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On a Generalization of a Covering Problem Called Single Cover on Undirected Flow Networks

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SUMMARY Location theory on networks is concerned with the problem of selecting the best location in a specified network for facilities. Many studies for the theory have been done. We have studied location theory from the standpoint of measuring the closeness between two vertices by the capacity (maximum flow value) between two vertices. In a previous paper, we have considered location problems, called covering problems and proposed polynomial time algorithms for these problems. These problems are applicable to assigning files to some computers in a computer network. This paper is concerned with a covering problem called the single cover problem defined in the previous paper. First, we define a generalized single cover problem and show that an algorithm proposed in the previous paper can be applicable to solving the generalized single cover problem. Then, we define a single cover problem satisfying cardinality constrains and show that the problem is solved in a polynomial time.

key words: *graphs and networks, flow network, location theory, covering problem*

1. Introduction

Location theory [1] on networks is concerned with the problem of selecting the best location in a specified network for facilities. Many studies for the theory have been done. We have studied location theory from the standpoint of measuring the closeness between two vertices by the capacity (maximum flow value) between two vertices [2]-[4]. In Ref. [3], we have considered location problems, called covering problems and defined two types of covering problems called the single cover problem and the plural cover problem. These problem can be solved in polynomial times and these problems are applicable to assigning files to some computers in a computer network. This paper is concerned with a covering problem called the single cover problem. We define a generalized single cover problem. We show that an algorithm proposed in Ref. [3] can be applicable to solving the generalized single cover problem. Then, we define a single cover problem satisfying cardinality constrains and show that the problem is solved in a polynomial time. For general terminology in graph theory, we refer the reader to Refs. [5] and [6].

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2. Definitions and a Fundamental Theorem

Let us consider an *undirected flow network* $N=(V, E, c_N)$ where V, E, and c_N are the vertex set, the edge set and the function assigning a positive real number $c_N(e)$ to each edge $e\in E$, respectively. $c_N(e)$ represents the edge capacity of e. The maximum flow value between two vertices u and v in N is called the *capacity* between u and v, denoted by $g_N(u, v)$. Especially, we define $g_N(v, v)=\infty$. In this paper, we attach a vertex weight function $h(\cdot)$ assigning positive real number to N. A subset U of V is called an *$h(\cdot)$-single cover* if $\max\{g_N(u, v)|u\in U\}\geqq h(v)$ for any $v\in V$. If U is an $h(\cdot)$-single cover having minimum cardinality, then U is called a *solution of the $h(\cdot)$-single cover problem.* In Ref. [3], we show that if the value of $h(v)$ is a constant for each vertex v, then a solution of the $h(\cdot)$-single cover problem can be obtained in a polynomial time. In Fig. 1, a number on each edge e represents $c_N(e)$ and a number in each vertex v represents $h(v)$. Let $U_1=\{v_1, v_4, v_6\}$. Since

$$\max\{g_N(u, v_2)|u\in U_1\}=g_N(v_1, v_2)=5 \geqq h(v_2)\ (=3),$$
$$\max\{g_N(u, v_3)|u\in U_1\}=g_N(v_4, v_3)=4 \geqq h(v_3)\ (=4),$$
$$\max\{g_N(u, v_5)|u\in U_1\}=g_N(v_4, v_5)=4$$

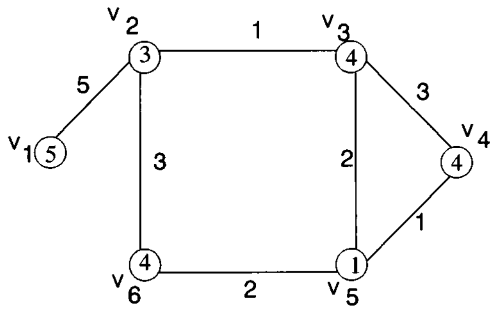

Fig. 1 A network N.

$$\geqq h(v_5)\,(=1),$$

$\max\{g_N(u, v)\,|\,u\in U_1\}\geqq h(v)$ for each $v\in V$. Therefore U_1 is an $h(\cdot)$-single cover. However $|U_1|$ is not minimum. $U_2=\{v_1, v_4\}$ is a solution of the $h(\cdot)$-single cover problem.

In a communication network, a vertex represents a terminal computer and an edge represents a link between computers. We assign a file to computers of the network. The file is copied and assigned to some computers. How do we assign these files ? We assume that the delay time to transport the data of a file can be ignored in this network. In this case, for each terminal computer pair, the number of links between the two computers is the measure representing the closeness between the two computers. Location theory on flow networks is applicable to the above case. If for each computer, the file which can be used by the computer is fixed, the $h(\cdot)$-single cover problem can be applied.

Let W be a non-empty subset of V. W is called a *stable set with respect to single cover* if $g_N(v_2, v_1)\geqq h(v_1)$ and $g_N(v_1, v_2)\geqq h(v_2)$ for any $v_1, v_2\in W$. We simply call a stable set with respect to single cover a *stable set* hereafter. And W is called a *maximal stable set* if it is not properly contained in any other stable set. Since $g_N(v, v)=\infty$, $\{v\}$ is a stable set for any $v\in V$. Therefore for any vertex v, there exists a maximal stable set containing v. For a maximal stable set W, let $h(W)=\max\{h(v)\,|\,v\in W\}$ in this paper. Let W_1 and W_2 be two maximal stable sets. We say that W_2 is *reliant* on W_1, if $g_N(v_1, v_2)\geqq h(v_2)$ for any $v_1\in W_1$, $v_2\in W_2$. If for any maximal stable set $W\,(\neq W_2)$, W_2 is not reliant on W, we call W_2 a *self-reliant set*. In Fig. 1, $\{v_1, v_2\}$ and $\{v_2, v_6\}$ are maximal stable sets and $\{v_2, v_6\}$ is reliant on $\{v_1, v_2\}$. $\{v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ are all self-reliant sets in Fig. 1.

The following theorem is a fundamental theorem [7] concerning maximum flow.

Theorem 1: Let N be an undirected flow network and let u, v and w be vertices. Then the following expressions hold.

(i) $g_N(u, v)=g_N(v, u)$,

(ii) $g_N(u, v)\geqq\min\{g_N(u, w), g_N(w, v)\}$. □

3. The $h(\cdot)$-Single Cover Problem

In this section, we give some properties concerned with the $h(\cdot)$-single cover problem and show that an algorithm proposed in Ref. [3] can be applicable to solving the $h(\cdot)$-single cover problem.

Lemma 1: Let N be an undirected flow network and let W_1 and W_2 be two distinct maximal stable sets. If W_2 is reliant on W_1, then W_1 is not reliant on W_2.

Proof: We assume that W_2 is reliant on W_1 and W_1 is reliant on W_2. Let v_1 be any vertex of W_1 and v_2 be any vertex of W_2. Since W_2 is reliant on W_1, $g_N(v_1, v_2)\geqq h(v_2)$. Since W_1 is reliant on W_2, $g_N(v_2, v_1)\geqq h(v_1)$. Therefore $W_1\cup W_2$ is a stable set. Since $W_1\neq W_2$, W_1 or W_2 is a proper subset of $W_1\cup W_2$. This contradicts the fact that W_1 and W_2 are maximal stable sets. □

Lemma 2: Let N be an undirected flow network and let W_1 and W_2 be two maximal stable sets. If W_2 is reliant on W_1, then $h(W_1)\geqq h(W_2)$.

proof: We assume that W_2 is reliant on W_1 and $h(W_1)<h(W_2)$. Let $w_i\in W_i$ where $h(w_i)=h(W_i)$ $(i=1, 2)$. Note that $w_2\notin W_1$ since $h(W_2)>h(W_1)$. Since W_2 is reliant on W_1, $g_N(v_1, w_2)\geqq h(w_2)$ for any $v_1\in W_1$. Since $h(w_2)>h(w_1)\geqq h(v_1)$, $g_N(v_1, w_2)>h(v_1)$. Therefore, $W_1\cup\{w_2\}$ is a stable set, contradicting the fact that W_1 is a maximal stable set. Therefore, $h(W_1)\geqq h(W_2)$. □

From the above discussion, it follows that the relation "is reliant on" is transitive.

Lemma 3: Let N be an undirected flow network and let W_1, W_2 and W_3 be three maximal stable sets. If W_2 is reliant on W_1 and W_3 is reliant on W_2, then W_3 is reliant on W_1.

Proof: We show that for any $v_1\in W_1$ and $v_3\in W_3$, $g_N(v_1, v_3)\geqq h(v_3)$. Let w_2 be a vertex in W_2 where $h(w_2)=h(W_2)$. Since W_2 is reliant on W_1, $g_N(v_1, w_2)\geqq h(w_2)$. Since W_3 is reliant on W_2, $g_N(w_2, v_3)\geqq h(v_3)$. From Lemma 2,

$$h(w_2)=h(W_2)\geqq h(W_3)\geqq h(v_3).$$

From Theorem 1,

$$g_N(v_1, v_3)\geqq\min\{g_N(v_1, w_2), g_N(w_2, v_3)\}\geqq h(v_3).$$

Therefore, W_3 is reliant on W_1. □

Now, we give a necessary and sufficient condition for U to be an $h(\cdot)$-single cover as follows.

Lemma 4: Let N be an undirected flow network and let U be a subset of V. A necessary and sufficient condition for U to be an $h(\cdot)$-single cover is

$$W\cap U\neq\phi \text{ for any self-reliant set } W.$$

Proof: First, we prove the necessity. Let U be an $h(\cdot)$-single cover and let W be a self-reliant set. We assume that $W\cap U=\phi$. Let w be a vertex of W where $h(w)=h(W)$. Since U is an $h(\cdot)$-single cover, there exists a vertex u_0 of U such that $g_N(u_0, w)\geqq h(w)$.

Now we assume that $h(w)\geqq h(u_0)$. Since $g_N(v, w)\geqq h(w)$ for any v of W,

$$g_N(v, u_0)\geqq\min\{g_N(v, w), g_N(w, u_0)\}\geqq h(w)$$
$$\geqq h(u_0).$$

Since $g_N(v, u_0)\geqq h(u_0)$ and $g_N(v, u_0)\geqq h(w)\geqq h(v)$, $W\cup\{u_0\}$ is a stable set. This contradicts the fact that W is a maximal stable set. Therefore, $h(w)<h(u_0)$.

Let W' be a maximal stable set containing u_0. Since $u_0\notin W$, $W'\neq W$. For any $v'\in W'$ and $v\in W$,

$$g_N(v'v)\geqq\min\{g_N(v', u_0), g_N(u_0, w), g_N(w, v)\}$$

$$\geq \min\{h(u_0), h(w), h(w)\} \geq h(w)$$
$$\geq h(v).$$

Therefore W is reliant on W'. This contradicts the fact that W is self-reliant. Hence, $W \cap U \neq \phi$.

Conversely, we assume that $W \cap U \neq \phi$ for any self-reliant set W of N. We show that for any $v \in V$, there exists $u \in U$ such that $g_N(u, v) \geq h(v)$. Let W_0 be a maximal stable set containing v.

If W_0 is self-reliant, then there exists a vertex $u \in W_0 \cap U$. From the definition of a stable set, $g_N(u, v) \geq h(v)$.

If W_0 is not self-reliant, then there exists a self-reliant set W' such that W_0 is reliant on W' from Lemma 1 and Lemma 3. From the assumption, there exists a vertex $u \in W' \cap U$. Since W_0 is reliant on W', $g_N(u, v) \geq h(v)$.

Therefore, U is an $h(\cdot)$-single cover. □

Lemma 5: Let N be an undirected flow network, and let W_1 and W_2 be two maximal stable sets and $W_1 \cap W_2 \neq \phi$. Then,

W_2 is reliant on W_1 if and only if
$$h(W_1) \geq h(W_2).$$

Proof: Let $v \in W_1 \cap W_2$. First, we assume that $h(W_1) \geq h(W_2)$. We show that W_2 is reliant on W_1. Let $w_i \in W_i$ where $h(w_i) = h(W_i)$ $(i=1, 2)$, and let v_1 be any vertex of W_1 and v_2 be any vertex of W_2. Since W_1 and W_2 are stable sets,

$$g_N(v, v_2) \geq h(v_2),\ g_N(v, w_1) \geq h(w_1) \text{ and}$$
$$g_N(v_1, w_1) \geq h(w_1).$$

Since $h(w_1) \geq h(w_2) \geq h(v_2)$,

$$g_N(v_1, v_2) \geq \min\{g_N(v_1, w_1), g_N(w_1, v), g_N(v, v_2)\}$$
$$\geq h(v_2)$$

from Theorem 1. Therefore, $g_N(v_1, v_2) \geq h(v_2)$, so that, W_2 is reliant on W_1.

Conversely, we assume that W_2 is reliant on W_1. From Lemma 2, $h(W_1) \geq h(W_2)$. □

Lemma 6: Let N be an undirected flow network. If W_1 and W_2 are two distinct self-reliant sets, then $W_1 \cap W_2 = \phi$.

Proof: We assume that $W_1 \cap W_2 \neq \phi$. From Lemma 5, W_1 is reliant on W_2 or W_2 is reliant on W_1. This contradicts the fact that W_1 and W_2 are self-reliant. Therefore $W_1 \cap W_2 = \phi$. □

From Lemmas 4 and 6, we obtain the following theorem.

Theorem 2: Let N be an undirected flow network. A necessary and sufficient condition to U to be a solution of the $h(\cdot)$-single cover problem is the following.

U is an $h(\cdot)$-single cover and there does not exist a proper subset U' of U such that U' is an $h(\cdot)$-single cover.

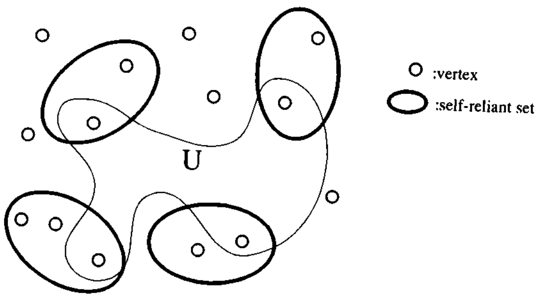

Fig. 2 Explanation for the proof of Theorem 2.

Proof: The necessity is clear.

From Lemma 4 and Lemma 6, U is a solution of the $h(\cdot)$-single cover problem if and only if for any self-reliant set W, $|U \cap W| = 1$ (see Fig. 2). So, the sufficiency is also clear. □

From the above theorem, a solution of the $h(\cdot)$-single cover problem can be obtained by the following simple algorithm. The following algorithm is striking similar to an algorithm for the original single cover problem [3].

Algorithm SINGLE_COVER_A

Input: an undirected flow network $N=(V, E, c_N)$ and a vertex weight function $h(\cdot)$

Output: a solution U, of the $h(\cdot)$-single cover problem

```
      begin
A1     U:=V;(* V={v1, ..., vn}*)
A2     for i=1 to n do
          begin
A3           if U-{vi} is an h(·)-single cover then
                U:=U-{vi}
          end
      end.
```

Obviously, an output U of the above algorithm is an $h(\cdot)$-single cover. We show that there does not exist a proper subset U' of U such that U' is an $h(\cdot)$-single cover. If U-$\{v_j\}$ is an $h(\cdot)$-single cover then v_j is deleted from U in the case of $i=j$ in the algorithm. So, U-$\{v_j\}$ is not an $h(\cdot)$-single cover. Clearly, any subset of U-$\{v_j\}$ is not an $h(\cdot)$-single cover. Therefore there does not exist a proper subset U' of U such that U' is an $h(\cdot)$-single cover. From Theorem 2, U is a solution of the $h(\cdot)$-single cover problem.

We obtain the capacities between all vertex pairs in $O(|V|s(|V|, |E|))$ time [8], where $s(|V|, |E|)$ is the time required to solve a maximum flow problem in N (the best time bound for $s(|V|, |E|)$ known to date is $O(|V||E|\log(|V|^2/|E|))$ [9]). Since judging whether U-$\{v_i\}$ is an $h(\cdot)$-single cover or not can be obtained in $O(|V|^2)$ time, the time complexity of Algorithm SINGLE_COVER_A is $O(|V|s(|V|, |E|) + |V|^3)$. However, using suitable data structures, judging whether U-$\{v_i\}$ is an $h(\cdot)$-single cover or not can be obtained in $O(|V|)$ time (see Appendix). In this case, a solution of the $h(\cdot)$-single cover problem can be obtained in $O(|V|s(|V|, |E|)+|V|^2) (=O(|V|s(|V|, |E|)))$ time.

4. A Single Cover Satisfying Cardinality Constraints

On the file assigning problem in a computer network discussed in Sect. 2, we had better avoid excessive accesses to a specific computer having the file. Therefore, we define a single cover satisfying cardinality constraints in this section. A similar problem on networks from the standpoint of measuring the closeness between two vertices by the distance between two vertices has been discussed in Ref. [10].

Let N be an undirected flow network and let W be a non-empty subset of V. W is called a *territory with respect to single cover* if there exists a vertex $w \in W$ such that for any $v' \in W$, $g_N(w, v') \geqq h(v')$. And such a w is called a *mother vertex* of W. We simply call a territory with respect to single cover a *territory* hereafter. Let $\mathcal{J}=\{W_1, \cdots, W_t\}$ be a vertex partition, namely for each i, j $(i \neq j)$. $W_i \cap W_j = \phi$, and $\bigcup_{i=1}^{t} W_i = V$. And let r be a positive integer. $\mathcal{J}$ is called an *$h(\cdot)$-single cover partition satisfying cardinality constraints* if W_i is a territory and $|W_i| \leqq r$ for each i. We simply call an $h(\cdot)$-single cover partition satisfying cardinality constraints an *$h(\cdot)$-single cover partition* hereafter. Note that there exist $h(\cdot)$-single cover partitions for any r since $\{v\}$ is a territory for each $v \in V$. $\mathcal{J}$ is called a *solution of the $h(\cdot)$-single cover partition problem* if $\mathcal{J}$ has minimum cardinality of all $h(\cdot)$-single cover partitions. For example, in Fig. 3, $\{v_1, v_2, v_3\}$ is a territory and v_1 is its mother vertex (v_2 is its mother vertex too). $\mathcal{J}=\{\{v_1, v_2, v_3, v_4\}, \{v_5\}, \{v_6, v_7\}, \{v_8, v_9\}\}$ is an $h(\cdot)$-single cover partition in the case of $r=4$. Note that any stable set is a territory and the set of mother vertices in an $h(\cdot)$-single cover partition is an $h(\cdot)$-single cover. On the file assigning problem in a computer network discussed in Sect. 2, mother vertices mean computers having the file and each vertex, that means a computer, in a territory accesses the mother vertex of the territory.

A territory has following properties.

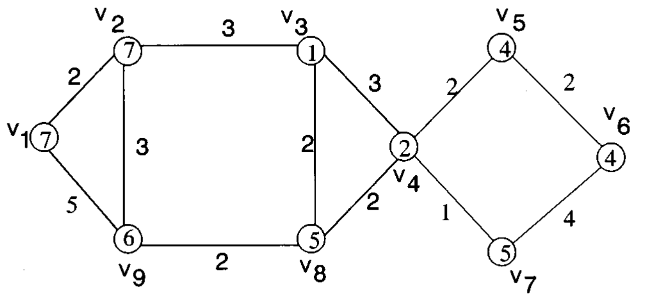

Fig. 3 A network N.

Lemma 7: Let N be an undirected flow network, let W be a territory and let v be a vertex where $v \notin W$. If there exists a vertex $v_0 \in W$ such that $h(v_0) \geqq h(v)$ and $g_N(v_0, v) \geqq h(v)$, then $W \cup \{v\}$ is a territory.

Proof: If v_0 is a mother vertex of W, then $W \cup \{v\}$ is clearly a territory. We assume that $w(\neq v_0)$ is a mother vertex of W.

$$g_N(w, v) \geqq \min\{g_N(w, v_0), g_N(v_0, v)\} \geqq h(v)$$

since $g_N(w, v_0) \geqq h(v_0)$, $g_N(v_0, v) \geqq h(v)$ and $h(v_0) \geqq h(v)$. Therefore, $W \cup \{v\}$ is a territory. □

Lemma 8: Let N be an undirected flow network. If W is a territory, then any non-empty subset W' of W is a territory.

Proof: Let w be a mother vertex of W and let $w' \in W'$ where $h(w') = \max\{h(v) | v \in W'\}$. We show that $g_N(w', v') \geqq h(v')$ for any $v' \in W'$. Since w is a mother vertex of W, $g_N(w, v') \geqq h(v')$ and $g_N(w, w') \geqq h(w')$. Since $h(w') \geqq h(v')$,

$$g_N(w', v') \geqq \min\{g_N(w', w), g_N(w, v')\} \geqq h(v').$$

Therefore, W' is a territory of N and w' is a mother vertex of W'. □

From the above lemmas, we obtain the following simple algorithm for the $h(\cdot)$-single cover partition problem.

Algorithm SINGLE_COVER_B

Input: an undirected flow network $N=(V, E, c_N)$, a vertex weight function $h(\cdot)$ and a positive integer r

Output: a solution $\mathcal{J}$, of the $h(\cdot)$-single cover partition problem

```
        begin
B1      sort all the vertices in order of h(·);(* let h(v1)
        ≧···≧h(vn)*)
B2      V':=V; J:=φ;
B3      while V'≠φ do
            begin
B4            let vi∈V' where  h(vi)=max{h(v)|v∈
              V'};
B5            Wi:={vi}; V':=V'-{vi};
B6            for j=1 to n do
                begin
B7                if vj∈V' then
                    begin
B8                    if gN(vi, vj)≧h(vj) and |Wi|r
                      then
                        begin
B9                        V':=V'-{vj};
B10                       Wi:=Wi∪{vj}
                        end
                    end
                end
B11           J:=J∪{Wi}
            end
      end.
```

We prove the correctness of the above algorithm.

Theorem 3: Algorithm SINGLE_COVER_B correctly finds a solution of the $h(\cdot)$-single cover partition problem.

Proof: Let $\mathcal{T}$ be an output of Algorithm SINGLE_COVER_B. Obviously, $\mathcal{T}$ is an $h(\cdot)$-single cover partition. We show that the $\mathcal{T}$ has minimum cardinality of all $h(\cdot)$-single cover partitions.

Now let $\mathcal{T}=\{W_{i_1}, W_{i_2}, \cdots\}$ where $i_1<i_2<\cdots$. Note that $v_{i_j}\in W_{i_j}$ for each j. We show that there exists a solution that contains $W_{i_1}, \cdots, W_{i_t}$ for each t.

First, we assume that there exists a solution that contains $W_{i_1}, \cdots, W_{i_k}(k\geqq 1)$. Under the assumption, we show that there exists a solution that contains $W_{i_1}, \cdots, W_{i_k}, W_{i_{k+1}}$. Let $\mathcal{T}_0$ be a solution of the $h(\cdot)$-single cover partition problem with the following properties.

1) $\mathcal{T}_0$ contains $W_{i_1}, \cdots, W_{i_k}$.

2) $W_{i_{k+1}}\cap W$, where $v_{i_{k+1}}\in W\in\mathcal{T}_0$, has maximum cardinality of all solutions that contain $W_{i_1}, \cdots, W_{i_k}$.

From a property of Algorithm SINGLE_COVER B,

$$h(v_{i_{k+1}})=\max\{h(v)\mid v\in V-W_{i_1}-\cdots-W_{i_k}\}.$$

Let w_0 be a mother vertex of W. For any $w\in W$, $g_N(w_0, w)\geqq h(w)$.
Since $g(w_0, v_{i_{k+1}})\geqq h(v_{i_{k+1}})$ and $h(v_{i_{k+1}})\geqq h(w)$,

$$g_N(v_{i_{k+1}}, w)\geqq\min\{g(v_{i_{k+1}}, w_0), g_N(w_0, w)\}$$
$$\geqq h(w).$$

So, for any $w\in W$, $g_N(v_{i_{k+1}}, w)\geqq h(w)$.

Now, we assume that $W\neq W_{i_{k+1}}$.

If $W_{i_{k+1}}\subset W$, then there exists a vertex $w\in W-W_{i_{k+1}}$, $w\in V-W_{i_1}-\cdots-W_{i_k}-W_{i_{k+1}}$, $g_N(v_{i_{k+1}}, w)\geqq h(w)$ and $|W_{i_{k+1}}|<r$, which contradicts the construction of $W_{i_{k+1}}$. Therefore, $W_{i_{k+1}}\not\subset W$.

Let $v\in W_{i_{k+1}}-W$. There exists $W'\in\mathcal{T}_0$ such that $v\in W'$. Note that $W'\neq W_{i_1}, \cdots, W_{i_k}$ (see Fig. 4). Since $g_N(v_{i_{k+1}}, v)\geqq h(v)$ and $h(v_{i_{k+1}})\geqq h(v)$, $W\cup\{v\}$ is a territory from Lemma 7.

We assume that $W\subset W_{i_{k+1}}$, namely $|W|<r$. If $W'-\{v\}\neq\phi$, then $W'-\{v\}$ is a territory from Lemma 8. Therefore, $(\mathcal{T}_0-\{W, W'\})\cup\{W\cup\{v\}, W'-\{v\}\}$ is also a solution of the $h(\cdot)$-single cover partition problem, contradicting the defining property of $\mathcal{T}_0$. If $W'-\{v\}=\phi$, then $(\mathcal{T}_0-\{W, W'\})\cup\{W\cup\{v\}\}$ is an $h(\cdot)$-single cover partition. This contradicts the fact that $\mathcal{T}_0$ is a solution of the $h(\cdot)$-single cover partition problem. Therefore, $W\not\subset W_{i_{k+1}}$.

Let $v'\in W-W_{i_{k+1}}$ (see Fig. 4). Since $v\in W_{i_{k+1}}$ and $v'\in W$,

$$g_N(v_{i_{k+1}}, v)\geqq h(v) \text{ and } g_N(v_{i_{k+1}}, v')\geqq h(v'),$$

Therefore, $h(v)\geqq h(v')$ from the construction of $W_{i_{k+1}}$ since $v\in W_{i_{k+1}}$ and $v'\notin W_{i_{k+1}}$.
Hence,

$$g_N(v, v')\geqq\min\{g_N(v, v_{i_{k+1}}), g_N(v_{i_{k+1}}, v')\}\geqq h(v').$$

From Lemma 7, $W'\cup\{v'\}$ is a territory. From Lemma 8, $(W\cup\{v\})-\{v'\}$ and $(W'\cup\{v'\})-\{v\}$ are territories. Therefore

$$(\mathcal{T}_0-\{W, W'\})$$
$$\cup\{(W\cup\{v\})-\{v'\}, (W'\cup\{v'\})-\{v\}\}$$

is also a solution of the $h(\cdot)$-single cover partition problem, contradicting the defining property of $\mathcal{T}_0$.

Hence $W=W_{i_{k+1}}$.

We can easily show that there exists a solution containing W_{i_1} by similar arguments. So, we omit the proof.

Therefore, for the output $\mathcal{T}=\{W_{i_1}, W_{i_2}, \cdots\}$ where $i_1<i_2<\cdots$, there exists a solution that contains $W_{i_1}, \cdots, W_{i_t}$ for each t, namely, $\mathcal{T}$ is a solution of the $h(\cdot)$-single cover partition problem. □

For example, let us consider N in Fig. 3. In the case of $r=4$, an output of Algorithm SINGLE_COVER_B is $\{\{v_1, v_2, v_9, v_8\}, \{v_7, v_6\}, \{v_5, v_4, v_3\}\}$ and mother vertices are v_1, v_7 and v_5, respectively.

The time complexity of Algorithm SINGLE COVER_B is clearly $O(|V|s(|V|, |E|)+|V|^2)$ $(=O(|V|s(|V|, |E|)))$.

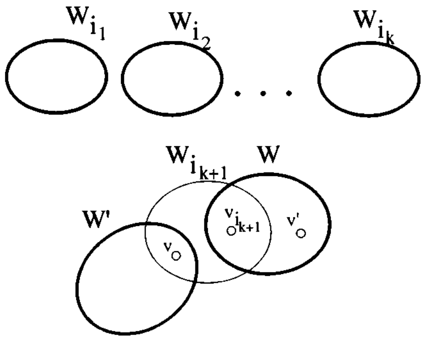

Fig. 4 Explanation for the proof of Theorem 3.

5. Conclusion

In this paper, we have generalized a single cover problem proposed in Ref. [3]. We have shown that a polynomial time algorithm proposed in Ref. [3] can be applicable to solving the generalized single cover problem. Then, we have defined a single cover problem satisfying cardinality constrains and we have shown that the problem is solved in a polynomial time.

On a file assigning problem in a computer network, if for each computer, the file which can be used by the computer is fixed, the $h(\cdot)$-single cover problem can be applied. If for each computer, the computer can use any file, a covering problem called the plural cover problem [3] can be applied. We plan to discuss on a generalized plural cover problem in another paper.

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Appendix

We propose a more efficient algorithm than SINGLE_COVER_A as follows.
Algorithm SINGLE_COVER_A'
Input: an undirected flow network $N=(V, E, c_N)$ and a vertex weight function $h(\cdot)$
Output: a solution U, of the $h(\cdot)$-single cover problem

begin
A'1 $U:=V;(*V=\{v_1, \cdots, v_n\}*)$
A'2 for $i=1$ to n do
A'3 $M_i:=\{v_j\in V|g_N(v_j, v_i)\geqq h(v_i)\}$;
A'4 for $i=1$ to n do
begin
A'5 if $M_k\neq\{v_i\}$ for any k then
begin
A'6 for $j=1$ to n do
A'7 if $v_i\in M_j$ then $M_j:=M_j-\{v_i\}$;
A'8 $U:=U-\{V_i\}$
end
else
A'9 for $j=1$ to n do
A'10 if $v_i\in M_j$ then $M_j:=\phi$
end
end.

In the loop of A'4-A'10, if $M_k=\phi$, there exists $i_0<i$ such that $v_{i_0}\in U$ and $g_N(v_{i_0}, v_k)\geqq h(v_k)$. Therefore, v_{i_0} is in the output of SINGLE_COVER_A'. And if $M_k\neq\phi$, there does not exist $i_0<i$ such that $v_{i_0}\in U$ and $g_N(v_{i_0}, v_k)\geqq h(v_k)$.

If $M_k\neq\{v_i\}$ for any k (A'5), then $U-\{v_i\}$ is obviously an $h(\cdot)$-single cover.

We assume that there exists k_0 such that $M_{k_0}=\{v_i\}$. Since there does not exist $i_0<i$ such that $v_{i_0}\in U$ and $g_N(v_{i_0}, v_{k_0})\geqq h(v_{k_0})$, $g_N(u, v_{k_0})<h(v_{k_0})$ for any $u\in U-\{v_i\}$. So, $U-\{v_i\}$ is not an $h(\cdot)$-single cover.

From the above discussion, in A'4-A'10, we can judge whether $U-\{v_i\}$ is an $h(\cdot)$-single cover or not. If we represent M_k using a list $L_k=(v_{k_1}, v_{k_2}, \cdots)$ $(k_1<k_2<\cdots)$, the loop of A'5-A'10 requires $O(|V|)$ time. Therefore, the time complexity of SINGLE_COVER_A' is $O(|V|s(|V|, |E|)+|V|^2)=O(|V|s(|V|, |E|))$.

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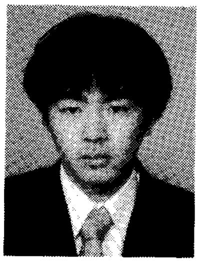

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