LETTER

The Periodicity of the Scattering Matrix and Its Application

Jian YANG†, *Regular Member*, Ying-Ning PENG†, *Nonmember*, Yoshio YAMAGUCHI††, Hiroyoshi YAMADA††, *Regular Members, and* Wolfgang-M. BOERNER†††, *Nonmember*

SUMMARY The periodicity of a target scattering matrix is studied when the target is rotated about the sight line of a monostatic radar. Except for the periodicity and invariance of the scattering matrix $diag(a,a)$, it is proved that only helixes have the quasi-invariance, and that only N-targets have the quasi-periodicity, demonstrating that a target with some angle rotation symmetry also has the scattering matrix form $diag(a,a)$. From this result, we conclude that it is impossible to extract the shape characteristics of a complex target from its scattering matrix or its Kennaugh matrix.

key words: *radar polarimetry, scattering matrix, invariance, target*

1. Introduction

For the monostatic radar case, if the reciprocity holds, every target corresponds to a symmetric scattering matrix in a coordinate system. When a target is rotated an angle Ψ about the sight line of the monostatic radar, the orientation angle of the target will increase or decrease Ψ, depending on the rotation directions. Usually, the scattering matrix of a radar target varies with Ψ [1], [2]. For some targets, however, the corresponding scattering matrices have invariance (e.g., the scattering matrix of a sphere $diag(1,1)$) or quasi-invariance (e.g., the scattering matrix of a left-helix or a right helix) when the targets are rotated. Here, the quasi-invariance means that the scattering matrix of a target does not vary with any Ψ except an absolute phase of the matrix. So, one interesting problem that faces us is what scattering matrices have the invariance or quasi-invariance. This is one of the problems we will solve in this letter. In addition, the concept of the quasi-periodicity will be introduced, and it will be proved that only N-targets [1] have the quasi-periodic property.

On the other hand, some targets, e.g., an electric fan with three vanes, have some rotation symmetry. If this kind of targets is rotated a special angle (e.g., 120 degrees for the electric fan with three vanes), the targets will be in the same location or position. This letter will present the form of the scattering matrices of these targets.

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†The authors are with the Department of Electronic Engineering, Tsinghua University, Beijing, 100084, P.R. China.

††The authors are with the Department of Information Engineering, Niigata University, Niigata University, Niigata-shi, 950-2181 Japan.

†††The author is with the Department of Electrical and Computer Engineering, University of Illinois at Chicago, Chicago, IL 60607-7053, USA.

2. Periodicity and Quasi-Periodicity

In a reciprocal isotropic medium for the monostatic radar case, let

$$[S] = \begin{bmatrix} s_{HH} & s_{HV} \\ s_{VH} & s_{VV} \end{bmatrix} \equiv \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \tag{1}$$

denote the symmetric scattering matrix of a radar target in the (H-V) polarization basis, here H and V denote the horizontal and vertical polarizations, respectively. If the target is rotated an angle Ψ about the sight line of the radar and its exposure is fixed, then the scattering matrix of the target in new position is given by [1], [2]

$$[S(\Psi)] = \begin{bmatrix} \cos\Psi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{bmatrix} \tag{2}$$

For any scattering matrix $[S]$, one knows that $[S(\Psi+\pi)] = [S(\Psi)]$. Seemingly the scattering matrix $[S(\Psi)]$ may be regarded as a period matrix of Ψ with the period π, but it is insignificant. In this letter, we define the periodicity and quasi-periodicity as follows.

For a target scattering matrix $[S]$, if there exists an angle θ $(0<\theta<\pi)$ such that

$$[S(\Psi+\theta)] = T(\theta)[S(\Psi)] \tag{3}$$

holds for any angle Ψ, here $T(\theta)$ is a scalar quantity, then $[S(\Psi)]$ is called a period matrix or quasi-period matrix (depending on $T(\theta)\equiv 1$ or not), and θ $(0<\theta<\pi)$ is called the period of $[S(\Psi)]$.

Now let us consider the forms of a period scattering matrix and a quasi-period scattering matrix. Letting $\Psi = 0$, then one can obtain from (2) and (3) that

$$s_1(\cos^2\theta - T(\theta)) + s_2 \sin 2\theta + s_3 \sin^2\theta = 0 \tag{4a}$$

$$-\frac{1}{2}s_1 \sin 2\theta + s_2(\cos 2\theta - T(\theta)) + \frac{1}{2}s_3 \sin 2\theta = 0 \tag{4b}$$

$$s_1 \sin^2\theta - s_2 \sin 2\theta + s_3(\cos^2\theta - T(\theta)) = 0 \tag{4c}$$

Note that not all of s_1, s_2 and s_3 are equal to zero. According to the theory of algebra [3], one knows that

$$\begin{vmatrix} \cos^2\theta - T(\theta) & \sin 2\theta & \sin^2\theta \\ -\frac{1}{2}\sin 2\theta & \cos 2\theta - T(\theta) & \frac{1}{2}\sin 2\theta \\ \sin^2\theta & -\sin 2\theta & \cos^2\theta - T(\theta) \end{vmatrix} = 0 \quad (5)$$

or

$$-T^3(\theta) + (1 + 2\cos 2\theta)T^2(\theta) - (1 + 2\cos 2\theta)T(\theta) + 1 = 0 \quad (6)$$

The solutions of the above equation are

$$T_1(\theta) = 1,\ T_2(\theta) = \cos 2\theta + i\sin 2\theta, \quad T_3(\theta) = \cos 2\theta - i\sin 2\theta \quad (7)$$

Substituting (7) into (4), we have the following results:
(i) $T_1(\theta) = 1$ corresponds to the solution $[S_0] = diag(a, a)$. The scattering matrix of a sphere or a plate has this form. For any θ and any Ψ, it is obvious that

$$[S_0(\Psi + \theta)] = [S_0(\Psi)] = [S_0] \quad (8)$$

demonstrating that $[S_0(\Psi)]$ is periodic with any θ. We say that $[S_0] = diag(a, a)$ has the invariance.
(ii) If $\sin 2\theta \neq 0$, $T_2(\theta) = \cos 2\theta + i\sin 2\theta$ and $T_3(\theta) = \cos 2\theta - i\sin 2\theta$ correspond to the scattering matrices of a left helix and a right helix:

$$[S_{h+}] = a\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

and

$$[S_{h-}] = a\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix},$$

respectively. For any θ and any Ψ, it is easy to check that

$$[S_{h\pm}(\Psi+\theta)] = e^{\pm 2\theta i}[S_{h\pm}(\Psi)] = e^{\pm 2(\theta+\Psi)i}[S_{h\pm}] \quad (9)$$

This equation implies that $[S_{h+}(\Psi)]$ and $[S_{h-}(\Psi)]$ have the quasi-periodicity with any θ. We say that $[S_{h+}] = a\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ and $[S_{h-}] = a\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ have the quasi-invariance.
(iii) If $\sin 2\theta = 0$, i.e., $\theta = \frac{\pi}{2}$, $T_2(\theta) = T_3(\theta) = -1$ corresponds to the solution $[S_N] = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. The targets having these scattering matrices except $diag(a, -a)$ are called N-targets by Huynen [1]. It should be pointed out that $diag(a, -a)$ corresponds to a diplane which was not regarded as an N-target by Huynen [1]. For convenience, however, this letter assumes that diplanes belong to N-targets. For any angle Ψ, one knows that

$$\left[S_N\left(\Psi + \frac{\pi}{2}\right)\right] = -[S_N(\Psi)] \quad (10)$$

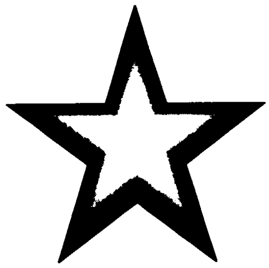

Fig. 1 Pentagram with $\frac{2\pi}{5}$ rotation symmetry.

which means that N-targets have the quasi-periodicity with the period $\frac{\pi}{2}$.

Note that the left helix and the right helix are two special cases of N-targets. From the above results, we conclude that only $[S_0(\Psi)]$ $(= [S_0])$ has the periodicity and that only $[S_N(\Psi)]$ has the quasi-periodicity.

3. Application

Now let us study the scattering matrix of a pentagram (Fig. 1). It is bulgy at the center. Obviously, the pentagram has $\frac{2\pi}{5}$ rotation symmetry, i.e., this target will be in the same location or position if it is rotated $\frac{2\pi}{5}$ about its symmetric axis. Let the pentagram be placed at some position which is far from a monostatic radar, and let the sight line of the radar be identical with the symmetric axis of this pentagram. After we select a coordinate system, the pentagram will have a scattering matrix $[S_p]$. Since this target has $\frac{2\pi}{5}$ rotation symmetry, $[S_p(\Psi)]$ should be periodic with the period $\frac{2\pi}{5}$. On the other hand, one knows from the above results that only $[S_0(\Psi)]$ $(= [S_0])$ has the periodicity. Therefore, the scattering matrix of the pentagram is the same as that of a sphere (or a plate), i.e., $[S_p] = [S_0] = diag(a, a)$.

Similarly, we conclude that the scattering matrices of all targets with α $(0 < \alpha < \pi)$ rotation symmetry have the form $diag(a, a)$ no matter how scraggly and screwy these targets are. Gear wheels and screw propellers are the typical examples of this kind of targets.

4. Conclusion

When a target is rotated, the change of its scattering matrix has been studied in this letter. It has been proved that only $diag(a, a)$ and $a\begin{bmatrix} 1 & \pm i \\ \pm i & -1 \end{bmatrix}$ have the invariance and quasi-invariance, respectively, and that only N-targets have the quasi-periodic property with the period $\frac{\pi}{2}$. Based on the periodicity and uniqueness of the scattering matrix $diag(a, a)$, a target with α $(0 < \alpha < \pi)$ rotation symmetry has been proved to have the scattering matrix form $diag(a, a)$ no matter how complex the target is. This is a new evidence for the conclusion that it is impossible to extract the shape characteristics of a complex target from its scattering matrix or its Kennaugh matrix. Therefore, it is a wrong viewpoint that the Kennaugh matrix (or the Huynen parameters [1]) can be used to describe the symmetry, structure, surface torsion and helicity of a radar target.

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