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Characterization of Minimum Route ETX in Multi-Hop Wireless Networks

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SUMMARY In multi-hop wireless networks, communication quality depends on the selection of a path between source and destination nodes from several candidate paths. Exploring how path selection affects communication quality is important to characterize the best path. To do this, in [1], we used expected transmission count (ETX) as a metric of communication quality and theoretically characterized minimum route ETX, which is the ETX of the best path, in a static one-dimensional random multi-hop network. In this paper, we characterize minimum route ETX in static two-dimensional multi-hop networks. We give the exact formula of minimum route ETX in a two-dimensional network, assuming that nodes are located with lattice structure and that the ETX function satisfies three conditions for simplifying analysis. This formula can be used as an upper bound of minimum route ETX without two of the three conditions. We show that this upper bound is close to minimum route ETX by comparing it with simulation results. Before deriving the formula, we also give the formula for a one-dimensional network where nodes are located at constant intervals. We also show that minimum route ETX in the lattice network is close to that in a two-dimensional random network if the node density is large, based on a comparison between the numerical and simulation results.

key words: *multi-hop wireless networks, expected transmission count, theoretical analysis*

1. Introduction

In multi-hop wireless networks [2], [3], source node S sends a packet to destination node D by a multi-hop path consisting of relay nodes. If candidate multi-hop paths exist between S and D, path selection from the candidates affects the communication quality as follows. If we select a path that minimizes the number of hops, links included in the path tend to be long. This tendency increases packet losses and retransmissions. As a result, the total number of transmissions until D receives a packet is increased. Conversely, if we select a path consisting of short links, the number of hops tends to increase. This tendency also increases the total number of transmissions. Therefore, to minimize transmissions, we should select a path that realizes an appropriately small number of hops and appropriately short links.

Exploring how path selection affects communication quality is important to characterize the best path. To do this, we use expected transmission count (ETX) [4], defined as the expected number of transmissions required to successfully send a packet through a link, and route ETX [4], which is the sum of the ETXs of all links in the selected path, as metrics of communication quality. In [1], we theoretically analyzed the minimum value of route ETX, which is called minimum route ETX, in a one-dimensional multi-hop network where nodes are randomly distributed, assuming that all nodes are stationary, and the ETX function is a convex monotonically increasing function as shown in Fig. 1, where the horizontal axis is the length of a link and the vertical axis is the ETX of a link. The ETX functions in Fig. 1 will be explained in Sect. 2. We characterized a path that approximately minimizes route ETX and gave a formula to compute the route ETX of this path. We also characterized minimum route ETX by comparisons with route ETXs for the Shortest Path Routing and the Longest Path Routing based on the results of theoretical analyses. From these characterizations, we can well understand how route ETX is minimized in a static one-dimensional network.

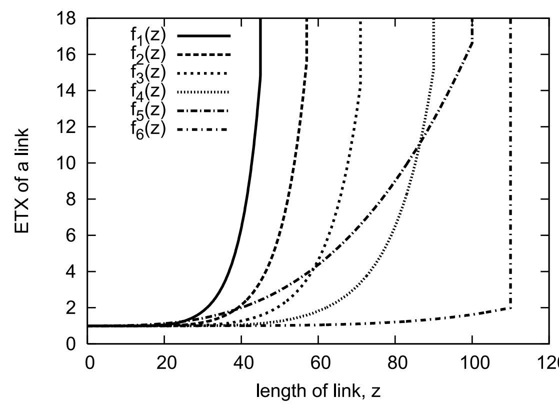

Fig. 1 Examples of ETX functions.

In this paper, we characterize how route ETX is minimized and compute minimum route ETX in static two-dimensional multi-hop networks. Theoretical analysis of a two-dimensional network is not easy if the nodes are randomly distributed, as assumed in [1]. Hence, we assume that nodes are located with lattice structure and that the ETX function satisfies three conditions for simplifying analysis. Under these assumptions, we characterize a path that minimizes route ETX in the lattice network and give a formula to compute minimum route ETX. This formula can be used as an upper bound of minimum route ETX without two of

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the three conditions. Then we show that this upper bound is close to minimum route ETX by comparing it with simulation results of minimum route ETX. Before deriving the formula, we also characterize minimum route ETX in a one-dimensional network where nodes are located at constant intervals. This characterization in a one-dimensional network is utilized in the above characterization in the lattice network. We also show that minimum route ETX in the lattice network is close to that in a two-dimensional random network if the node density is large by comparing the numerical and simulation results.

The rest of this paper is organized as follows. In Sect. 2, we explain our definitions and assumptions. In Sect. 3, we theoretically analyze minimum route ETX in a one-dimensional network where nodes are located at constant intervals. In Sect. 4, we theoretically analyze minimum route ETX in a two-dimensional network where nodes are located with lattice structure and compare the numerical results of minimum route ETX in the lattice network with simulation results in a two-dimensional random network. Section 5 concludes this paper.

2. Definitions and Assumptions

In this paper, we theoretically compute minimum route ETXs in static one-dimensional and lattice networks. Denote by $u(z)$ the ETX of a link with length z. Suppose that $0 < u(z) < \infty$ for $0 \le z \le d$, and $u(z) = \infty$ for $z > d$, where d is a constant and means the maximum transmitting range. Note that we assume that any pair of nodes has a link between them to explain the relation between a pair of nodes in the analysis. Therefore, a pair of nodes whose distance is greater than d cannot directly exchange any packets even though they are assumed to have a link.

We use six ETX functions denoted by $f_1(z)$ to $f_6(z)$. These functions, defined in Appendix A, are depicted in Fig. 1, where it is assumed that $d = 45$ for $f_1(z)$, $d = 57$ for $f_2(z)$, $d = 71$ for $f_3(z)$, $d = 90$ for $f_4(z)$, $d = 100$ for $f_5(z)$, and $d = 110$ for $f_6(z)$. $f_1(z)$ to $f_4(z)$ are computed from the relation between the packet reception rate and ETX, a channel model, a radio receiver model, and other parameters given in [4]–[6]. As seen from this figure, $f_1(z)$ to $f_4(z)$ are convex monotonically increasing functions. In this paper we assume that $u(z)$ is a convex monotonically increasing function based on these facts. Also, $f_5(z)$ and $f_6(z)$ are defined as convex monotonically increasing functions, as in Appendix A.

Consider the following three conditions:

A_1**:** $u(z)$ is a convex monotonically increasing function.
A_2**:** $1 < \frac{zu''(z)}{u'(z)} < 5$ for $0 \le z \le d$, where $u'(z)$ and $u''(z)$ are the first and second derivatives of $u(z)$, respectively.
A_3**:** $u(d) > 2u\left(\frac{d}{\sqrt{2}}\right)$.

$f_1(z)$ to $f_6(z)$ satisfy A_1, as mentioned. Figure 2 shows the numerical examples of $\frac{zu''(z)}{u'(z)}$ for $f_1(z)$ to $f_6(z)$. From Figs. 1 and 2, $f_1(z)$ to $f_4(z)$ do not satisfy A_2 but A_3. $f_5(z)$ was

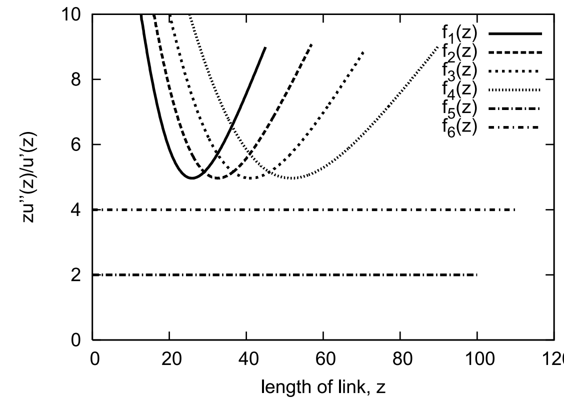

Fig. 2 Numerical examples of $\frac{zu''(z)}{u'(z)}$.

originally defined as a function that satisfies both A_2 and A_3. From the definition, $f_6(z)$ does not satisfy A_3 but A_2. The above three conditions and these differences between the ETX functions will be used to analyze minimum route ETX in the lattice network.

Let $U(r)$ be the route ETX of route r. Let R_k be the set of all k-hop paths. Let $r_{O,k}$ be the k-hop path that minimizes route ETX in R_k. Let r_{OR} be the path that minimizes route ETX in all paths. Namely, minimum route ETX is represented as $U(r_{OR}) = \min_k U(r_{O,k})$.

3. Analysis of a One-Dimensional Network

In this section, we consider a one-dimensional network where the nodes are located at coordinate ia for all i, where i is an integer, a is a positive real number, and $a \le d$. Suppose that S and D are located at 0 and L, where L is a multiple of a and $L \ge 0$. Suppose that all nodes including S and D are stationary. Figure 3 shows an example of the network.

Consider k-hop path r_1 that consists of k links whose lengths are $\ell_1, ..., \ell_k$, and another k-hop path r_2 that consists of k links whose lengths are $\ell'_1, ..., \ell'_k$. If we can construct a sequence identical to $\ell_1, ..., \ell_k$ by rearranging $\ell'_1, ..., \ell'_k$, then we regard r_1 and r_2 as identical.

We can exactly compute $U(r_{OR})$ by the following theorems:

Theorem 1. *Define $n_{1,k}$, $n_{2,k}$, $\ell_{1,k}$, and $\ell_{2,k}$ as*

$$n_{1,k} = k - n_{2,k}, \tag{1}$$

$$n_{2,k} = \frac{L}{a} - k\left\lfloor \frac{L}{ka} \right\rfloor, \tag{2}$$

$$\ell_{1,k} = \left\lfloor \frac{L}{ka} \right\rfloor a, \tag{3}$$

$$\ell_{2,k} = \ell_{1,k} + a. \tag{4}$$

Let $r_{1,k}$ be the k-hop path that consists of $n_{1,k}$ and $n_{2,k}$ links whose lengths are $\ell_{1,k}$ and $\ell_{2,k}$, respectively. $r_{1,k}$ always exists, and $r_{O,k} = r_{1,k}$.

Theorem 2. *Let n_0 be a positive integer that minimizes $\frac{u(n_0 a)}{n_0 a}$. Then*

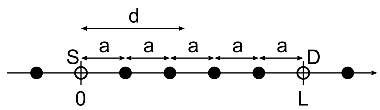

Fig. 3 One-dimensional multi-hop wireless network.

$$U(r_{OR}) = \begin{cases} U(r_{O,1}), & L \le n_0 a, \\ \min\left\{ U\left(r_{O,\left\lfloor \frac{L}{n_0 a} \right\rfloor}\right), U\left(r_{O,\left\lceil \frac{L}{n_0 a} \right\rceil}\right) \right\}, & \\ & L > n_0 a. \end{cases} \quad (5)$$

Theorem 1 means that minimum route ETX is realized by a path in which there are at most two kinds of link lengths and the difference between them is a. In [1], we characterized minimum route ETX in a one-dimensional random network and showed that we can construct a path which approximately minimizes route ETX by making the lengths of all links in the path close to each other. From these facts, we can see that Theorem 1 supports this characterization in [1]. Theorem 2 gives two candidates for the number of hops that realize minimum route ETX.

Theorems 1 and 2 analytically give minimum route ETX in a one-dimensional multi-hop wireless network without executing shortest path algorithms like Dijkstra algorithm [7] assuming that nodes are located at constant intervals along a straight line, link cost (i.e., ETX) is represented as a function of link length, and the function is a convex monotonically increasing function. These theorems can be used not only in such a wireless network but in general networks as long as the above assumptions can be made. Namely, links do not have to be wireless and can represent various things with the above assumptions.

3.1 Proof of Theorem 1

To prove Theorem 1, we use Lemmas 1 and 2.

Lemma 1. *If* $z_1 < z_2$*, then* $u(z_1+a)-u(z_1) < u(z_2+a)-u(z_2)$.

Proof. Let $u'(z)$ be the derivative of $u(z)$. Then

$$\{u(z_2 + a) - u(z_2)\} - \{u(z_1 + a) - u(z_1)\} = \int_0^a u'(z + z_2)dz - \int_0^a u'(z + z_1)dz > 0 \quad (6)$$

because $u'(z + z_2) - u'(z + z_1) > 0$ because $u(z)$ is a convex monotonically increasing function. □

Lemma 2. *Suppose that* r *is a* k*-hop path consisting of* k *links whose lengths are* $\ell_1, ..., \ell_k$*. If a pair of integers* i *and* j *exists such that* $\ell_j > \ell_i + a$*, then* $r \neq r_{O,k}$.

Proof. Without loss of generality, we assume that $\ell_2 > \ell_1 + a$. Let r' be a k-hop path consisting of links whose lengths are $\ell_1 + a, \ell_2 - a, \ell_3, ..., \ell_k$. Then we have

$$U(r)-U(r') = \{u(\ell_2)-u(\ell_2-a)\}-\{u(\ell_1+a)-u(\ell_1)\}. \quad (7)$$

From Eq. (7) and Lemma 1, we have $U(r) > U(r')$. This inequality means that $r \neq r_{O,k}$. □

Using these lemmas, we have Theorem 1 as follows: In $r_{1,k}$, the total length of the links equals L from Eqs. (1) to (4). Therefore, $r_{1,k}$ always exists. Let r be a k-hop path different from $r_{1,k}$. Suppose that the lengths of links included in r are $\ell_1, \ell_2, ..., \ell_k$. Then there must be integer i such that $\ell_i < \ell_{1,k}$ or $\ell_i > \ell_{2,k}$ because only $r_{1,k}$ consists of two kinds of links whose lengths are $\ell_{1,k}$ and $\ell_{2,k}$. If $\ell_i < \ell_{1,k}$, then there must be integer j such that $\ell_j > \ell_{1,k}$ so that $\ell_1+\ell_2+...+\ell_k = L$, and we have $\ell_j > \ell_i + a$. Hence, from Lemma 2, $r \neq r_{O,k}$. In the same manner, we can prove that if $\ell_i > \ell_{2,k}$, then $r \neq r_{O,k}$. Therefore, $r \neq r_{O,k}$ for all $r \in R_k - \{r_{1,k}\}$. Consequently, $r_{O,k} = r_{1,k}$.

3.2 Proof of Theorem 2

In this subsection, we represent $r_{O,k}$ as $r_{O,k}(L)$ to explicitly represent it with parameter L. Let n_0 be a positive integer that minimizes $\frac{u(n_0 a)}{n_0 a}$. Let k_O be the number of hops of r_{OR}. To prove Theorem 2, we use Lemmas 3 to 6.

Lemma 3.

$$U(r_{O,k}(L + a)) = U(r_{O,k}(L)) - u\left(\left\lfloor \frac{L}{ka} \right\rfloor a\right) + u\left(\left\lfloor \frac{L}{ka} \right\rfloor a + a\right). \quad (8)$$

Proof. If $L + a$ is not a multiple of ka, then $\left\lfloor \frac{L+a}{ka} \right\rfloor = \left\lfloor \frac{L}{ka} \right\rfloor$. If $L+a$ is a multiple of ka, then $\left\lfloor \frac{L+a}{ka} \right\rfloor = \frac{L+a}{ka}$ and $\left\lfloor \frac{L}{ka} \right\rfloor = \frac{L+a}{ka} - 1$. In both cases, from Theorem 1, Eq. (8) holds. □

Lemma 4. *Let* k_i *be the number of hops of* r_{OR} *for* $L = ia$*, where* i *is an integer and* $i \ge 0$*. If* $i < j$*, then* $k_i \le k_j$.

Proof. First, we consider the case where $j = i + 1$. In this case, from Lemma 3,

$$U(r_{O,k_i}(ja)) = U(r_{O,k_i}(ia)) - u\left(\left\lfloor \frac{ia}{k_i a} \right\rfloor a\right) + u\left(\left\lfloor \frac{ia}{k_i a} \right\rfloor a + a\right), \quad (9)$$

$$U(r_{O,k_j}(ja)) = U(r_{O,k_j}(ia)) - u\left(\left\lfloor \frac{ia}{k_j a} \right\rfloor a\right) + u\left(\left\lfloor \frac{ia}{k_j a} \right\rfloor a + a\right). \quad (10)$$

Assume that $k_i > k_j$. Then $\left\lfloor \frac{ia}{k_i a} \right\rfloor a \le \left\lfloor \frac{ia}{k_j a} \right\rfloor a$ and

$$u\left(\left\lfloor \frac{ia}{k_i a} \right\rfloor a + a\right) - u\left(\left\lfloor \frac{ia}{k_i a} \right\rfloor a\right) \le u\left(\left\lfloor \frac{ia}{k_j a} \right\rfloor a + a\right) - u\left(\left\lfloor \frac{ia}{k_j a} \right\rfloor a\right) \quad (11)$$

from Lemma 1. Also, from the definition of k_i, we have

$$U(r_{O,k_i}(ia)) < U(r_{O,k_j}(ia)). \quad (12)$$

From Eqs. (9), (10), (11), and (12), we have

$$U(r_{O,k_i}(ja)) < U(r_{O,k_j}(ja)), \quad (13)$$

which is a contradiction. Therefore, $k_i \le k_j$. Next, when $j = i + m$, where m is an integer and $m \ge 2$, from the above result, we obviously have $k_i \le k_{i+1} \le k_{i+2} \le ... \le k_{i+m} = k_j$. Then this lemma holds. □

Lemma 5. *$U(r) \ge \frac{u(n_0 a)}{n_0 a} L$ for all r.*

Proof. Suppose that r is a k-hop path that consists of k links whose lengths are $\ell_1, ..., \ell_k$. Then

$$U(r) = \sum_{i=1}^{k} u(\ell_i) \ge \sum_{i=1}^{k} \frac{u(n_0 a)}{n_0 a} \ell_i = \frac{u(n_0 a)}{n_0 a} L \tag{14}$$

since $\frac{u(\ell_i)}{\ell_i} \ge \frac{u(n_0 a)}{n_0 a}$ for $1 \le i \le k$. □

Lemma 6. *If L is a multiple of $n_0 a$, then $k_O = \frac{L}{n_0 a}$.*

Proof. We have $U\left(r_{O,\frac{L}{n_0 a}}\right) = \frac{u(n_0 a)}{n_0 a} L$ from Theorem 1, and $U(r) \ge U\left(r_{O,\frac{L}{n_0 a}}\right)$ for all r from Lemma 5. Then $k_O = \frac{L}{n_0 a}$. □

Using these lemmas, we have Theorem 2 as follows: If $L = n_0 a$, then $k_O = 1$ from Lemma 6. From this result and Lemma 4, $k_O = 1$ if $L < n_0 a$. Hence, $U(r_{OR}) = U(r_{O,1})$ if $L \le n_0 a$. Next, we consider the case where $L > n_0 a$. Suppose that L' is a multiple of a and $L' > n_0 a$. From Lemma 6, $k_O = \left\lfloor \frac{L'}{n_0 a} \right\rfloor$ if $L = \left\lfloor \frac{L'}{n_0 a} \right\rfloor n_0 a$, and $k_O = \left\lceil \frac{L'}{n_0 a} \right\rceil$ if $L = \left\lceil \frac{L'}{n_0 a} \right\rceil n_0 a$. Since $\left\lfloor \frac{L'}{n_0 a} \right\rfloor n_0 a \le L' \le \left\lceil \frac{L'}{n_0 a} \right\rceil n_0 a$, we have $k_O = \left\lfloor \frac{L'}{n_0 a} \right\rfloor$ or $k_O = \left\lceil \frac{L'}{n_0 a} \right\rceil$ if $L = L'$ from Lemma 4. Consequently, $U(r_{OR}) = \min\left\{ U\left(r_{O,\left\lfloor \frac{L}{n_0 a} \right\rfloor}\right), U\left(r_{O,\left\lceil \frac{L}{n_0 a} \right\rceil}\right) \right\}$ if $L > n_0 a$.

3.3 Numerical Results

We show some numerical results of minimum route ETX in a one-dimensional network, where $a = 10$. We compute minimum route ETX by Eq. (5) for ETX functions $f_1(z)$, $f_2(z)$, $f_3(z)$, and $f_4(z)$ depicted in Fig. 1. We set d to 45 for $f_1(z)$, 57 for $f_2(z)$, 71 for $f_3(z)$, and 90 for $f_4(z)$. Figure 4 shows the numerical results with simulation results computed by the Dijkstra algorithm [7]. From Fig. 4, we can confirm that the numerical results completely agree with the simulation results.

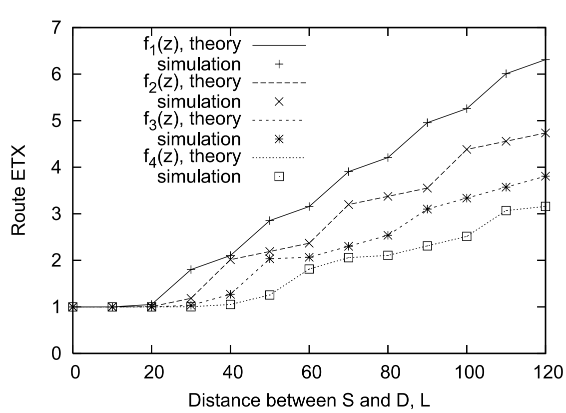

Fig. 4 Minimum route ETX, where $a = 10$.

4. Analysis of Two-Dimensional Networks

In this section, we consider a two-dimensional network where nodes are located at coordinates (ia, ja) for all integers i and j, where a is a positive real number and $a \le d$. Suppose that S and D are located at $(0, 0)$ and (L_x, L_y), respectively, where L_x and L_y are multiples of a, $L_x \ge 0$, and $L_y \ge 0$. Suppose that all nodes including S and D are stationary. Figure 5 shows an example of the network. In this section, we represent a link from a node at (x_1, y_1) to a node at (x_2, y_2) as vector $\boldsymbol{\ell} = (x_2 - x_1, y_2 - y_1)$.

Consider k-hop path r_1 consisting of k links $\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_k$, and another k-hop path r_2 consisting of k links $\boldsymbol{\ell}'_1, ..., \boldsymbol{\ell}'_k$. If we can construct a sequence identical to $\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_k$ by rearranging $\boldsymbol{\ell}'_1, ..., \boldsymbol{\ell}'_k$, then we regard r_1 and r_2 as identical.

Consider the following three conditions on $u(z)$:

- A_1**:** $u(z)$ is a convex monotonically increasing function.
- A_2**:** $1 < \frac{z u''(z)}{u'(z)} < 5$ for $0 \le z \le d$, where $u'(z)$ and $u''(z)$ are the first and second derivatives of $u(z)$, respectively.
- A_3**:** $u(d) > 2u\left(\frac{d}{\sqrt{2}}\right)$†.

If $u(z)$ satisfies A_1 to A_3, we can exactly compute $U(r_{OR})$ by the following theorems:

Theorem 3. *Assume that $u(z)$ satisfies A_1 and A_2. Define $n_{3,k}$ to $n_{6,k}$ and $\boldsymbol{\ell}_{3,k}$ to $\boldsymbol{\ell}_{6,k}$ as*

$$n_{3,k} = k - n_{4,k} - n_{5,k} - n_{6,k}, \tag{15}$$

$$n_{4,k} = \frac{L_x}{a} - k \left\lfloor \frac{L_x}{ka} \right\rfloor - n_{6,k}, \tag{16}$$

$$n_{5,k} = \frac{L_y}{a} - k \left\lfloor \frac{L_y}{ka} \right\rfloor - n_{6,k}, \tag{17}$$

$$n_{6,k} = \max\left\{0, \frac{L_x + L_y}{a} - k \left\lfloor \frac{L_x}{ka} \right\rfloor - k \left\lfloor \frac{L_y}{ka} \right\rfloor - k \right\}, \tag{18}$$

$$\boldsymbol{\ell}_{3,k} = \left(\left\lfloor \frac{L_x}{ka} \right\rfloor a, \left\lfloor \frac{L_y}{ka} \right\rfloor a \right), \tag{19}$$

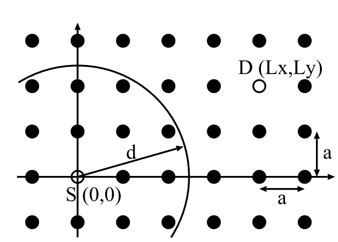

Fig. 5 Lattice multi-hop wireless network.

†We used an equation $u(d_{s,opt}) = 2u\left(\frac{d_{s,opt}}{2}\right)$ and a condition $u(d) \ge 2u\left(\frac{d}{2}\right)$, which are similar to A_3, in [1]; however, these are used just for obtaining $d_{s,opt}$, which is an important parameter in [1], and have no direct relation with A_3.

$$\boldsymbol{\ell}_{4,k} = \boldsymbol{\ell}_{3,k} + (a, 0), \tag{20}$$
$$\boldsymbol{\ell}_{5,k} = \boldsymbol{\ell}_{3,k} + (0, a), \tag{21}$$
$$\boldsymbol{\ell}_{6,k} = \boldsymbol{\ell}_{3,k} + (a, a). \tag{22}$$

Let $r_{2,k}$ be the k-hop path consisting of $n_{3,k}$ links, $n_{4,k}$ links, $n_{5,k}$ links, and $n_{6,k}$ links represented as $\boldsymbol{\ell}_{3,k}$, $\boldsymbol{\ell}_{4,k}$, $\boldsymbol{\ell}_{5,k}$, and $\boldsymbol{\ell}_{6,k}$, respectively. If $U(r_{2,k}) < \infty$, then $U(r_{O,k}) = U(r_{2,k})$.

Theorem 4. *Assume that $u(z)$ satisfies A_1 to A_3. Define $r_{2,k}$ in the same manner as in Theorem 3. Then*

$$U(r_{OR}) = \begin{cases} u(0), & L_x = 0 \text{ and } L_y = 0, \\ \min\limits_{1 \le k \le \frac{L_x+L_y}{a}, U(r_{2,k})<\infty} U(r_{2,k}), & \\ & L_x > 0 \text{ or } L_y > 0. \end{cases} \tag{23}$$

We can exactly compute $U(r_{OR})$ by Eq. (23) if $u(z)$ satisfies A_1 to A_3. Theorems 3 and 4 analytically give minimum route ETX in a lattice multi-hop wireless network without executing shortest path algorithms like Dijkstra algorithm assuming that nodes are located with lattice structure, link cost (i.e., ETX) is represented as a function of link length, and the function satisfies A_1 to A_3. These theorems give an analytical framework for solutions to shortest path problems in general lattice networks as long as the above assumptions can be made.

As can be easily expected, the ETX functions do not always satisfy these conditions, especially A_2 and A_3. Even if $u(z)$ does not satisfy A_1 to A_3, the right side of Eq. (23) can be used as an upper bound of minimum route ETX. Namely, we generally have

$$U(r_{OR}) \le \begin{cases} u(0), & L_x = 0 \text{ and } L_y = 0, \\ \min\limits_{1 \le k \le \frac{L_x+L_y}{a}, U(r_{2,k})<\infty} U(r_{2,k}), & \\ & L_x > 0 \text{ or } L_y > 0. \end{cases} \tag{24}$$

The proof of Eq. (24) is provided in Appendix B. We numerically evaluate the relation between this upper bound and minimum route ETX in Sect. 4.3. We also compare the numerical results of Eq. (23) with the simulation results in a two-dimensional random network to investigate whether the above theorems can be applied to random networks.

Before proving Theorems 3 and 4, we explain the meaning of condition A_2. Analysis of the lattice network considered here is an extended version of that of the one-dimensional network in Sect. 3. We showed how to construct $r_{O,k}$ in a one-dimensional network when $u(z)$ satisfies A_1 in Sect. 3. We intuitively expected that $r_{2,k}$ defined in Theorem 3 equals $r_{O,k}$ in the lattice network when $u(z)$ satisfies A_1 and tried to prove that $r_{2,k}$ and $r_{O,k}$ are identical only with the assumption A_1 because x-components of links in $r_{2,k}$ equal links included in $r_{O,k}$ of the one-dimensional network with $L = L_x$, and y-components equal those with $L = L_y$. Unfortunately, however, we found that $r_{2,k}$ and $r_{O,k}$ are not always identical if $u(z)$ is a function that increases too rapidly or too slowly. Then we added condition A_2, which means that $u(z)$ does not increase too rapidly or too slowly. As will be proved in the following subsection, while A_1 and A_2 hold, $r_{2,k}$ and $r_{O,k}$ are always identical. Next, we explain the meaning of A_3. Suppose that $r_{OR} = r_{O,k}$. This means that a k-hop path minimizes route ETX. We found that, even in such a case, $U(r_{2,k})$ can be infinity if $u(d)$ is very small. This means that $U(r_{2,k})$ and $U(r_{OR})$ are not identical in such a case. Therefore, we added A_3 to prevent a situation where $u(d)$ is too small to make $U(r_{2,k})$ infinity.

4.1 Proof of Theorem 3

To prove Theorem 3, we use the following function:

$$\hat{u}(z) = \begin{cases} u(z), & 0 \le z \le d, \\ \dfrac{u'(d)}{4d^3}(z^4 - d^4) + u(d), & z > d. \end{cases} \tag{25}$$

As easily proved, if $u(z)$ satisfies A_1 and A_2, then $\hat{u}(z)$ satisfies the following conditions:

B_1: $\hat{u}(z)$ monotonically increases with respect to z for $z \ge 0$.
B_2: $\frac{\hat{u}'(z)}{z}$ monotonically increases with respect to z for $z \ge 0$, where $\hat{u}'(z)$ is the derivative of $\hat{u}(z)$.
B_3: $\frac{\hat{u}'(z)}{z^5}$ monotonically decreases with respect to z for $z \ge 0$.

Let $\hat{U}(r)$ be the route ETX computed using $\hat{u}(z)$ as a link ETX function instead of $u(z)$. Note that $\hat{U}(r) = U(r)$ if $U(r) < \infty$. Also, in general, $\hat{U}(r) \le U(r)$ for any route r. We also use the following function:

$$\hat{u}_p(g, h, c) = \hat{u}\left(\sqrt{g^2 + h^2 + 2ghc}\right) + \hat{u}\left(\sqrt{g^2 + h^2 - 2ghc}\right), \tag{26}$$

where $g \ge 0$, $h \ge 0$, and $0 \le c \le 1$. Also, if $\hat{u}(z)$ satisfies B_1 to B_3, we have

P_1: If $h' \le h$, then $\hat{u}_p(g, h', c) \le \hat{u}_p(g, h, c)$.
P_2: If $c' \le c$, then $\hat{u}_p(g, h, c') \le \hat{u}_p(g, h, c)$. Especially if $g > 0$, $h > 0$, and $c' < c$, then $\hat{u}_p(g, h, c') < \hat{u}_p(g, h, c)$.
P_3: If $h > 0$, then $\hat{u}_p(g, h, 1) < \hat{u}_p\left(g, \sqrt{5}h, 0\right)$.
P_4: If $h > 0$ and $h \ge \sqrt{5}h'$, then $\hat{u}_p(g, h', c') < \hat{u}_p(g, h, c)$.

The proofs of P_1 to P_4 are provided in Appendix C.

To prove Theorem 3, we use Lemmas 7 to 10.

Lemma 7. *Suppose that x_i, y_i, x_j, and y_j are multiples of a. Let $\boldsymbol{\ell}_i$ and $\boldsymbol{\ell}_j$ be (x_i, y_i) and (x_j, y_j), respectively. Let $\boldsymbol{\ell}'_i$ and $\boldsymbol{\ell}'_j$ be (x'_i, y'_i) and (x'_j, y'_j), respectively, where $x'_i = \left[\frac{x_i+x_j}{2a}\right]a$, $y'_i = y_i + y_j - y'_j$, $x'_j = x_i + x_j - x'_i$, and $y'_j = \left[\frac{y_i+y_j}{2a}\right]a$, where $[\cdot]$ is the integer part of $\cdot$. If $|x_j - x_i| \ge 2a$ or $|y_j - y_i| \ge 2a$, then $|\boldsymbol{\ell}_i + \boldsymbol{\ell}_j| = |\boldsymbol{\ell}'_i + \boldsymbol{\ell}'_j|$, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| > 0$, and $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \ge \sqrt{5}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|$.*

Proof. Clearly, if $|x_j - x_i| \ge 2a$ or $|y_j - y_i| \ge 2a$, then $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| > 0$. Also, from the definitions of $\boldsymbol{\ell}'_i$ and $\boldsymbol{\ell}'_j$, $|\boldsymbol{\ell}_i + \boldsymbol{\ell}_j| = |\boldsymbol{\ell}'_i + \boldsymbol{\ell}'_j|$. If $|x_j - x_i| \ge 2a$, then

1) $|x_j - x_i| = 2a$ and $|y_j - y_i| = 0$,

2) $|x_j - x_i| = 2a$ and $|y_j - y_i| \geq a$,
3) $|x_j - x_i| \geq 3a$ and $|y_j - y_i| = 0$, or
4) $|x_j - x_i| \geq 3a$ and $|y_j - y_i| \geq a$.

In 1), $|x'_j - x'_i| = 0$ and $|y'_j - y'_i| = 0$; therefore, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| = 2a$ and $|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i| = 0$. In 2), $|x'_j - x'_i| = 0$ and $|y'_j - y'_i| \leq a$; therefore, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{5}a$ and $|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i| \leq a$. In 3), $|x'_j - x'_i| \leq a$ and $|y'_j - y'_i| = 0$; therefore, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq 3a$ and $|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i| \leq a$. In 4), $|x'_j - x'_i| \leq a$ and $|y'_j - y'_i| \leq a$; therefore, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{10}a$ and $|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i| \leq \sqrt{2}a$. Consequently, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{5}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|$ in 1) to 4). In the same manner, we can prove that if $|y_j - y_i| \geq 2a$, then $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{5}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|$. □

Lemma 8. *If* $|\boldsymbol{\ell}_i + \boldsymbol{\ell}_j| = |\boldsymbol{\ell}'_i + \boldsymbol{\ell}'_j|$, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| > 0$, *and* $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{5}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|$, *then* $\hat{u}(|\boldsymbol{\ell}_i|) + \hat{u}(|\boldsymbol{\ell}_j|) > \hat{u}(|\boldsymbol{\ell}'_i|) + \hat{u}(|\boldsymbol{\ell}'_j|)$.

Proof. Define g, h, c, g', h', and c' as

$$g = \frac{1}{2}|\boldsymbol{\ell}_j + \boldsymbol{\ell}_i|, \tag{27}$$

$$h = \frac{1}{2}|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i|, \tag{28}$$

$$c = \frac{|\boldsymbol{\ell}_j + \boldsymbol{\ell}_i|^2 + |\boldsymbol{\ell}_j - \boldsymbol{\ell}_i|^2 - 4\min\{|\boldsymbol{\ell}_i|, |\boldsymbol{\ell}_j|\}^2}{2|\boldsymbol{\ell}_j + \boldsymbol{\ell}_i||\boldsymbol{\ell}_j - \boldsymbol{\ell}_i|}, \tag{29}$$

$$g' = \frac{1}{2}|\boldsymbol{\ell}'_j + \boldsymbol{\ell}'_i|, \tag{30}$$

$$h' = \frac{1}{2}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|, \tag{31}$$

$$c' = \frac{|\boldsymbol{\ell}'_j + \boldsymbol{\ell}'_i|^2 + |\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|^2 - 4\min\{|\boldsymbol{\ell}'_i|, |\boldsymbol{\ell}'_j|\}^2}{2|\boldsymbol{\ell}'_j + \boldsymbol{\ell}'_i||\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|}. \tag{32}$$

From the assumptions, $g = g'$, $h > 0$, and $h \geq \sqrt{5}h'$. Then $\hat{u}_p(g', h', c') < \hat{u}_p(g, h, c)$ from P_4. Also, by substituting Eqs. (27) to (32) into Eq. (26), we have

$$\hat{u}_p(g, h, c) = \hat{u}(|\boldsymbol{\ell}_i|) + \hat{u}(|\boldsymbol{\ell}_j|), \tag{33}$$

$$\hat{u}_p(g', h', c') = \hat{u}(|\boldsymbol{\ell}'_i|) + \hat{u}(|\boldsymbol{\ell}'_j|). \tag{34}$$

Consequently, $\hat{u}(|\boldsymbol{\ell}_i|) + \hat{u}(|\boldsymbol{\ell}_j|) > \hat{u}(|\boldsymbol{\ell}'_i|) + \hat{u}(|\boldsymbol{\ell}'_j|)$. □

Lemma 9. *Let r be a k-hop path consisting of k links* $\boldsymbol{\ell}_1 = (x_1, y_1)$, ..., $\boldsymbol{\ell}_k = (x_k, y_k)$. *If a pair of integers i and j exists such that* $|x_j - x_i| \geq 2a$, $|y_j - y_i| \geq 2a$, *or* ($|x_j| = |x_i| + a$ *and* $|y_j| = |y_i| + a$), *then k-hop path r' exists such that* $\hat{U}(r') < \hat{U}(r)$.

Proof. Let $\boldsymbol{\ell}'_i$ and $\boldsymbol{\ell}'_j$ be (x'_i, y'_i) and (x'_j, y'_j), respectively, where $x'_i = \left\lceil \frac{x_i + x_j}{2a} \right\rceil a$, $y'_i = y_i + y_j - y'_j$, $x'_j = x_i + x_j - x'_i$, and $y'_j = \left\lceil \frac{y_i + y_j}{2a} \right\rceil a$. Let r' be the k-hop path made by replacing $\boldsymbol{\ell}_i$ and $\boldsymbol{\ell}_j$ of r with $\boldsymbol{\ell}'_i$ and $\boldsymbol{\ell}'_j$. First, suppose that $|x_j - x_i| \geq 2a$ or $|y_j - y_i| \geq 2a$. From Lemma 7, $|\boldsymbol{\ell}_i + \boldsymbol{\ell}_j| = |\boldsymbol{\ell}'_i + \boldsymbol{\ell}'_j|$, $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| > 0$, and $|\boldsymbol{\ell}_j - \boldsymbol{\ell}_i| \geq \sqrt{5}|\boldsymbol{\ell}'_j - \boldsymbol{\ell}'_i|$. Then from Lemma 8, $\hat{u}(|\boldsymbol{\ell}_i|) + \hat{u}(|\boldsymbol{\ell}_j|) > \hat{u}(|\boldsymbol{\ell}'_i|) + \hat{u}(|\boldsymbol{\ell}'_j|)$. Hence, $\hat{U}(r') < \hat{U}(r)$. Next, suppose that $|x_j| = |x_i| + a$ and $|y_j| = |y_i| + a$. In this case, $\boldsymbol{\ell}'_i = (x_i, y_j)$ and $\boldsymbol{\ell}'_j = (x_j, y_i)$. If we define g, h, c, g', h', and c' as Eqs. (27) to (32), we have $g = g' > 0$, $h = h' > 0$, and $c > c'$. Hence, $\hat{u}_p(g', h', c') < \hat{u}_p(g, h, c)$ from P_2. From this relation and Eqs. (33) and (34), $\hat{u}(|\boldsymbol{\ell}_i|) + \hat{u}(|\boldsymbol{\ell}_j|) > \hat{u}(|\boldsymbol{\ell}'_i|) + \hat{u}(|\boldsymbol{\ell}'_j|)$. Therefore, $\hat{U}(r') < \hat{U}(r)$. □

Lemma 10. $\hat{U}(r_{2,k}) = \min_{r \in R_k} \hat{U}(r)$.

Proof. Let r be a k-hop path different from $r_{2,k}$. Suppose that r consists of links $\boldsymbol{\ell}_1 = (x_1, y_1)$, ..., $\boldsymbol{\ell}_k = (x_k, y_k)$. First, suppose integer i such that $\boldsymbol{\ell}_i$ is not equal to $\boldsymbol{\ell}_{3,k}$, $\boldsymbol{\ell}_{4,k}$, $\boldsymbol{\ell}_{5,k}$, or $\boldsymbol{\ell}_{6,k}$. In this case, $x_i < \left\lfloor \frac{L_x}{ka} \right\rfloor a$, $x_i > \left\lfloor \frac{L_x}{ka} \right\rfloor a + a$, $y_i < \left\lfloor \frac{L_y}{ka} \right\rfloor a$, or $y_i > \left\lfloor \frac{L_y}{ka} \right\rfloor a + a$. If $x_i < \left\lfloor \frac{L_x}{ka} \right\rfloor a$, then there must be integer j such that $x_j > \left\lfloor \frac{L_x}{ka} \right\rfloor a$ so that $x_1 + x_2 + ... + x_k = L_x$, and we have $x_j - x_i \geq 2a$. Hence, from Lemma 9, k-hop path r' exists such that $\hat{U}(r') < \hat{U}(r)$. In the same manner, we can prove that if $x_i > \left\lfloor \frac{L_x}{ka} \right\rfloor a + a$, $y_i < \left\lfloor \frac{L_y}{ka} \right\rfloor a$, or $y_i > \left\lfloor \frac{L_y}{ka} \right\rfloor a + a$, then k-hop path r' exists such that $\hat{U}(r') < \hat{U}(r)$. Next, suppose that $\boldsymbol{\ell}_i$ is equal to $\boldsymbol{\ell}_{3,k}$, $\boldsymbol{\ell}_{4,k}$, $\boldsymbol{\ell}_{5,k}$, or $\boldsymbol{\ell}_{6,k}$ for all i. In this case, r includes both $\boldsymbol{\ell}_{3,k}$ and $\boldsymbol{\ell}_{6,k}$ since only $r_{2,k}$ consists of three kinds of links: $\{\boldsymbol{\ell}_{3,k}, \boldsymbol{\ell}_{4,k}, \boldsymbol{\ell}_{5,k}\}$ or $\{\boldsymbol{\ell}_{4,k}, \boldsymbol{\ell}_{5,k}, \boldsymbol{\ell}_{6,k}\}$. If $\boldsymbol{\ell}_i = \boldsymbol{\ell}_{3,k}$ and $\boldsymbol{\ell}_j = \boldsymbol{\ell}_{6,k}$, then $|x_j| = |x_i| + a$ and $|y_j| = |y_i| + a$. Hence, from Lemma 9, k-hop path r' exists such that $\hat{U}(r') < \hat{U}(r)$. From these results, k-hop path r' exists such that $\hat{U}(r') < \hat{U}(r)$ for all $r \in R_k - \{r_{2,k}\}$. Therefore, $\hat{U}(r_{2,k}) = \min_{r \in R_k} \hat{U}(r)$. □

Generally, $\hat{U}(r_{O,k}) \leq U(r_{O,k}) \leq U(r_{2,k})$. From Lemma 10, $\hat{U}(r_{2,k}) \leq \hat{U}(r_{O,k})$. If $U(r_{2,k}) < \infty$, then $\hat{U}(r_{2,k}) = U(r_{2,k})$. From these relations, Theorem 3 holds.

4.2 Proof of Theorem 4

To prove Theorem 4, we use Lemmas 11 to 13.

Lemma 11. $\hat{u}(z) > 2\hat{u}\left(\frac{z}{\sqrt{2}}\right)$ *for* $z > d$.

Proof. Let $b(z)$ be $\hat{u}(z) - 2\hat{u}\left(\frac{z}{\sqrt{2}}\right)$. Then

$$\frac{db(z)}{dz} = z \cdot \left\{ \frac{\hat{u}'(z)}{z} - \frac{\hat{u}'\left(\frac{z}{\sqrt{2}}\right)}{\frac{z}{\sqrt{2}}} \right\}. \tag{35}$$

We have $b(d) > 0$ from A_3, and $\frac{db(z)}{dz} > 0$ from Eq. (35) and B_2. From these relations, $b(z) > 0$ for $z > d$. Therefore, $\hat{u}(z) > 2\hat{u}\left(\frac{z}{\sqrt{2}}\right)$ for $z > d$. □

Lemma 12. *Suppose that x and y are multiples of a,* $x \geq 0$, $y \geq 0$, *and* $\sqrt{x^2 + y^2} > a$. *If* $x_1 = \left\lfloor \frac{x}{2a} \right\rfloor a$, $x_2 = \left\lceil \frac{x}{2a} \right\rceil a$, $y_1 = \left\lceil \frac{y}{2a} \right\rceil a$, *and* $y_2 = \left\lfloor \frac{y}{2a} \right\rfloor a$, *then* $\sqrt{x_1^2 + y_1^2} \leq \sqrt{\frac{x^2+y^2}{2}}$ *and* $\sqrt{x_2^2 + y_2^2} \leq \sqrt{\frac{x^2+y^2}{2}}$.

Proof. If $\sqrt{x^2 + y^2} > a$, then 1) $y \neq a$, 2) $y = a$ and $x = a$, or 3) $y = a$ and $x \geq 2a$. In 1), we have $y_1 \leq \frac{2}{3}y$ and $x_1 \leq \frac{x}{2}$; therefore, $\sqrt{x_1^2 + y_1^2} \leq \sqrt{\frac{x^2}{4} + \frac{4y^2}{9}} \leq \sqrt{\frac{x^2+y^2}{2}}$. In

2), $\sqrt{x_1^2+y_1^2} = \sqrt{\frac{x^2+y^2}{2}}$ because $x_1 = 0$ and $y_1 = a$. In 3), $y_1 = a$ and $x_1 \le \frac{x}{2}$; therefore, $\frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x^2+y^2}} \le \frac{\sqrt{\left(\frac{x}{2}\right)^2+a^2}}{\sqrt{x^2+a^2}}$. Also, in 3), $\frac{\sqrt{\left(\frac{x}{2}\right)^2+a^2}}{\sqrt{x^2+a^2}} \le \frac{\sqrt{\left(\frac{2a}{2}\right)^2+a^2}}{\sqrt{(2a)^2+a^2}} \le \frac{1}{\sqrt{2}}$ because $\frac{\sqrt{\left(\frac{x}{2}\right)^2+a^2}}{\sqrt{x^2+a^2}}$ monotonically decreases with respect to x and $x \ge 2a$. Hence, $\sqrt{x_1^2+y_1^2} \le \sqrt{\frac{x^2+y^2}{2}}$ in 3). In the same manner, we can prove that $\sqrt{x_2^2+y_2^2} \le \sqrt{\frac{x^2+y^2}{2}}$. □

Lemma 13. *If $U(r_{2,k}) = \infty$, then $r_{O,k} \ne r_{OR}$.*

Proof. Suppose that $U(r_{2,k}) = \infty$. In this case, $r_{2,k}$ includes a link whose length is greater than d. Denote this link by $\boldsymbol{\ell} = (x, y)$. Note that $x \ge 0$ and $y \ge 0$ because (x, y) is included in $r_{2,k}$ and we have $\sqrt{x^2+y^2} > a$ if $|\boldsymbol{\ell}| > d$. We build $k+1$-hop path $r_{3,k+1}$ by replacing $\boldsymbol{\ell}$ of $r_{2,k}$ with two vectors denoted by $\boldsymbol{\ell}_1 = (x_1, y_1)$ and $\boldsymbol{\ell}_2 = (x_2, y_2)$. Suppose that $x_1 = \left\lfloor \frac{x}{2a} \right\rfloor a$, $x_2 = \left\lceil \frac{x}{2a} \right\rceil a$, $y_1 = \left\lceil \frac{y}{2a} \right\rceil a$, and $y_2 = \left\lfloor \frac{y}{2a} \right\rfloor a$. From Lemma 12, $|\boldsymbol{\ell}_1| \le \frac{1}{\sqrt{2}}|\boldsymbol{\ell}|$ and $|\boldsymbol{\ell}_2| \le \frac{1}{\sqrt{2}}|\boldsymbol{\ell}|$. Then from Lemma 11, $\hat{u}(|\boldsymbol{\ell}_1|) + \hat{u}(|\boldsymbol{\ell}_2|) \le 2\hat{u}\left(\frac{1}{\sqrt{2}}|\boldsymbol{\ell}|\right) < \hat{u}(|\boldsymbol{\ell}|)$. This means that $\hat{U}(r_{3,k+1}) < \hat{U}(r_{2,k})$. From this relation and Lemma 10, we have $\hat{U}(r_{2,k+1}) < \hat{U}(r_{2,k})$.

Let k_{min} be the minimum value of k' such that $k' > k$ and $U(r_{2,k'}) < \infty$. From the above relation, $\hat{U}(r_{2,k}) > \hat{U}(r_{2,k+1}) > \ldots > \hat{U}(r_{2,k_{min}})$. Also, $U(r_{O,k}) \ge \hat{U}(r_{O,k}) \ge \hat{U}(r_{2,k})$ from Lemma 10, and $U(r_{O,k_{min}}) = \hat{U}(r_{2,k_{min}})$ from Theorem 3. From these relations, $U(r_{O,k}) > U(r_{O,k_{min}})$. This inequality means that $r_{O,k} \ne r_{OR}$. □

Using these lemmas, we have Theorem 4 as follows. Clearly, if $L_x = 0$ and $L_y = 0$, then $U(r_{OR}) = u(0)$. If $L_x > 0$ or $L_y > 0$, then we have

$$U(r_{OR}) = \min_{1 \le k \le \frac{L_x+L_y}{a}} U(r_{O,k}) \qquad (36)$$

since the number of hops of r_{OR} is not greater than $\frac{L_x+L_y}{a}$. From Eq. (36) and Lemma 13, we have

$$U(r_{OR}) = \min_{1 \le k \le \frac{L_x+L_y}{a}, U(r_{2,k})<\infty} U(r_{O,k}). \qquad (37)$$

Therefore, from Eq. (37) and Theorem 3, Eq. (23) holds.

4.3 Numerical Results

We show some numerical examples of Eq. (23). We use functions $f_1(z)$ to $f_6(z)$ defined in Sect. 2 as $u(z)$, and show the numerical results of Eq. (23) for these functions. The numerical results of Eq. (23) for $f_5(z)$ are the exact values of the minimum route ETX, and those for $f_1(z)$ to $f_4(z)$ and $f_6(z)$ are the upper bounds of the minimum route ETX because $f_1(z)$ to $f_4(z)$ do not satisfy A_2 and $f_6(z)$ does not satisfy A_3. Figures 6 and 7 show the numerical results of Eq. (23) for $f_5(z)$ and $f_6(z)$, respectively. In Fig. 6, $a = 20$ and $d = 100$, and $a = 20$ and $d = 110$ in Fig. 7. Figure 8 shows those for $f_1(z)$, $f_2(z)$, $f_3(z)$, and $f_4(z)$, where $a = 10$, $d = 45$ for $f_1(z)$, $d = 57$ for $f_2(z)$, $d = 71$ for $f_3(z)$, and $d = 90$ for $f_4(z)$. Suppose again that source node S is assumed to be at $(0, 0)$. In these figures, the x-, y-, and z-axes represent L_x, L_y, and the minimum route ETX for the destination node at (L_x, L_y), respectively. In Figs. 6 and 7, $L_x = 0, a, 2a, \ldots, 12a$, and $L_y = 0, a, 2a, \ldots, 12a$. In Fig. 8, we show the minimum route ETXs for three values of L_x to make this figure more visible. Then $L_x = 0, 6a, 12a$, and $L_y = 0, a, 2a, \ldots, 12a$. These figures also show the simulation

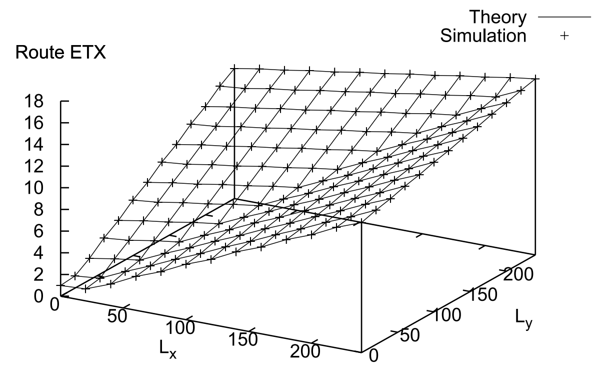

Fig. 6 Minimum route ETX, where $a = 20$, $u(z) = f_5(z)$, and $d = 100$.

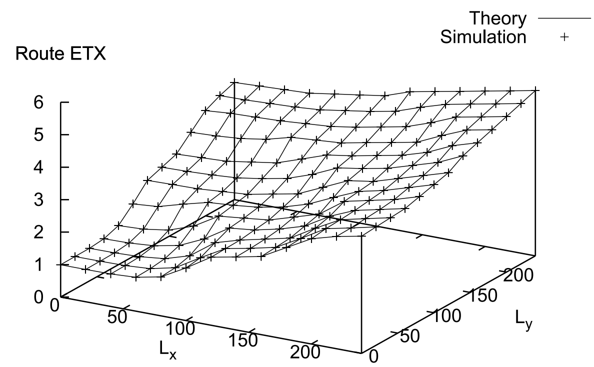

Fig. 7 Minimum route ETX, where $a = 20$, $u(z) = f_6(z)$, and $d = 110$.

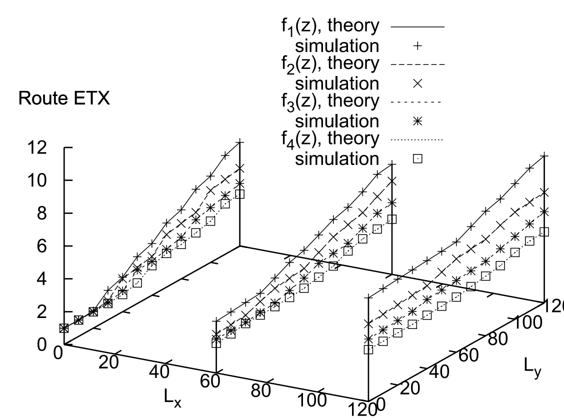

Fig. 8 Minimum route ETX, where $a = 10$, $u(z) = f_1(z)$, $f_2(z)$, $f_3(z)$, $f_4(z)$, $d = 45$ for $f_1(z)$, $d = 57$ for $f_2(z)$, $d = 71$ for $f_3(z)$, and $d = 90$ for $f_4(z)$.

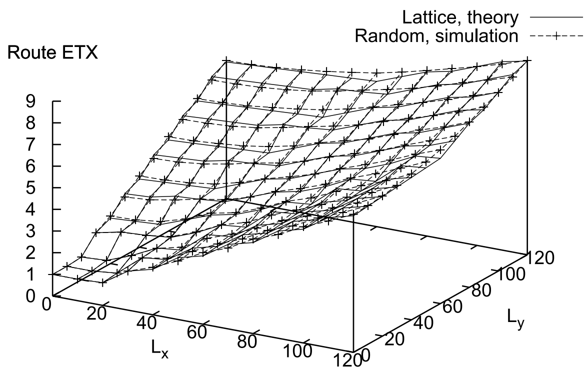

Fig. 9 Minimum route ETX, where $a = 10$, $u(z) = f_1(z)$, and $d = 45$.

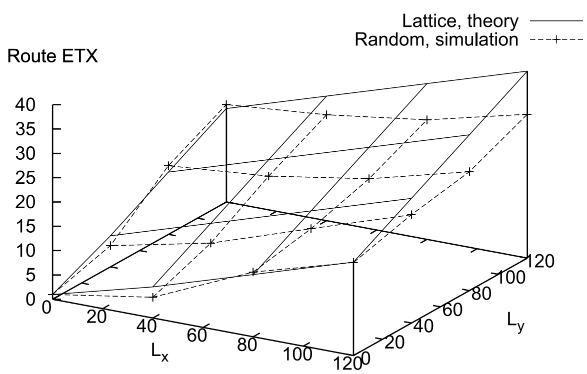

Fig. 10 Minimum route ETX, where $a = 40$, $u(z) = f_1(z)$, and $d = 45$.

results of the minimum route ETX computed by the Dijkstra algorithm. From Fig. 6, we can confirm that the numerical results completely agree with the simulation results because $f_5(z)$ satisfies A_1, A_2, and A_3. Figure 7 shows that the upper bounds are close to the simulation results even though $f_6(z)$ does not satisfy A_3. As mentioned in Appendix A, we define $f_1(z)$ to $f_4(z)$ assuming that transmitters and receivers are mica2 mote nodes. As a result, unfortunately, these functions do not satisfy A_2; however, Fig. 8 shows that the upper bounds are close to the simulation results. From these results, Eq. (23) gives a good approximation to minimum route ETX even if A_2 or A_3 is not satisfied although derivation of $u(z)$ based on actual communication nodes, such as mica2 mote, sometimes causes violation of A_2 and A_3.

Next, we compare the numerical results of Eq. (23) with the simulation results of the minimum route ETX in a two-dimensional network where the nodes are randomly distributed. In computer simulations, we randomly distribute N nodes on a $[-200, 200] \times [-200, 200]$ square area, where N is decided based on a Poisson distribution with intensity $\frac{1}{a^2}$. The densities of the nodes are the same in both networks. We locate S and D at $(0, 0)$ and (L_x, L_y), respectively. If there is at least one path whose ETX is finite between S and D, we construct a path that minimizes route ETX using the Dijkstra algorithm and compute the minimum route ETX. Otherwise, we decide N and distribute N nodes again because S and D have no path in the random network. We repeat these procedures 1000 times and compute the mean of minimum route ETX. Figures 9 and 10 show the numerical and simulation results for $a = 10$ and $a = 40$, respectively. $f_1(z)$ is used as $u(z)$, and d is set to 45. These figures show that the numerical results of Eq. (23) are close to those in the random network for $a = 10$, although there is a difference between them for $a = 40$. These results indicate that Eq. (23) can be used as a good approximation to minimum route ETX even in a random network when the density of nodes is large.

5. Conclusions

In this paper, we theoretically and precisely analyzed minimum route ETX and characterized path $r_{O,k}$, which realizes minimum route ETX, in a static one-dimensional multi-hop wireless network where nodes are located at constant intervals. Using a similar idea to that in the one-dimensional case, we also characterized $r_{O,k}$ in a static lattice multi-hop wireless network and derived a formula to compute minimum route ETX if the ETX function satisfies three given conditions and an upper bound of minimum route ETX otherwise. From numerical results, we showed that the formula can be used as an approximation of minimum route ETX even if these conditions are not satisfied. We also compared minimum route ETX in the lattice network with that in a two-dimensional random network and showed that the formula for the lattice network can be used for the two-dimensional random network if the density of nodes is large. The analyses in this paper focus on ETX, and we assume that the ETX function is a convex monotonically increasing function with respect to the length of a link. If we use metrics that do not satisfy this assumption, we cannot always use the results of this paper and may need other approaches to analysis of the communication quality. For example, medium time metric (MTM) [8] is a typical metric that does not satisfy this assumption because MTM is represented as a step-like function with respect to the length of a link. Hence, future problems include the analysis of communication quality using other metrics like MTM. Analyses of minimum route ETX considering interferences and other distributions of nodes are also our future problems.

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Appendix A: ETX Functions

In [4], the ETX of a link whose length is z, $u(z)$, is assumed to be $u(z) = \frac{1}{p(z)^2}$, where $p(z)$ is the packet reception rate (PRR) at a receiver z away from a sender. Since PRR is generally represented as a random variable, we use its expected value as $p(z)$. Then

$$p(z) = \int_{-\infty}^{\infty} \Psi(\gamma_{dB}) f(\gamma_{dB}, z) d\gamma_{dB}, \tag{A· 1}$$

where $\Psi(\gamma_{dB})$ is the PRR for the signal to noise ratio (SNR) γ_{dB}, and $f(\gamma_{dB}, z)$ is the probability density function of SNR for link length z. From a radio receiver model in [5], we can compute $\Psi(\gamma_{dB})$ for some common modulation and encoding schemes. In this paper, we assume non-coherent frequency shift keying modulation and non-return-to-zero encoding. Then

$$\Psi(\gamma_{dB}) = \left\{1 - \frac{1}{2}\exp\left(-\frac{10^{\frac{\gamma_{dB}}{10}}}{2} \cdot \frac{B_N}{\Delta}\right)\right\}^{8\alpha}, \tag{A· 2}$$

where α is the packet size (in bytes), B_N is the noise bandwidth (in Hz), and Δ is the bit data rate (in bps). Also, from a channel model in [5], SNR obeys a normal distribution with mean μ and variance σ^2, where

$$\mu = P_t - PL(z_0) - 10\eta \log_{10}\left(\frac{z}{z_0}\right) - P_n, \tag{A· 3}$$

where P_t is the transmitting power (in dB), $PL(z_0)$ is the propagation loss (in dB) at reference length z_0, η is the path loss exponent, and P_n is the noise floor (in dB). Namely, $f(\gamma_{dB}, z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\gamma_{dB}-\mu)^2}{2\sigma^2}}$. [5] suggests to use the empirical data of σ, η, and $PL(z_0)$ shown in [6]. Hence, we assume a parking structure as a propagation environment and set these parameters to $\sigma = 3.95$, $\eta = 3$, and $PL(1) = 34.35$ dB [6]. In [5], as an example of the set of parameters needed for the above computation, the following values are given: $P_n = -105$ dBm, $z_0 = 1$ m, $\alpha = 50$ bytes, $B_N = 30$ kHz, $\Delta = 19.2$ kbps, and -20 dBm $< P_t <$ 5 dBm. To determine these parameters, mica2 mote is assumed as a wireless node. This paper uses these values. We examine four values for P_t as follows: $P_t = -15$ dBm for $f_1(z)$, $P_t = -12$ dBm for $f_2(z)$, $P_t = -9$ dBm for $f_3(z)$, and $P_t = -6$ dBm for $f_4(z)$.

We also define $f_5(z)$ and $f_6(z)$ as follows:

$$f_5(z) = \begin{cases} \left(\frac{z}{40}\right)^3 + 1, & z \le 100, \\ \infty, & z > 100, \end{cases} \tag{A· 4}$$

$$f_6(z) = \begin{cases} \left(\frac{z}{110}\right)^5 + 1, & z \le 110, \\ \infty, & z > 110. \end{cases} \tag{A· 5}$$

Appendix B: Proof of Eq. (24)

Clearly, if $L_x = 0$ and $L_y = 0$, then $U(r_{OR}) \le u(0)$. If $L_x > 0$ or $L_y > 0$, then $\frac{L_x+L_y}{a}$ is a positive integer. On the other hand, $U(r_{OR}) \le U(r_{2,k})$ for any positive integer k because $r_{2,k}$ is one of the paths that connect S and D while r_{OR} is the path that minimizes route ETX in all paths which connect S and D. Therefore, Eq. (24) holds.

Appendix C: Proofs of P_1 to P_4

By partially differentiating Eq. (26) with respect to h,

$$\frac{\partial \hat{u}_p(g,h,c)}{\partial h} = \frac{(h+gc)\hat{u}'\left(\sqrt{g^2+h^2+2ghc}\right)}{\sqrt{g^2+h^2+2ghc}} + \frac{(h-gc)\hat{u}'\left(\sqrt{g^2+h^2-2ghc}\right)}{\sqrt{g^2+h^2-2ghc}}. \tag{A· 6}$$

From Eq. (A· 6), B_1, and B_2, $\frac{\partial \hat{u}_p(g,h,c)}{\partial h} \ge 0$. Hence, P_1 holds.

By partially differentiating Eq. (26) with respect to c,

$$\frac{\partial \hat{u}_p(g,h,c)}{\partial c} = \frac{gh\hat{u}'\left(\sqrt{g^2+h^2+2ghc}\right)}{\sqrt{g^2+h^2+2ghc}} - \frac{gh\hat{u}'\left(\sqrt{g^2+h^2-2ghc}\right)}{\sqrt{g^2+h^2-2ghc}}. \tag{A· 7}$$

From Eq. (A· 7) and B_2, $\frac{\partial \hat{u}_p(g,h,c)}{\partial c} \ge 0$. In addition, if $g > 0$, $h > 0$, and $c > 0$, then $\frac{\partial \hat{u}_p(g,h,c)}{\partial c} > 0$. Hence, P_2 holds.

To prove P_3, from Eq. (26), we only have to prove inequality $2\hat{u}\left(\sqrt{g^2+5h^2}\right) - \hat{u}(|g-h|) - \hat{u}(g+h) > 0$. In general, we have

$$\hat{u}\left(\sqrt{g^2+5h^2}\right) - \hat{u}(|g-h|) = \int_{|g-h|}^{\sqrt{g^2+5h^2}} \hat{u}'(z)dz. \tag{A· 8}$$

By substituting $z = t^{\frac{1}{6}}$ into Eq. (A· 8), we have

$$\hat{u}\left(\sqrt{g^2+5h^2}\right) - \hat{u}(|g-h|) = \int_{(g-h)^6}^{(g^2+5h^2)^3} \frac{\hat{u}'(t^{1/6})}{6t^{5/6}} dt. \tag{A· 9}$$

From B_3, $\frac{\hat{u}'(t^{1/6})}{6t^{5/6}} \ge \frac{\hat{u}'\left(\sqrt{g^2+5h^2}\right)}{6\left(\sqrt{g^2+5h^2}\right)^5}$ for $t \le (g^2+5h^2)^3$. Then we have

$$\hat{u}\left(\sqrt{g^2+5h^2}\right)-\hat{u}(|g-h|)$$
$$\geq \left\{(g^2+5h^2)^3-(g-h)^6\right\}\cdot\frac{\hat{u}'\left(\sqrt{g^2+5h^2}\right)}{6\left(\sqrt{g^2+5h^2}\right)^5}. \quad (\text{A}\cdot 10)$$

In the same manner, we have

$$\hat{u}\left(\sqrt{g^2+5h^2}\right)-\hat{u}(g+h)$$
$$\geq \left\{(g^2+5h^2)^3-(g+h)^6\right\}\cdot\frac{\hat{u}'\left(\sqrt{g^2+5h^2}\right)}{6\left(\sqrt{g^2+5h^2}\right)^5}. \quad (\text{A}\cdot 11)$$

From Eqs. (A· 10) and (A· 11), we have

$$2\hat{u}\left(\sqrt{g^2+5h^2}\right)-\hat{u}(|g-h|)-\hat{u}(g+h)$$
$$\geq (120g^2h^4+248h^6)\cdot\frac{\hat{u}'\left(\sqrt{g^2+5h^2}\right)}{6\left(\sqrt{g^2+5h^2}\right)^5}. \quad (\text{A}\cdot 12)$$

The right side of Eq. (A· 12) is greater than 0 because $h > 0$ and $\hat{u}'\left(\sqrt{g^2+5h^2}\right) > 0$, which is derived from B_1. Therefore, P_3 holds.

By applying P_2, P_3, P_2, and P_1 in this order to $\hat{u}_p(g, h', c')$, we have

$$\hat{u}_p(g,h',c') \leq \hat{u}_p(g,h',1) < \hat{u}_p\left(g, \sqrt{5}h', 0\right)$$
$$\leq \hat{u}_p(g,h,0) \leq \hat{u}_p(g,h,c). \quad (\text{A}\cdot 13)$$

Then P_4 holds.

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