On Characteristic Polarization States in the Cross-Polarized Radar Channel

Yoshio Yamaguchi, Wolfgang-M. Boerner, Hyo J. Eom, Masakazu Sengoku, Seiichi Motooka, and Takeo Abe

Abstract—**The characteristic polarization states in the cross-polarized radar channel for the monostatic reciprocal case is derived based on a Stokes vector formulation. The optimization procedure for the channel power leads to an eigenvalue equation which explains the characteristic polarization state properties mathematically and physically. The method is illustrated for a hypothetical target.**

I. Introduction

As regards the characteristic polarization states of a radar target for the completely polarized wave case, Boerner *et al.* [1], [2] have already derived eight characteristic polarization states based on the polarization transformation ratio, for which a radar receives optimum power. These states are two copolarization maximums (CO-POL Maxs), two copolarization nulls (CO-POL Nulls), two cross-polarization maximums (X-POL Maxs), two cross-polarization saddles (X-POL Saddles), and two cross-polarization nulls (X-POL Nulls). Since the pair of X-POL Nulls and CO-POL Maxs is identical, there exists a total of eight physical characteristic polarization states. It is known that three pairs of CO-POL Maxs, CO-POL Nulls, and X-POL Maxs form a "POLARIZATION FORK" on one great circle, the target characteristic circle, on the polarization (POINCARÉ) sphere.

The purpose of this letter is to present an alternative method for deriving the characteristic polarization states in the cross-polarized radar channel for the monostatic reciprocal case based on a Stokes vector formulation [3]–[5]. The Stokes vector formulation has an advantage in its applicability for finding solutions for both completely polarized wave and partially polarized wave cases. In the following, we show that the optimization procedure to the cross-channel power for the coherent case leads to an eigenvalue equation which explains the characteristic polarization state properties mathematically and physically. Some numerical examples are considered to demonstrate the validity of this method.

II. Cross-Polarized Channel Power

Consider the case for which a monostatic radar transmits a completely polarized (coherent) wave and receives a coherent scattered wave from a target. Assuming that the transmitted wave has unit magnitude, the wave can be expressed in terms of Stokes vector as follows

$$\boldsymbol{g}_{tr}^T = (1, x_1, x_2, x_3) \tag{6}$$

Manuscript received July 17, 1992; revised December 2, 1992; rerevised April 24, 1992. This work was supported in part by The Japan Securities Scholarship Foundation.

Y. Yamaguchi and M. Sengoku are with the Department of Information Engineering, Faculty of Engineering, Niigata University, Ikarashi 2-8050, Niigata-shi, 950-21, Japan.

W. M. Boerner is with the Department of Electrical Engineering and Computer Science, University of Illinois at Chicago, Chicago, IL 60680.

H. J. Eom is with the Department of Electrical Engineering, Korea Advanced Institute of Science and Technology, 400 Kusung-dong Yusung-gu, Taejon, Korea.

S. Motooka and T. Abe are with the Department of Electronic Engineering, Chiba Institute of Technology, Chiba, 275 Japan.

IEEE Log Number 9201564.

where T denotes transpose, and $x_i (i = 1, 2, 3)$ is the component of Stokes vector $\boldsymbol{g}_{tr}$ which constitutes sub-Stokes vector $\tilde{\boldsymbol{X}}$.

$$\tilde{\boldsymbol{X}}^T = (x_1, x_2, x_3). \tag{7}$$

The radar is assumed to have two polarimetric receiving channels; the first channel has a copolarized receiving antenna whose polarization state is the same as that of the transmitting antenna, the other has a cross-polarized antenna whose polarization state is orthogonal. The channel power depends on the transmitting polarization state and target scattering property such as shape, orientation, size, material, etc. Since the scattering property of a target cannot be controlled, we obtain the target information by changing polarization state of transmitting wave in a polarization agile radar. The problem here is to find polarization states for which the cross-polarized channel power is optimal for a given target. If we concentrate on polarimetric information excluding amplitude dependency due to path length in a scattered wave, the power in the cross-polarized radar channel in terms of transmitting Stokes vector is expressed as follows [3]

$$\begin{aligned} P_x &= \frac{1}{2}\boldsymbol{g}_{tr}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [M]\boldsymbol{g}_{tr} \\ &= \frac{1}{2}\left(-\tilde{\boldsymbol{X}}^T [N]\tilde{\boldsymbol{X}} + m_{00}\right) \end{aligned} \tag{8}$$

where $[M]$ is defined as Mueller matrix representing scattering property of target

$$[M] = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{9}$$

and we define

$$[N] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ -m_{31} & -m_{32} & -m_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & -m_{33} \end{bmatrix}. \tag{10}$$

It should be noted that the matrix $[N]$ is symmetric for the monostatic reciprocal case.

III. Eigenvalue Problem and Characteristic Polarization States

We optimize the X-POL power (3) subject to the constraint

$$\Phi = \sqrt{x_1^2 + x_2^2 + x_3^2} - 1 = 0 \tag{11}$$

This constraint is due to an assumption that the transmitted wave is coherent. The optimization procedure employing Lagrangian method with multiplier μ

$$\frac{\partial P_x}{\partial x_i} - \mu\frac{\partial \Phi}{\partial x_i} = 0 \quad (i = 1, 2, 3) \tag{12}$$

leads to the following matrix equation.

$$-\begin{bmatrix} m_{11} + \mu & m_{12} & m_{13} \\ m_{12} & m_{22} + \mu & m_{23} \\ m_{13} & m_{23} & \mu - m_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

This equation reduces to an eigenvalue equation of general form.

$$[-N]\tilde{\boldsymbol{X}} = \lambda\tilde{\boldsymbol{X}} \tag{14}$$

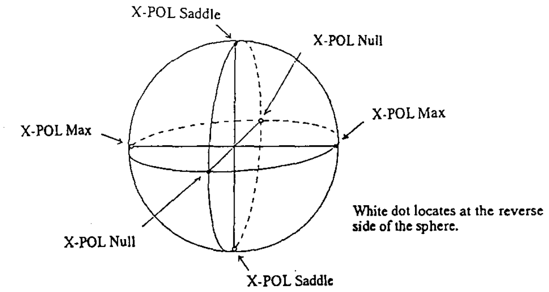


Fig. 1. Location of characteristic polarization states on the Poincaré sphere in the cross-polarized radar channel. Stationary points of P_x appear at the intersection of axes and the sphere surface.

where λ is the eigenvalue of $\tilde{\boldsymbol{X}}$and is equal to μ in this case. Mathematically, P_x in (3) is essentially of Hermitian form, hence the optimization of P_x leads to the eigenvalue equation (9). Since $[-N]$ is also real and symmetric, we find from mathematical point of view that

1) This 3×3 matrix equation has three real eigenvalues, λ_1, λ_2, and λ_3 ($\lambda_1 > \lambda_2 \geq \lambda_3$) including degeneracy of $\lambda_2 = \lambda_3$.
2) The eigenvectors corresponding to the eigenvalues are orthogonal to each other.

The solutions to (9) provide stationary points in P_x and the solution sub-Stokes vector correspond to characteristic polarization states in the cross-polarized channel for a given target $[M]$. From the first property and from the fact that the matrix $[-N]$ is of Hermitian form associated with physical power, the largest eigenvalue λ_1 gives the largest power (X-POL Max), the smallest eigenvalue λ_3 gives the minimal (X-POL Null), while the intermediate eigenvalue λ_2 gives the second max or the second min (X-POL Saddle) when $\lambda_2 \neq \lambda_3$. For the case of $\lambda_2 = \lambda_3$, X-POL Saddle points vanish and hence the corresponding power does not exist. The power in the characteristic polarization state associated with the eigenvalue is given by

$$P_x = \frac{1}{2}(\lambda_i + m_{00}). \tag{15}$$

On the other hand, we understand from the second property that the eigenvectors are orthogonal to each other, which, in turn, implies these solution sub-Stokes vectors are orthogonal. Since the unit vectors of sub-Stokes vector constitute three rectangular coordinate axes of the Poincaré sphere, solution vectors constitutes a new frame of rectangular coordinate in the Poincaré sphere [2] due to this spatial orthogonality property. Hence, this spatial orthogonality of characteristic polarization states on the Poincaré sphere always holds for the property of the characteristic polarization states for the cross-polarized radar channel, because characteristic polarization states correspond to solution vectors. General location manner of characteristic polarization states is depicted on the Poincaré sphere in Fig. 1

For a given eigenvalue, say, we obtain two solution vectors under the condition (6). That is, if $\boldsymbol{X}_1^T = (x_1, x_2, x_3)$is a solution vector, then $\boldsymbol{X}_2^T = (-x_1, -x_2, -x_3)$also becomes the solution vector. The condition $\boldsymbol{X}_1^T \boldsymbol{X}_2 = -1$is the polarimetric orthogonality condition for two polarization states. Since the tip of a solution vector on the Poincaré sphere surface represents a characteristic polarization state, the tips of these two solution vectors must locate on the antipodal points on the Poincaré sphere representing orthogonal polarization states to each other. Even though the polarization states are orthogonal, they produce the same power because the pair solution vectors are determined from the same eigenvalue.

TABLE I
EIGENVALUES, EIGENVECTORS, AND POWER FOR CASE A

eigenvalue	power	x_1	x_2	x_3	characteristic pol. state
λ_1=1.75	2.25	0.0	1.0	0.0	Max
	2.25	0.0	−1.0	0.0	
λ_2=−1.75	0.5	0.707	0.0	-0.707	Saddle
	0.5	-0.707	0.0	0.707	
λ_3=−2.75	0.0	0.707	0.0	0.707	Null
	0.0	-0.707	0.0	-0.707	

IV. NUMERICAL EXAMPLES

In this section, we show the characteristic polarization states using a few examples. The first example (case A) is quoted from [1] for the sake of comparison.

Case A

If a Mueller matrix $[M]$ is given as

$$[M] = \begin{bmatrix} 2.75 & 1.50 & 0.0 & 1.50 \\ 1.50 & 2.25 & 0.0 & 0.50 \\ 0.0 & 0.0 & -1.75 & 0.0 \\ -1.5 & -0.5 & 0.0 & -2.25 \end{bmatrix}$$

then the eigenvalue equation becomes

$$\begin{bmatrix} -2.25 & 0 & -0.5 \\ 0 & 1.75 & 0 \\ -0.5 & 0 & -2.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{17}$$

The eigenvalues and solution vectors are listed in Table I. The power spectrum as a function of the transmitting polarization state, tilt angle τ and ellipticity angle ϵ, is illustrated in Fig. 2(a). One can find six stationary points in Fig. 2(a) which correspond to the characteristic polarization states (X-POL Maxs, X-POL Nulls, and X-POL Saddles) in the cross-polarized channel. These points are displayed on the Poincaré polarization sphere in Fig. 2(b). It should be noted in Fig. 2(b) that each pair locates antipodal points on the sphere and that three lines connecting each pair intersect at the origin at right angle with each other as shown in Fig. 1. These results are in complete agreement with [1].

Case B

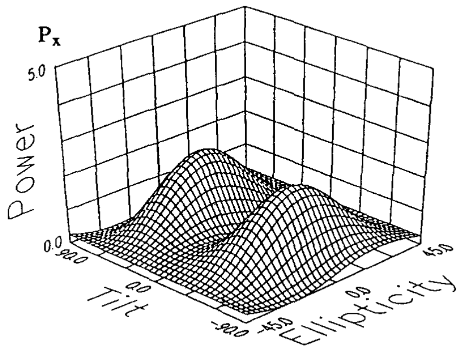
For the second example, we choose the Mueller matrix as

$$[M] = \begin{bmatrix} 0.729 & 0.156 & 0.175 & 0.025 \\ 0.156 & 0.209 & 0.525 & -0.425 \\ 0.175 & 0.525 & 0.198 & 0.438 \\ -0.025 & 0.425 & -0.438 & -0.323 \end{bmatrix}. \tag{18}$$

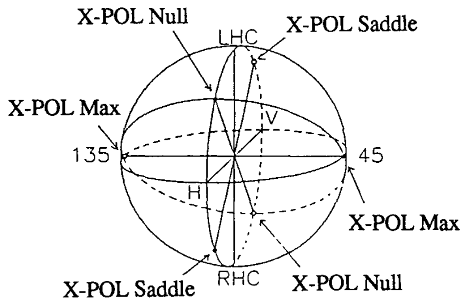
This Mueller matrix is derived [3] from a Sinclair scattering matrix $[S]$

$$[S] = \begin{bmatrix} 0.25 + j0.75 & -0.1 + j0.5 \\ -0.1 + j0.5 & 0.5 - j0.25 \end{bmatrix}. \tag{19}$$

The resultant power plot and the characteristic polarization states on the Poincaré sphere are shown in Figs. 3(a) and (b), while the solution vectors are listed in Table II. The property of these states is

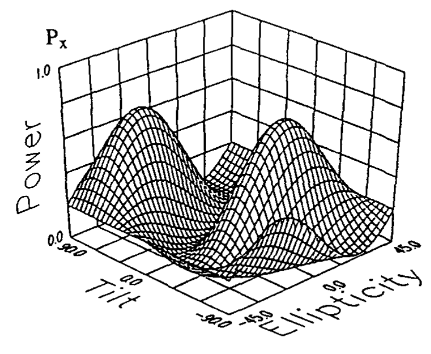


(a)

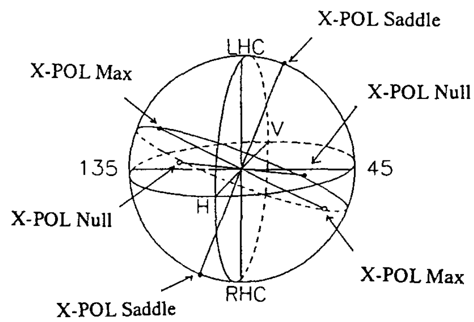


(b)

Fig. 2. Example of characteristic polarization states for case A. (a) Cross-polarized channel power as a function of transmitting polarization state (tilt angle τ and ellipticity angle e) (b) Poincaré sphere representation of characteristic polarization states.



(a)



(b)

Fig. 3. Example of characteristic polarization states for case B. (a) Cross-polarized channel power as a function of transmitting polarization state (tilt angle τ and ellipticity angle ϵ) (b) Poincaré sphere representation of characteristic polarization states.

the same as in Fig. 1. In the above cases (A) and (B), we adopted a hypothetical Mueller matrix and a scattering matrix to show the validity of this approach. This approach holds for any coherent case if a scattering matrix or a corresponding Mueller matrix which may be either numerical or experimental one is given.

V. Conclusion

A total of six characteristic polarization states in the cross polarized radar channel is derived based on the Stokes vector formalism for the monostatic reciprocal case. These states are a pair of X-POL Maxs, a pair of X-POL Saddles, and a pair of X-POL Nulls. Although these characteristic polarization states derived by this method are exactly the same as those derived by Sinclair matrix optimization method using the polarization transformation ratio [2], the formulation associated with an eigenvalue equation provides a comprehensive physical and mathematical approach. An explanation is given to the properties of polarization state from the mathematical and physical point of view. For the case of copolarized case, it still needs to be investigated with other methods and the result will be reported in the near future.

TABLE II
Eigenvalues, Eigenvectors, and Power for Case B

eigenvalue	power	x_1	x_2	x_3	characteristic pol. state
λ_1=0.69	0.709	-0.601	0.610	-0.516	Max
	0.709	0.601	-0.610	0.516	
λ_2=-0.69	0.02	-0.448	0.278	0.850	Saddle
	0.02	0.448	-0.278	-0.850	
λ_3=-0.73	0.0	0.662	0.742	0.106	Null
	0.0	-0.662	-0.742	-0.106	

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