PAPER

Stable Decomposition of Mueller Matrix

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SUMMARY Huynen has already provided a method to decompose a Mueller matrix in order to retrieve detailed target information in a polarimetric radar system. However, this decomposition sometimes fails in the presence of small error or noise in the elements of a Mueller matrix. This paper attempts to improve Huynen's decomposition method. First, we give the definition of stable decomposition and present an example, showing a problem of Huynen's approach. Then two methods are proposed to carry out stable decompositions, based on the nonlinear least square method and the Newton's method. Stability means the decomposition is not sensitive to noise. The proposed methods overcomes the problems on the unstable decomposition of Mueller matrix, and provides correct information of a target.

key words: *radar polarimetry, scattering matrix, Mueller matrix, target classification, decomposition*

1. Introduction

In the early radar systems, only the amplitude information of EM wave was used, and any information on the phase was ignored. In the modern polarimetric radar systems, however, the complete electromagnetic vector wave information (both the amplitude and phase, as well as the relations between vector components) is the key factor for advanced sensing. Now radar polarimetry has become an indispensable tool in modern radar and imaging technology. "Among the engineering scientists who have most decisively contributed toward forefront advances for the 'Development of Polarimetric Radar Theory, Techniques and Target Phenomenology,' J. R. Huynen stands out as one of the towering giants" [1]. The main contribution of Huynen on radar polarimetry consists of three parts – "the structural target diagram," "the Huynen parameters" and "target decomposition theorem" [2]-[6].

One of the important problems in radar polarimetry is target decomposition. Recently, This problem has attracted attention with the advent of the full polarimetric radar systems. Since Huynen's original work [2], various decomposition techniques have been proposed. Based on the work of light scattering by small anisotropic particles [7], Cloude and Mishchenko decomposed the scattering matrix into orthogonal component matrices, which defined a vector to form a covariance matrix for decomposition [8]-[12]. By solving the eigenvalue problem, Holm and Barnes decomposed the covariance matrix into three components corresponding to the pure target and two noise terms [13]. A similar method was also considered by Cloude [9], [11], too. Krogager [14], [15] decomposed the scattering matrix into three coherent components, which have physical interpretation in terms of diplane (dihedral corner), sphere and helical targets. Among the methods of target decomposition, Huynen's method on the Mueller matrix decomposition is one of the basic techniques as regards the uniqueness and the physical relevance of certain mathematical procedures [1]-[6]. The definitions and the relations of the scattering matrix, the Mueller matrix and the covariance matrix are described in [1]-[8], and [20]-[21].

The basic idea of Huynen's approach to the Mueller matrix decomposition is to distinguish a wanted target (pure single target) from its clutter environment. However, Huynen's approach, as pointed out in the Sect. 2.2, is not a stable decomposition. Here, stable decomposition means that the extracted Mueller matrix corresponding to a scattering matrix of a pure single target is not sensitive to noise. The strict definition will be given in the Sect. 2.2. We will find out that the concept of stability is very important in the target decomposition techniques. Unfortunately, sometimes Huynen's decomposition cannot be used to extract the wanted target information from the measured Mueller matrix because Huynen's approach is not stable.

This paper tries to improve Huynen's decomposition of Mueller matrix. First, one example is presented, showing the importance of a stable decomposition and a disadvantage of Huynen's approach. Then based on the nonlinear least square method and the Newton's method, two numerical methods will be proposed for decomposing the Mueller matrix or covariance matrix. These approaches can be proved as stable decompositions. Finally, a numerical example is given, showing the validity of our methods.

2. Huynen's Decomposition of Mueller Matrix and Concept of Stability

2.1 Huynen's Decomposition of Mueller Matrix

The Mueller matrix is very important in radar polarimetry, because it relates the incident Stokes vector to the scattered Stokes vector, and represents the information of a polarimetric target [1]-[6]. For an averaged Mueller matrix in the time-varying case or a Mueller matrix which contains

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some noise, there exist incoherent components. We denote this kind of Mueller matrices as $<M>$. Huynen's approach is to decompose $<M>$ into M_o and M_n which represent an equivalent single target and a N-target (non-symmetric target and noise), respectively. Mathematically, Huynen's decomposition is expressed as

$$<M> = M_o + M_n \,, \tag{1}$$

where

$$M_o = \begin{bmatrix} A_o + B_o^s & C & H & F^s \\ C & A_o + B^s & E^s & G \\ H & E^s & A_o - B^s & D \\ F^s & G & D & B_o^s - A_o \end{bmatrix}$$

and

$$M_n = \begin{bmatrix} B_o^n & 0 & 0 & F^n \\ 0 & B^n & E^n & 0 \\ 0 & E^n & -B^n & 0 \\ F^n & 0 & 0 & B_o^n \end{bmatrix}.$$

The parameters A_o, C, H, G and D are fixed, and the other parameters in M_o are chosen to satisfy the constraint so that M_o corresponds to a coherent target. The superscripts "s" and "n" denote single target and N-target, respectively. One can obtain the detailed "physical interpretation" of this decomposition and the above parameters by [2]-[6]. Note that the above Mueller matrix has several different forms. In this paper, the formulation of Mueller matrix is identical with [1] and [8]. The relations between the Mueller matrix and the scattering matrix are described in [1]-[8] and [20]-[21].

2.2 Concept of Stable Decomposition and an Example

Definition 1. Let $\langle M \rangle$ denote a measured Mueller matrix which is a sum of two components M^0 and ΔM:

$$\langle M \rangle = M^0 + \Delta M \,, \tag{2}$$

where M^0 denotes a Mueller matrix corresponding to a coherent scattering matrix and ΔM denotes a 4×4 noise matrix. By one method of decomposition, the measured Mueller matrix $\langle M \rangle$ is decomposed into two parts as (1), where M_0 can also be corresponding to a coherent scattering matrix. We define the number

$$a = \| M^0 - M_0 \| / \| \Delta M \| \tag{3}$$

as the *noise sensitivity factor* (associated with M^0 and ΔM) of the method. If there exists one positive number A such that $\| M^0 - M_0 \| \le A \| \Delta M \|$ holds for all M^0 and ΔM, then the smallest number A_{min} is called as *stability infimum*. If A_{min} is not large (e.g. $A_{min} \le 5$), then the decomposition is called as a *stable decomposition* of Mueller matrix, where $\| * \|$ denotes matrix's norm †.

Definition 2. If there exist one Mueller matrix M^0 which corresponds to a coherent scattering matrix and one 4×4 noise matrix ΔM such that the corresponding noise sensitivity factor is very large (e.g. $a \ge 50$), then the decomposition is called as an *unstable decomposition*.

The concept of stable decomposition means that the extracted Mueller matrix is close to the Mueller matrix of the practical target if there are small errors contained in a measured Mueller matrix. Since the measured data usually contains some errors, a stable decomposition is very important. Unfortunately, Huynen's approach is not a stable decomposition as shown below.

If a scattering matrix

$$S = \begin{bmatrix} 1 & 0.1i \\ 0.1i & -0.99 + 0.02i \end{bmatrix}$$

is given, the corresponding Mueller matrix becomes

$$M = \begin{bmatrix} 1.00025 & 0.00975 & 0.002 & -0.199 \\ 0.00975 & 0.98025 & -0.002 & -0.001 \\ 0.002 & -0.002 & -0.98 & -0.02 \\ -0.199 & -0.001 & -0.02 & 1 \end{bmatrix}.$$

Now let us consider an averaged Mueller matrix $\langle M \rangle$ which is similar to M but has small errors in the 1×1, 3×3 and 4×4 elements

$$\langle M \rangle = \begin{bmatrix} 1.00125 & 0.00975 & 0.002 & -0.199 \\ 0.00975 & 0.98025 & -0.002 & -0.001 \\ 0.002 & -0.002 & -0.9805 & -0.02 \\ -0.199 & -0.001 & -0.02 & 1.0005 \end{bmatrix}.$$

Using Huynen's method, we can obtain the following decomposition result. (The detailed description of this model will be given in the Sect. 4.)

$$\begin{aligned} \langle M \rangle &= M_0 + M_n \\ &= \begin{bmatrix} 0.33372 & -0.00975 & 0.00200 & -0.06633 \\ -0.00975 & 0.32706 & -0.00067 & -0.00100 \\ 0.00200 & -0.00067 & -0.32631 & -0.02000 \\ -0.06633 & -0.00100 & -0.02000 & 0.33297 \end{bmatrix} \\ &+ \begin{bmatrix} 0.66753 & 0 & 0 & -0.13267 \\ 0 & 0.65319 & -0.00133 & 0 \\ 0 & -0.00133 & -0.65319 & 0 \\ -0.13267 & 0 & 0 & 0.66753 \end{bmatrix}, \end{aligned}$$

† Mathematically, there are many kinds of norms. For the decomposition problem of the Mueller matrix, it is convenient to select the matrix's norm as the square root of the sum of all squared elements. In this paper, we will use this definition of norm.

where the first term denotes the extracted Mueller matrix of the single target from $\langle M \rangle$. The scattering matrix corresponding to the Mueller matrix M_o is

$$S_0 = \begin{bmatrix} 0.58321 & 0.00114+0.05772i \\ 0.00114+0.05772i & -0.56521+0.03429i \end{bmatrix}.$$

Obviously, this is a tremendous error in this illustrative example, and the noise sensitivity factor a is 1088.84540. Hence, we can not use Huynen's method to extract the wanted target correctly. According to Definition 2, we can know Huynen's approach is unstable. Therefore, it is necessary to improve Huynen's approach to the decomposition of Mueller matrix.

3. Decomposition Techniques of Mueller Matrix

3.1 Proposed Methods

Let S and M denote a scattering matrix and the corresponding Mueller matrix, respectively. Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}. \tag{4}$$

Using the method of Ref. [16], we can express Mueller matrix in terms of scattering matrix's elements as

$$M = \sum_{n=1}^{3} \sum_{k=1}^{3} Q_{nk} s_n s_k^* .$$

where

$$Q_{11} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_{12} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & 1 & -i \\ 1 & 1 & 0 & 0 \\ i & i & 0 & 0 \end{bmatrix},$$

$$Q_{13} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & -i \\ 0 & i & i & -1 \end{bmatrix}, \quad Q_{22} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$Q_{23} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & i \\ 1 & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \end{bmatrix}, \quad Q_{33} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$Q_{21} = Q_{12}^*$, $Q_{31} = Q_{13}^*$, $Q_{32} = Q_{23}^*$.

Conversely, if a measured Mueller matrix $\langle M \rangle = (m_{nk})_{4\times4}$ containing some errors is given, we can also express the Mueller matrix as

$$\langle M \rangle = \sum_{n=1}^{3} \sum_{k=1}^{3} c_{nk} Q_{nk} , \tag{6}$$

where

$$c_{11} = m_{01} + \frac{1}{2}(m_{00} + m_{11})$$

$$c_{22} = \frac{1}{2}(m_{00} - m_{11})$$

$$c_{33} = \frac{1}{2}(m_{00} + m_{11}) - m_{01}$$

$$c_{12} = \frac{1}{2}(m_{02} + m_{12}) + \frac{1}{2}(m_{03} + m_{13})i$$

$$c_{13} = \frac{1}{2}(m_{11} - m_{00}) + m_{22} + m_{23}\, i$$

$$c_{23} = \frac{1}{2}(m_{02} - m_{12}) + \frac{1}{2}(m_{03} - m_{13})i$$

$c_{21} = c_{12}^*, c_{31} = c_{13}^*$ and $c_{32} = c_{23}^*$.

c_{nk} (n, k = 1, 2, 3) is the elements of the covariance matrix [1]-[2]. According to Eqs. (5) and (6), we can apply the following approaches to decompose the Mueller matrix.

Method 1

Step 1 From (6), express the Mueller matrix $\langle M \rangle$ as the following form

$$\langle M \rangle = \sum_{n=1}^{3} \sum_{k=1}^{3} |c_{nk}| \exp(i\varphi_{nk}) Q_{nk} . \tag{7}$$

Step 2 Let $s_k = r_k \exp(i\,\theta_k)$ (k = 1, 2, 3) denote the elements of the scattering matrix of the wanted target, then r_k (k = 1, 2, 3) can be obtained by the following method:

(i) Let $r_k^0 = \sqrt{|c_{kk}|}$ (k = 1, 2, 3).

(ii) After r_k^m (k = 1, 2, 3) have been obtained, $r_k^{m+1} = r_k^m + \Delta r_k^m$ (k = 1, 2, 3) can be calculated by

$$\sum_{k=1}^{3}\left(\left(r_k^m\right)^2 \Delta r_n^m + \left(2 r_k r_n - |c_{nk}|\right)\Delta r_k^m \right) = \sum_{k=1}^{3}\left(r_k^m |c_{nk}| - r_n^m \left(r_k^m\right)^2 \right) \quad \text{for } n = 1, 2, 3. \tag{8}$$

which is derived from the Newton's method and the following nonlinear least square method:

$$\min\left\{ \sum_{n=1}^{3} \sum_{k=1}^{3} \left(|c_{nk}| - r_k r_n \right)^2 \right\}. \tag{9}$$

If $\Delta r_1^m + \Delta r_2^m + \Delta r_3^m < \varepsilon m_{00}$, where ε is very small (for example, ε = 10^{-10}), we can regard r_k^{m+1} $(k = 1, 2, 3)$ as the modulus of the scattering matrix's elements approximately.

Step 3 Let $s_k = r_k \exp(i\,\theta_k)$ (k = 1, 2, 3). If $r_1 \neq 0$, let θ_1=0. (For the case of r_1 = 0, let θ_2 or θ_3=0.) Then θ_2, θ_3 can be obtained by

$$\theta_2 = \frac{1}{r_1 + r_2 + r_3}\left(r_3 \varphi_{23} - (r_1 + r_2)\varphi_{12} - r_3 \varphi_{13} \right), \tag{10a}$$

$$\theta_3 = \frac{-1}{r_1 + r_2 + r_3}\left(r_2 \varphi_{23} + r_2 \varphi_{12} + (r_1 + r_3)\varphi_{13} \right), \tag{10b}$$

which are derived from the linear least square method:

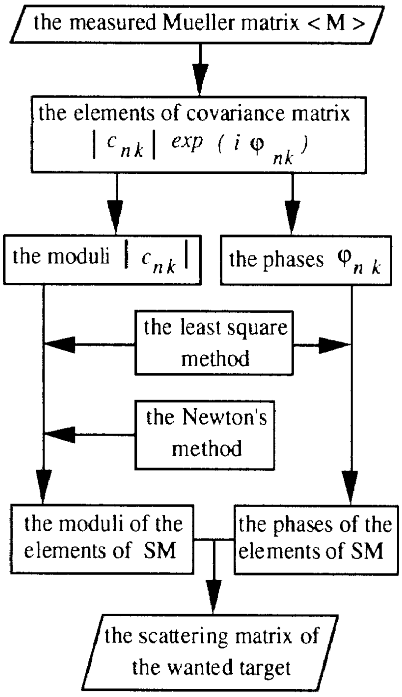

Fig. 1 The procedure for Method 1.

$$min\left\{r_1r_2(\varphi_{12}+\theta_2)^2+r_1r_3(\varphi_{13}+\theta_3)^2+r_2r_3(\varphi_{23}-\theta_2+\theta_3)^2\right\}. \tag{11}$$

Note that the phase of the complex number "zero" can be any value. So, the phase of c_{nk} is very sensitive to errors if $|c_{nk}|$ is very small. Because of this reason, we use the above method for obtaining the phases of elements of the wanted scattering matrix. According to the results of some numerical examples, we have found that (11) is superior to the following expression:

$$min\left\{(\varphi_{12}+\theta_2)^2+(\varphi_{13}+\theta_3)^2+(\varphi_{23}-\theta_2+\theta_3)^2\right\}. \tag{12}$$

Figure 1 shows the procedure for the Method 1.

From the above process of the calculation, we know Method 1 can also be used to decompose the covariance matrix because the scattering matrix of the single target can be obtained from the covariance matrix's elements c_{nk} exp (i φ_{nk}) directly by Step 2 and Step 3. In other words, if we have known the covariance matrix $\langle C\rangle = (c_{nk}\exp(i\varphi_{nk}))_{3\times3}$ which contains some errors, we can also use the above method to extract a single target's covariance matrix C_0

$$\langle C\rangle = C_0 + C_N\ . \tag{13}$$

Method 2

Let M_o denote the extracted Mueller matrix from $\langle M\rangle$. Method 2 is to find M_o such that

$$min\left(\left\|\langle M\rangle-M_0\right\|\right)^2\ . \tag{14}$$

Because there are five parameters in (14), it is a complicated task to obtain the scattering matrix corresponding to M_o . This paper presents the following numerical method:

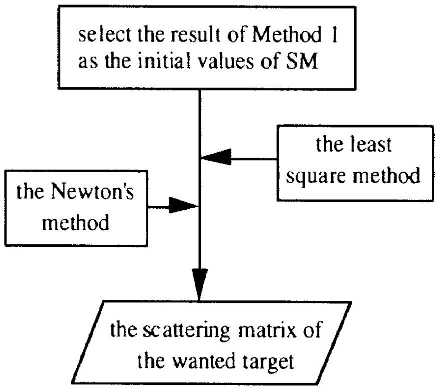

Fig. 2 The procedure for Method 2.

Step 1 After r_k exp (i θ_k) (k = 1, 2, 3) are obtained by Method 1, we denote $r_1 = x_1^0$, $r_2\exp(i\theta_2)=x_2^0+x_3^0 i$ and $r_3\exp(i\theta_3)=x_4^0+x_5^0 i$.

Step 2 If $x_i^n (i = 1, 2, \ldots, 5)$ have been known, Δx_k^n can be calculated by

$$W\Delta\vec{x^n}=\vec{b}\ , \tag{15}$$

where

$$W=\left(\frac{\partial^2 w}{\partial x_i\partial x_j}\right)_{5\times5},$$

$$\vec{\Delta x^n}=\left(\Delta x_1^n, \Delta x_2^n, \ldots \Delta x_5^n\right)^t,$$

$$\vec{b}=\left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \ldots \frac{\partial w}{\partial x_5}\right)^t$$

and

$$\begin{aligned} w &= w(x_1, x_2, \ldots, x_5) = \left(\left\|\langle M\rangle - M_o\right\|\right)^2 \\ &= \left(m_{00}-\frac{1}{2}\left(x_1^2+2x_2^2+2x_3^2+x_4^2+x_5^2\right)\right)^2 \\ &+2\left(m_{10}-\frac{1}{2}\left(x_1^2-x_4^2-x_5^2\right)\right)^2 \\ &+2\left(m_{20}-\left(x_1x_2+x_2x_4+x_3x_5\right)\right)^2 \\ &+2\left(m_{30}+\left(x_1x_3+x_2x_5-x_3x_4\right)\right)^2 \\ &+\left(m_{11}-\frac{1}{2}\left(x_1^2-2x_2^2-2x_3^2+x_4^2+x_5^2\right)\right)^2 \\ &+2\left(m_{21}-\left(x_1x_2-x_2x_4-x_3x_5\right)\right)^2 \\ &+2\left(m_{31}+\left(x_1x_3-x_2x_5+x_3x_4\right)\right)^2 \\ &+\left(m_{22}-\left(x_1x_4+x_2^2+x_3^2\right)\right)^2 \\ &+2\left(m_{23}+x_1x_5\right)^2 \\ &+\left(m_{33}-\left(x_2^2+x_3^2-x_1x_4\right)\right)^2 . \end{aligned} \tag{16}$$

If $\Delta x_1^n + \Delta x_2^n + ... + \Delta x_5^n < \varepsilon m_{00}$, we can regard $x_1^n, x_2^n + x_3^n i$ and $x_4^n + x_5^n i$ approximately as the scattering matrix's elements s_1, s_2, s_3, respectively. Note that because of the complicated expression of $w = (x_1, x_2, ..., x_5)$, it is very tedious to express $\frac{\partial w}{\partial x_k}$ and $\frac{\partial^2 w}{\partial x_i \partial x_j}$ in the analytic forms. So we use the following approximate values of derivatives

$$\frac{\partial w}{\partial x_k} \approx \frac{w(x_1, x_2, ..., x_k + h_s, ..., x_5) - w(x_1, x_2, ..., x_k + h_s, ..., x_5)}{2h_s} \tag{17a}$$

$$\frac{\partial^2 w}{\partial x_k^2} \approx \frac{w(x_1, x_2, ..., x_k + h_s, ..., x_5) - w(x_1, x_2, ..., x_k + h_s, ..., x_5)}{h_s^2} - \frac{2w(x_1, x_2, ..., x_k, ..., x_5)}{h_s^2} \tag{17b}$$

$$\frac{\partial^2 w}{\partial x_i \partial x_j} \approx \frac{w(x_1 ..., x_i + h_s, ..., x_j + h_s, ..., x_5) - w(x_1 ..., x_i + h_s, ..., x_j + h_s, ..., x_5)}{4h_s^2} - \frac{w(x_1 ..., x_i + h_s, ..., x_j + h_s, ..., x_5) - w(x_1 ..., x_i + h_s, ..., x_j + h_s, ..., x_5)}{h_s^2} \tag{17c}$$

Step 3 Let $x_k^{n+1} = x_k^n - r_{co} \Delta x_k^n$, where r_{co} is one number of 0.0, 0.1, 0.2, ..., 1.0 and such that the Mueller matrix corresponding to $x_k^{n+1} = x_k^n - r_{co} \Delta x_k^n$ is the most approximate to $< M >$.

The procedure for the Method 2 is shown in Fig. 2.

3.2 The Problem of Stability

The advantage of Method 1 is the simplicity of calculation. Note that the covariance matrix is equivalent to the Mueller matrix because the elements of the covariance matrix and the elements of the Mueller matrix can be expressed by each other in forms of linear combination. Furthermore, both of Method 1 and Method 2 are based on the least square method. So the calculation results by both methods are very close. Usually, the result calculated by Method 2 is slightly better than that by Method 1. This conclusion has been proved by some numerical results. Therefore, we need to prove the stability of Method 2 only.

Let M^0 denote an arbitrary Mueller matrix corresponding to a coherent scattering matrix, and let

$$\langle M \rangle = M^0 + \Delta M \ , \tag{18}$$

where ΔM denotes an arbitrary 4×4 noise matrix. Assume that by Method 2, $\langle M \rangle$ is decomposed as

$$\langle M \rangle = M_o + M_n \ . \tag{19}$$

Note that both Mueller matrices, M^0 in (18) and M_0 in (19), are corresponding to coherent scattering matrices. According to Method 2 (Expression (14)), we can know

$$| M_n | \le | \Delta M | .$$

So, we have

$$\begin{aligned} \|M^0 - M_0\| &= \|M^0 + \Delta M - M_0 - \Delta M\| \\ &= \|M_0 + M_n - M_0 - \Delta M\| \\ &\le \|M_n\| + \|\Delta M\| \le 2\|\Delta M\|. \end{aligned} \tag{20}$$

According to the definition of stability, we can know the stability infimum satisfies $A_{min} \le 2.0$, which means that the stability of Method 2 is very good. From this result, we can know that if the errors contained in the measured Mueller matrix are small, the extracted scattering matrix can always be close to the real scattering matrix no matter what the form of the measured Mueller matrix is.

Note that 2.0 is only an upper bound of stability infimum as shown in (20). In the practical problems, the noise sensitivity factors of our methods are usually smaller than 2.0 We will show this conclusion in the next Section.

4. Numerical Example

Consider a scattering matrix which contains some noise

$$S_r = S + \Delta SN = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} \Delta sn_1 & \Delta sn_2 \\ \Delta sn_2 & \Delta sn_3 \end{bmatrix} \tag{21}$$

According to the practical problems, some authors [17]-[19] assumed that the complex noise terms Δsn_k (k = 1, 2, 3) are independent of signals s_l (l = 1, 2, 3), characterized with zero-mean, Gaussian distributions, and with the following properties:

$$\langle \Delta sn_k s_l^* \rangle = 0 \quad (k, l = 1, 2, 3.) \tag{22a}$$

$$\langle \Delta sn_k \rangle = 0 \quad (k = 1, 2, 3.) \tag{22b}$$

$$\langle \Delta sn_k \Delta sn_k^* \rangle = \sigma_k \quad (k = 1, 2, 3.) \tag{22c}$$

$$\langle \Delta sn_k \Delta sn_l^* \rangle = 0 \quad (k \ne l). \tag{22d}$$

In this paper, we assume $\sigma_k = \sigma$ (k = 1, 2, 3.). Consider a concrete scattering matrix

$$S^r = \begin{bmatrix} 1 + \Delta sn_1 & 0.1i + \Delta sn_2 \\ 0.1i + \Delta sn_2 & -0.99 + 0.02i + \Delta sn_3 \end{bmatrix}.$$

Using the above assumption, we can obtain the corresponding averaged Mueller matrix as

$$\langle M \rangle = \begin{bmatrix} 1.00025 + 2\sigma & 0.00975 & 0.002 & -0.199 \\ 0.00975 & 0.98025 & -0.002 & -0.001 \\ 0.002 & -0.002 & -0.98 + \sigma & -0.02 \\ -0.199 & -0.001 & -0.02 & 1 + \sigma \end{bmatrix}.$$

Table 1 Errors of calculation by three methods.

methods / cases	Huynen's method	Method 1	Method 2
σ = 0.0005	0.92641	0.00027	0.00025
σ = 0.001	1.21233	0.00055	0.00049
σ = 0.005	1.72316	0.00274	0.00268
σ = 0.01	1.87049	0.00549	0.00530
σ = 0.05	2.08498	0.02734	0.02674

Table 2 Noise sensitivity factors of three methods.

methods / cases	Huynen's method	Method 1	Method 2
σ = 0.0005	1088.84	0.41013	0.30738
σ = 0.001	653.267	0.43028	0.34862
σ = 0.005	155.543	0.43029	0.39672
σ = 0.01	79.6692	0.43030	0.40270
σ = 0.05	16.2558	0.43030	0.40894

Note that in practical problems, we do not know the above scattering matrix. But it is possible to derive the above Mueller matrix from the measured data. The purpose we used the above scattering matrix is for comparison of calculation results only.

In the Sect. 2, we have known Huynen's method could not be used to extract the wanted target for the case of σ = 0.0005. Now, by use of Huynen's method, Method 1 and Method 2, we can obtain the following results:

Huynen's Method:

$$S_0=\begin{bmatrix} 0.58321 & 0.00114+0.05772i \\ 0.00114+0.05772i & -0.56521+0.03429i \end{bmatrix},$$

Method 1:

$$S_0=\begin{bmatrix} 1.00012 & 0.10001i \\ 0.10001i & -0.99012+0.02000i \end{bmatrix},$$

Method 2:

$$S_0=\begin{bmatrix} 1.00004 & -0.00003+0.09998i \\ -0.00003+0.09998i & -0.99013+0.01997i \end{bmatrix}.$$

Table 1 shows the errors of calculation by three methods for the cases of σ = 0.0005, 0.001, 0.005, 0.01 and 0.05, where the error is defined as

$$error=\left|s_1-s_1^0\right|+2\left|s_2-s_2^0\right|+\left|s_3-s_3^0\right|,$$

s_i and s_i^0 are the elements of S and S_0, respectively. Obviously, Huynen's method can not be used to extract the wanted target from above averaged Mueller matrix, but the presented methods of this paper can provide very good results. Table 2 shows the noise sensitivity factors of three methods for the cases of σ = 0.0005, 0.001, 0.005, 0.01 and 0.05, showing our methods are not sensitive to noise. Note that in this example, the noise sensitivity factors of our methods are smaller than theoretical value 2.0 of an upper bound which was proved in the Sect. 3. Because Method 1 is much simpler than Method 2, and the calculation results by both methods are almost the same, it is better to use Method 1 for extracting the wanted information from a measured Mueller matrix, although the results of calculation by Method 2 are slightly better than Method 1.

Table 3 Errors of calculation for typical targets by Method 1.

targets / cases	sphere $\begin{bmatrix}1&0\\0&1\end{bmatrix}$	wire $\begin{bmatrix}1&0\\0&0\end{bmatrix}$	diplane $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	helix $\begin{bmatrix}1&\pm i\\\pm i&-1\end{bmatrix}$
σ = 0.0005	0.00025	0.00025	0.00025	0.00033
σ = 0.001	0.00050	0.00050	0.00050	0.00067
σ = 0.005	0.00250	0.00250	0.00250	0.00333
σ = 0.01	0.00499	0.00499	0.00499	0.00666
σ = 0.05	0.02485	0.02470	0.02485	0.03320

Table 4 Noise sensitivity factors for typical targets by Method 1.

targets / cases	sphere $\begin{bmatrix}1&0\\0&1\end{bmatrix}$	wire $\begin{bmatrix}1&0\\0&0\end{bmatrix}$	diplane $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	helix $\begin{bmatrix}1&\pm i\\\pm i&-1\end{bmatrix}$
σ = 0.0005	0.40825	0.40827	0.40824	0.54373
σ = 0.001	0.40817	0.40822	0.40817	0.54405
σ = 0.005	0.40822	0.40824	0.40822	0.54423
σ = 0.01	0.40823	0.40825	0.40823	0.54429
σ = 0.05	0.40825	0.40825	0.40825	0.54432

Next, let us consider the effectiveness of the proposed decomposition (Method 1) for some typical targets: sphere (plane), wire, diplane and helix. Using the above assumptions (21) and (22), we can obtain the calculation errors as shown in Table 3 and the noise sensitivity factors as shown in Table 4. From these tables, one can find that the proposed decomposition is very effective. Furthermore, from (20) we can know that the proposed decomposition is effective for any kind of targets if the errors contained in the measured Mueller matrix are small. So the proposed method can always be used to extract the wanted scattering matrix very well from the measured data.

5. Conclusion

Since the measured Mueller matrix contains some errors, a stable decomposition is very important for extracting the wanted target's scattering matrix from the measured Mueller matrix. Unfortunately, Huynen's approach is not a stable decomposition. In this paper, we first introduced the definition of stable decomposition. Then the authors provided two numerical methods for decomposing the Mueller matrix or covariance matrix, based on the nonlinear Least Squares Method and the Newton's Method. These proposed approaches were proved as stable decompositions and the stability infimum satisfies $A_{min} \leq 2.0$. According to these results, if the errors contained in the measured Mueller matrix are small, the extracted scattering matrix can always be close to the real scattering matrix no matter what the measured Mueller matrix is. By numerical example, the effectiveness of the proposed methods was shown. So, it is possible to apply the proposed methods to the measured data. Finally, it should be pointed out that the Method 1 is much simpler than the Method 2. So it is better to use Method 1 for extracting the wanted target's scattering matrix from the measured Mueller matrix.

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