarrow i + 1$ and go to Step 4.**Step 7:** Return $H_{C_\omega}(p)$.

The frequency of evaluating the value of each $H_{c_i}(k)$ is $O(\omega p)$. Therefore the complexity of Algorithm 2 is $O(\omega p)$ times of the complexity of Algorithm 1. Hence we obtain the following theorem.

Theorem 4: The DSR p -collection problem belongs to the class P.

The obtained results so far are brought together in Table 1.

5. Conclusion

In this paper we obtained complexity results of some subproblems of the p -collection problem. We proved that the decision problem corresponding to the p -collection problem is strongly NP-complete, and that the decision problem in a tree network is weakly NP-complete. And we showed a polynomial time algorithm for the restricted p -collection problem such that the degree sum of vertices with degree equal or more than 3 is bound to some constant $K \geq 0$. Although the restricted problem is in P, the algorithm may be practically useless unless K is enough small because it is exponential in terms of K . So we need to decrease its complexity. We will consider an approximation algorithm for the p -collection problem.

References

- [1] K. Watanabe, H. Tamura, and M. Sengoku, "Problems of where to locate p -sinks in a flow network," IEICE Trans., vol. J78-A, no. 8, pp. 938–946, 1995.
- [2] O. Kariv and S.L. Hakimi, "An algorithmic approach to network location problems. I: The p -centers," SIAM J. Appl. Math., vol. 37, no. 3, pp. 513–538, 1979.
- [3] O. Kariv and S.L. Hakimi, "An algorithmic approach to network location problems. II: The p -medians," SIAM J. Appl. Math., vol. 37, no. 3, pp. 539–560, 1979.
- [4] H. Tamura, M. Sengoku, S. Shinoda, and T. Abe, "Location problems on undirected flow networks," IEICE Trans., vol. E73, no. 12, pp. 1989–1993, 1990.
- [5] M.R. Garey and D.S. Johnson, "The rectilinear Steiner tree problem is NP-complete," SIAM J. vol. 32, pp. 826–834, 1977.
- [6] M.R. Garey and D.S. Johnson, "Computers and Intractability: A Guide to the Theory of NP-completeness," Freeman,

San Francisco, 1979.

- [7] A.V. Goldberg and R.E. Tarjan, "A new approach to the maximum flow problem," Proc. the 18th Annual ACM Symposium on Theory of Computing, pp.136–146, 1986.
- [8] R.A. Ahuja, T.L. Magnanti, and J.B. Orlin, "Network Flows," Prentice Hall, 1993.
- [9] M. Iri, et al., "Graph Theory with Exercises," Corona Publ., 1983.

Appendix A: Checking Eq. (2)

For any $1 \leq i \leq m$, from $W_i = W' \cap V/E_i$,

$$\begin{aligned} |Y' \cap V/E_i| - |W_i| \\ = |Y' \cap V/E_i| - |W' \cap V/E_i|, \end{aligned}$$

by $W' \subseteq Y'$

$$\begin{aligned} &= |(Y' - W') \cap V/E_i| \\ &= |Y' \cap (V/E_i - W')|. \end{aligned}$$

By Lemma 5(I), for any $1 \leq i, j \leq m$ ($i \neq j$), we know that $V/E_i - W'$ and $V/E_j - W'$ are disjoint, and so $Y' \cap (V/E_i - W')$ and $Y' \cap (V/E_j - W')$ are disjoint. Hence we get

$$\begin{aligned} \sum_i (|Y' \cap V/E_i| - |W_i|) + |W'| \\ = \sum_i (|Y' \cap (V/E_i - W')|) + |W'| \\ = |\bigcup_i (Y' \cap (V/E_i - W'))| + |W'| \\ = |Y' \cap (\bigcup_i V/E_i - W')| + |W'|, \end{aligned}$$

by Lemma 5(III)

$$= |Y' \cap (V/E' - W')| + |W'|,$$

as $Y' \cap (V/E' - W')$ and W' are disjoint

$$\begin{aligned} &= |(Y' \cap (V/E' - W')) \cup W'| \\ &= |(Y' \cup W') \cap ((V/E' - W') \cup W')|, \end{aligned}$$

by $W' \subseteq Y'$

$$= |Y' \cap (V/E' \cup W')|,$$

by $Y' \subseteq V/E'$

$$= |Y' \cap V/E'| = |Y'| = k.$$

Appendix B: Checking Eq. (3)

By Lemma 5(III) we know $W' = \bigcup_i W_i$. Hence $W' = \bigcup_i W_i \subseteq \bigcup_i Y'_i$. Thus

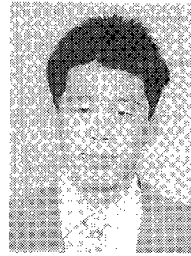
$$\begin{aligned} Y' &= \bigcup_i Y'_i \\ &= (\bigcup_i Y'_i - W') \cup W' \\ &= \bigcup_i (Y'_i - W') \cup W'. \end{aligned}$$

Since $Y'_1 - W', \dots, Y'_m - W'$ and W' are disjoint, we get

$$\begin{aligned} |Y'| &= |\bigcup_i (Y'_i - W') \cup W'| \\ &= \sum_i |Y'_i - W'| + |W'| \\ &= \sum_i |Y'_i - W' \cap Y'_i| + |W'| \\ &= \sum_i (|Y'_i| - |W' \cap Y'_i|) + |W'|. \end{aligned}$$

Since $W_i \subseteq W'$ and $W_i \subseteq Y'_i$, we obtain $W_i \subseteq W' \cap Y'_i$. However, since $Y'_i \subseteq V/E_i$, we obtain $W' \cap Y'_i \subseteq W' \cap V/E_i = W_i$. Thus $W_i = W' \cap Y'_i$. Hence

$$\begin{aligned} |Y'| &= \sum_i (|Y'_i| - |W_i|) + |W'| \\ &= \sum_i (|Y'_i| - |W' \cap Y'_i|) + |W'| \\ &= k. \end{aligned}$$



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