Development of Target Null Theory

Jian Yang, Yoshio Yamaguchi, *Senior Member, IEEE*, Hiroyoshi Yamada, *Member, IEEE*, Wolfgang-Martin Boerner, *Fellow, IEEE*, Harold Mott, *Senior Member, IEEE*, and Yingning Peng, *Senior Member, IEEE*

***Abstract*—In a co- or cross-polarized channel, the polarization states of the transmitting and receiving antennas are the same or orthogonal, and the corresponding target nulls (i.e., the CO-POL Nulls or X-POL Nulls) are defined as the polarization states of the transmitting antenna such that the received power equals zero. However, no systematic studies have been carried out to solve the problem of the corresponding target nulls if the polarization states of the transmitting and receiving antennas are independent. In this paper, the target null theory is extended to the case of two independent polarization states. For two arbitrary independent symmetric scattering matrices, it is proved that there exists only one pair of polarization states such that both of the received powers equal zero. This polarization states' pair is called the co-null of the two targets, which can easily be obtained by solving an eigenvalue problem. Based on this concept and algebraic theory, the concept of the co-null space is introduced for the symmetric scattering matrix case, and many important results are presented, e.g., the relations between the co-null and the CO-POL/X-POL Nulls, the properties of the co-null space, and the relation between the co-null and target decomposition. Finally, the co-null for the asymmetric scattering matrix case is studied. The concepts of the mono-co-null space and the bi-co-null space are introduced, and the relations between both spaces are presented.**

***Index Terms*—Polarization, radar polarimetry, scattering matrix, target null.**

I. INTRODUCTION

IN RADAR polarimetry, an important concept is the characteristic polarization states first considered by Kennaugh [1] in the early 1950s. Later, Huynen [2] redeveloped Kennaugh's work, and introduced his famous "polarization fork" concept illustrating that the characteristic polarization states in the co-polarized channel form a fork on the Poincaré sphere. Since the 1980s, the optimization procedures for various cases were presented for obtaining characteristic polarization states, notably by Boerner and his collaborators [3]–[9], Mott [10], Van Zyl [11], [12], and Lüneburg [29], [30]. In a co-polarized or cross-polarized channel, the polarization states of the transmitting and receiving antennas are the same or orthogonal, and the characteristic polarization states are defined as the polarization states of the transmitting antenna such that the received power equals extreme values. For the symmetric coherent scattering matrix case, it has been known that there exist in total five pairs of characteristic polarization states in general. Among these characteristic polarization states, the CO-POL Nulls and X-POL Nulls are very important and useful. For example, Yamaguchi *et al.* [13]–[16] utilized the CO-POL Nulls and X-POL Nulls of clutter for polarimetric detection of objects buried in snowpack and sandy ground.

For the cases of the co-polarized and cross-polarized channels, the target nulls (i.e., the CO-POL Nulls and X-POL Nulls) have been studied in detail [2]–[11], [17]–[19]. However, no systematic studies have been carried out to solve the problem of the corresponding target nulls if the polarization states of the transmitting and receiving antennas are independent. We know nothing except that there exist infinite pairs of transmitting and receiving polarization states such that the received power equals zero [20], [21]. Therefore, it is necessary to systematically study the target null problem for the case of two independent polarization states. On the other hand, group theory has been employed by Cloude [24]–[26] to simplify the algebraic notation and gain some geometrical insight into the complexities of polarimetry. These works [24]–[26] have attracted many polarimetrists' attention. Recently, the authors introduced the concepts of the co-null and the co-null Abelian group for extension of the target nulls [22]. Following this work, this paper systematically studies the target nulls for the case of two independent antenna polarization states.

In Section II, we succinctly state some basic results on the CO-POL Nulls and X-POL Nulls which are necessary for the following sections. Then we present the concept of the co-null of two targets for the symmetric scattering matrix case in Section III. It is proved that there exists only one pair of polarization states such that the received powers of two independent targets equal zero (here, "two independent targets" means that the scattering matrices of the two targets are linearly independent). This polarization states' pair is called the co-null of the two targets, which can easily be obtained by solving an eigenvalue problem. From the concept of a co-null, another concept, the co-null space, is introduced, and many interesting results are presented, e.g., the relations between a co-null and the CO-POL/X-POL Nulls, the properties of a co-null space, the relation between a co-null of two targets, and target decomposition. Using the concepts of the co-null and the co-null space, we obtain two important classes of targets: the classes of the symmetric targets and H-targets.

In Section IV, we introduce the concept of the co-nulls of two targets in the general scattering matrix case (including the asymmetric scattering matrix and symmetric scattering matrix). For

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J. Yang and Y. Peng are with the Department of Electronic Engineering, Tsinghua University, Beijing, 100084 China.

Y. Yamaguchi and H. Yamada are with the Department of Information Engineering, Niigata University, Niigata-shi, 950-2181 Japan (e-mail: yamaguch@ie.niigata-u.ac.jp).

W.-M. Boerner is with the Department of Electrical Engineering and Computer Science, University of Illinois, Chicago, IL 60607-7018 USA.

H. Mott is with the Department of Electrical Engineering, University of Alabama, Huntsville, AL 35405 USA.

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two asymmetric scattering matrices, this paper points out that there exist two co-nulls, which can also be obtained by solving an eigenvalue problem. Then we define the mono-co-null space and the bi-co-null space and present some properties of the two spaces, e.g., the relations between a mono-co-null space and a bi-co-null space, and the relation between a co-null and target decomposition for the asymmetric scattering matrix case.

In this paper, the proofs of some evident results are omitted.

II. CO-POL NULLS AND X-POL NULLS

In the polarization basis H–V (horizontal and vertical polarizations), the Sinclair scattering matrix of a target in the backscattering case is expressed as

$$[S] = \begin{bmatrix} s_{hh} & s_{hv} \\ s_{vh} & s_{vv} \end{bmatrix} \tag{1}$$

where $s_{xy}(x, y = h, v)$ denotes the scattering element of y transmitting and x receiving polarizations. For the reciprocal backscattering case, $s_{hv} = s_{vh}$, i.e., the scattering matrix is symmetric. For the bistatic radar case, however, two coordinate systems are necessary [10], and the corresponding scattering matrix of a target may be asymmetric.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ denote the polarization states of the transmitting and receiving antennas, respectively. Then the received power is expressed as

$$P = |[S]\boldsymbol{a} \bullet \boldsymbol{b}|^2 = \left|\boldsymbol{b}^t[S]\boldsymbol{a}\right|^2 \tag{2}$$

where $\bullet$ denotes the dot product of vectors, and the superscript t denotes the transpose. Without loss of generality, this paper assumes that $||\boldsymbol{a}|| = ||\boldsymbol{b}|| = 1$.

In the co-polarized channel case, $\boldsymbol{b} = \boldsymbol{a}$, and the CO-POL Nulls are defined as the transmitting polarization states such that $P_c = |[S]\boldsymbol{a} \bullet \boldsymbol{a}|^2 = 0$. Usually, there exist two CO-POL Nulls. For the symmetric scattering matrix case, the CO-POL Nulls are determined by [2], [19]

$$[S]\boldsymbol{a} = \lambda \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}. \tag{3}$$

In the cross-polarized channel case, $\boldsymbol{b} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\boldsymbol{a}^*$, where the superscript $*$ denotes the complex conjugate. The X-POL Nulls are defined as the transmitting polarization states such that $P_x = |[S]\boldsymbol{a} \bullet \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\boldsymbol{a}^*|^2 = 0$. For the symmetric scattering matrix case, there exist two X-POL Nulls in general which are determined by

$$[S]\boldsymbol{a} = \lambda \boldsymbol{a}^*. \tag{4}$$

Let $\boldsymbol{a}_{x\,1}$ and $\boldsymbol{a}_{x\,2}$ denote the two X-POL Nulls, then it is easy to prove that

$$\boldsymbol{a}_{x\,2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_{x\,1}^*. \tag{5}$$

III. CO-NULL OF TARGETS FOR THE SYMMETRIC SCATTERING MATRIX CASE

In this section, we only consider the symmetric scattering matrix case. If it is not specially mentioned, the H–V polarization basis is implied in this section. The target nulls for the asymmetric scattering matrix case will be studied in the next section. Note that although some results in this section have been published in part [22], it seems still necessary to restate them here for systematically developing the target null theory because we add some proofs and interpretations of these results which are important for understanding the properties of the co-null.

A. Co-Null

Definition [22]: Let $[S_1]$ and $[S_2]$ denote the scattering matrices of target 1 and target 2. If $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the equations $[S_1]\boldsymbol{a} \bullet \boldsymbol{b} = 0$ and $[S_2]\boldsymbol{a} \bullet \boldsymbol{b} = 0$, then we call $(\boldsymbol{a}, \boldsymbol{b})$ the co-null (pair) of the targets 1 and 2, or the co-null of $[S_1]$ and $[S_2]$. Furthermore, if $\boldsymbol{a}$ and $\boldsymbol{b}$ also satisfy the equation $[S_3]\boldsymbol{a} \bullet \boldsymbol{b} = 0$, we call $(\boldsymbol{a}, \boldsymbol{b})$ the co-null of $[S_1]$, $[S_2]$, and $[S_3]$.

From this definition, we have the following results:

R1) $[S_1]$ and $[S_2]$ are linearly independent. $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is the co-null of $[S_1]$ and $[S_2]$ if and only if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are two eigenvectors of the eigenvalue equation $[S_1]\boldsymbol{a} = \lambda[S_2]\boldsymbol{a}$.

Proof: Let $(\boldsymbol{a}, \boldsymbol{b})$ be the co-null of $[S_1]$ and $[S_2]$. Then we have from $[S_1]\boldsymbol{a} \bullet \boldsymbol{b} = 0$ and $[S_2]\boldsymbol{a} \bullet \boldsymbol{b} = 0$ that

$$[S_1]\boldsymbol{a} = \lambda_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{b} \tag{6.1a}$$

$$[S_2]\boldsymbol{a} = \lambda_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{b} \tag{6.1b}$$

which leads to

$$[S_1]\boldsymbol{a} = \lambda[S_2]\boldsymbol{a}. \tag{6.2a}$$

Similarly, by use of the symmetry of two scattering matrices we have

$$[S_1]\boldsymbol{b} = \lambda[S_2]\boldsymbol{b}. \tag{6.2b}$$

From (6.2a) and (6.2b), we therefore derive the conclusion R1). #

R1 provides us not only a method for obtaining the co-null of two targets but also a new physical meaning of the eigenvalue equation $[S_1]\boldsymbol{a} = \lambda[S_2]\boldsymbol{a}$. In particular, letting $[S] = [S_1]$ and $[S_2] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, one observes that the eigenvectors of $[S]\boldsymbol{a} = \lambda\boldsymbol{a}$ form the co-null of the two targets: the target corresponding to the scattering matrix $[S]$ and the sphere target (corresponding to the unit scattering matrix). This is a new interpretation of the eigenvalue equation $[S]\boldsymbol{a} = \lambda\boldsymbol{a}$.

For the symmetric scattering matrix case, the order of the transmitting and receiving polarization states ($\boldsymbol{a}$ and $\boldsymbol{b}$) in the equation $[S]\boldsymbol{a} \bullet \boldsymbol{b} = 0$ can be ignored because the reciprocity theorem holds. From this, we conclude according to R1 that two arbitrary independent symmetric scattering matrices have and only have one co-null pair $(\boldsymbol{a}, \boldsymbol{b})$ if the magnitude of the polarization state is not considered.

TABLE I

	Wire $\begin{bmatrix}1&0\\0&0\end{bmatrix}$	Sphere $\begin{bmatrix}1&0\\0&1\end{bmatrix}$	Diplane $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	Left helix $\begin{bmatrix}1&j\\j&-1\end{bmatrix}$	Right helix $\begin{bmatrix}1&-j\\-j&-1\end{bmatrix}$
Wire $\begin{bmatrix}1&0\\0&0\end{bmatrix}$		$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$
Sphere $\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$		$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$
Diplane $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$		$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$
Left helix $\begin{bmatrix}1&j\\j&-1\end{bmatrix}$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$		$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$
Right helix $\begin{bmatrix}1&-j\\-j&-1\end{bmatrix}$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$	$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{j}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-j}{\sqrt{2}}\end{bmatrix}\right)$	

Using the above result, one easily obtains the co-null of two targets. Table I shows the co-nulls for some typical targets. From the equation

$$(\gamma_1[S_1] + \gamma_2[S_2])\boldsymbol{a} \bullet \boldsymbol{b} = \gamma_1[S_1]\boldsymbol{a} \bullet \boldsymbol{b} + \gamma_2[S_2]\boldsymbol{a} \bullet \boldsymbol{b} \qquad (7)$$

it is straightforward to obtain the following result:

R2) If $(\boldsymbol{a}, \boldsymbol{b})$ is the co-null of $[S_1]$ and $[S_2]$, then $(\boldsymbol{a}, \boldsymbol{b})$ is the co-null of $[S_1]$, $[S_2]$ and $\gamma_1[S_1]+\gamma_2[S_2]$ too, where γ_1 and γ_2 are two arbitrary complex numbers.

According to the definition of the co-null, one easily observes that

R3) The co-null of two different rank-1 scattering matrices

$$\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & \rho_1^2 \end{bmatrix} \text{ and } \begin{bmatrix} \rho_2^2 & \rho_2 \\ ;\rho_2 & 1 \end{bmatrix}$$

is

$$\left(\frac{1}{\sqrt{1+|\rho_1|^2}} \begin{bmatrix} -\rho_1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{1+|\rho_2|^2}} \begin{bmatrix} 1 \\ -\rho_2 \end{bmatrix} \right).$$

This result points out that there exists a simple relation between two rank-1 scattering matrices and their co-null $(\boldsymbol{a}, \boldsymbol{b})$, when $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. Furthermore, we have that

R4) $(\boldsymbol{a}, \boldsymbol{b})$ is the co-null of $[S_1]$ and $[S_2]$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, there exist two rank-1 scattering matrices which also have the co-null $(\boldsymbol{a}, \boldsymbol{b})$.

Proof: Express $(\boldsymbol{a}, \boldsymbol{b})$ as the form

$$\left(\frac{1}{\sqrt{1+|\rho_1|^2}} \begin{bmatrix} -\rho \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{1+|\rho_2|^2}} \begin{bmatrix} 1 \\ -\rho_2 \end{bmatrix} \right).$$

Using R3, we can easily obtain two desired rank-1 scattering matrices that have the co-null $(\boldsymbol{a}, \boldsymbol{b})$. #

B. Co-Null Space

Next, we will introduce the definition of the co-null space and present its properties.

R5) If $[S_0] = \left[\begin{smallmatrix}0 & 0\\0 & 0\end{smallmatrix}\right]$ is regarded as a special scattering matrix, then all scattering matrices having the null $(\boldsymbol{a}, \boldsymbol{b})$ form a two-dimensional (2-D) linear space, called the co-null space. In this case, the null $(\boldsymbol{a}, \boldsymbol{b})$ is also called the co-null of the space.

Proof: Denote the set of the scattering matrices that have the null $(\boldsymbol{a}, \boldsymbol{b})$ as G. From R2, one can conclude that G is a linear space. On the other hand, every scattering matrix in the set G satisfies $[S]\boldsymbol{a} \bullet \boldsymbol{b} = 0$, i.e., a scattering matrix in the set G has only two independent complex elements (variables). Therefore, G is a 2-D linear space. #

R5 demonstrates that all targets in a co-null space share the same null $(\boldsymbol{a}, \boldsymbol{b})$. From R5 and the definition of the linear space [27], we know that there are infinite targets in a co-null space. If we select the co-null of the space as the polarization states of the transmitting and receiving antennas, then an echo from any target in the co-null space can be suppressed. This conclusion is potentially useful for target discrimination in clutter, or polarimetric contrast enhancement.

R6) For an arbitrary co-null space, two linearly independent nonsingular scattering matrices exist such that every scattering matrix in the space can be expressed as the linear combination of them.

Proof: Let $[S_1]$ and $[S_2]$ denote two linearly independent matrices in a co-null space.

1) If $[S_1]$ and $[S_2]$ are nonsingular, R6 holds evidently.
2) If $[S_1]$ and $[S_2]$ are singular, without loss of generality, we assume that

$$[S_1] = \begin{bmatrix} c_1 & c_1\rho_1 \\ c_1\rho_1 & c_1\rho_1^2 \end{bmatrix} \quad \text{and} \quad [S_2] = \begin{bmatrix} c_2\rho_2^2 & c_2\rho_2 \\ c_2\rho_2 & c_2 \end{bmatrix}.$$

Because $[S_1]$ and $[S_2]$ are linearly independent, we know that $c_1c_2 \neq 0$ and $\rho_1\rho_2 \neq 1$. Consider

$$\begin{aligned} |[S_1]+[S_2]| &= \begin{vmatrix} c_1 + c_2\rho_2^2 & c_1\rho_1 + c_2\rho_2 \\ c_1\rho_1 + c_2\rho_2 & c_1\rho_1^2 + c_2 \end{vmatrix} \\ &= c_1c_2(\rho_1\rho_2 - 1)^2 \neq 0 \end{aligned}$$

which means that $[S_1]+[S_2]$ is nonsingular. Similarly, $[S_1]-[S_2]$ can also be proved to be nonsingular. Note that $[S_1]+[S_2]$ and $[S_1]-[S_2]$ are linearly independent because $[S_1]$ and $[S_2]$ are two linearly independent matrices. Therefore, R6 is true.
3) If $[S_1]$ is nonsingular and $[S_2]$ is singular, then it is easy to prove that there exists at least one number c such that $c[S_1]+[S_2]$ is nonsingular. Since $[S_1]$ and $[S_2]$ are linearly independent, $[S_1]$ and $c[S_1]+[S_2]$ are also linearly independent. It follows that R6 is true. #

Summing up the above cases, one deduces that R6 holds.

R7) Let $(\boldsymbol{a}, \boldsymbol{b})$ be the co-null of a co-null space. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then two rank-1 scattering matrices exist such that every scattering matrix in the space can be expressed as the linear combination of them.

Proof: Without loss of generality, we assume that

$$\boldsymbol{a} = \frac{1}{\sqrt{1+|\rho_1|^2}} \begin{bmatrix} \rho_1 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{b} = \frac{1}{\sqrt{1+|\rho_2|^2}} \begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} \qquad (\rho_1\rho_2 \neq 1).$$

Then it is evident that

$$\begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & \rho_1^2 \end{bmatrix} \text{ and } \begin{bmatrix} \rho_2^2 & -\rho_2 \\ -\rho_2 & 1 \end{bmatrix}$$

are two independent matrices that have the co-null $(\boldsymbol{a}, \boldsymbol{b})$. From R2 and R5

$$G = \left\{ c_1 \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & \rho_1^2 \end{bmatrix} + c_2 \begin{bmatrix} \rho_2^2 & -\rho_2 \\ -\rho_2 & 1 \end{bmatrix} \right\}$$

is the co-null space which has the co-null $(\boldsymbol{a}, \boldsymbol{b})$. #

R7 points out the simple relation between a co-null space and its co-null $(\boldsymbol{a}, \boldsymbol{b})$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent. By using the proof of R7, we can straightforwardly write the general form of a co-null space from its co-null. Therefore, a rank-1 scattering matrix is very important in a co-null space.

R8) If a space has the co-null $(\boldsymbol{a}, \boldsymbol{a})$, then there exist a rank-1 scattering matrix and a nonsingular scattering matrix such that every scattering matrix in the space can be expressed as the linear combination of them.

Proof: Denote the co-null space as G. According to R6, we know that there exist infinite nonsingular scattering matrices in G. Let $[S]$ denote one of them.

On the other hand, one can easily obtain a rank-1 scattering matrix $[S_1]$ in G from its co-null [see the proof of R7)]. Obviously, $[S]$ and $[S_1]$ are linearly independent. Note that the dimension of G is 2. So every scattering matrix in the space G can be expressed as the linear combination of $[S]$ and $[S_1]$. #

R9) Let G_1 and G_2 be two co-null spaces which have the co-nulls $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$, respectively. If

$$\boldsymbol{a}_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \boldsymbol{a}_1$$

and

$$\boldsymbol{b}_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \boldsymbol{b}_1$$

then for an arbitrary scattering matrix $[S_1] \in G_1$, there exists a scattering matrix $[S_2]$ in G_2 such that

$$[S_2] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} [S_1] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In this case, the space G_2 is called the rotation space of the space G_1 with the rotation angle θ denoted as $G_2 = G_1(\theta)$.

R10) If a sphere belongs to a co-null space G, then $G(\pm(\pi/2)) = G$.

Proof: Let $[S] = \left[\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right]$ be a scattering matrix in the space G. Since the sphere target belongs to the space G, i.e., $\left[\begin{smallmatrix} s_1+s_3 & 0 \\ 0 & s_1+s_3 \end{smallmatrix}\right] \in G$, we conclude from the definition of the linear space that

$$\begin{bmatrix} s_3 & -s_2 \\ -s_2 & s_1 \end{bmatrix} = \begin{bmatrix} s_1+s_3 & 0 \\ 0 & s_1+s_3 \end{bmatrix} - \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \in G. \quad (8)$$

The above result means that, for an arbitrary scattering matrix $[S]$, its rotation scattering matrix $\left[\begin{smallmatrix} s_3 & -s_2 \\ -s_2 & s_1 \end{smallmatrix}\right]$ with $\pm(\pi/2)$ rotation angle also belongs to the space G. Therefore, $G(\pm(\pi/2)) = G$. #

Conversely, if $G(\pm(\pi/2)) = G$, one can easily prove that $\left[\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right] \in G$ by using the above method.

From the assumption of R10, we know that the co-null $(\boldsymbol{a}, \boldsymbol{b})$ of the space G satisfies

$$\boldsymbol{a} \bullet \boldsymbol{b} = 0$$

or

$$\boldsymbol{b} = \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a} \tag{9}$$

which means that the polarization states of the receiving and transmitting antennas have the same ellipticity angle, and have two orientation angles with $\pi/2$ difference.

Example 1: From Table I, it is known that $(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])$ is the co-null of a wire, sphere (or plate), and diplane. Furthermore, $(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])$ is the co-null of the space $G_s = \{\left[\begin{smallmatrix}a & 0\\0 & b\end{smallmatrix}\right]\}$. In this space, all scattering matrices can be expressed as the combination of two wires: $(\left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right])$ and $(\left[\begin{smallmatrix}0 & 0\\0 & 1\end{smallmatrix}\right])$. According to Huynen's phenomenological theory [2], we know that $\mathbf{U}_{-\pi/2\leq\theta<\pi/2} G_s(\theta)$ forms the class of the symmetric targets. Note that $\left[\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right]$ (sphere or plate) belongs to every rotation space $G_s(\theta)$. From R10, we show that $\mathbf{U}_{-\pi/2\leq\theta<0} G_s(\theta) = \mathbf{U}_{0\leq\theta<\pi/2} G_s(\theta)$. Therefore, the class of the symmetric targets can also be expressed as $\mathbf{U}_{0\leq\theta<(\pi/2)} G_2(\theta)$.

Another important space is G_H which possesses the co-null

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right).$$

It is easy to prove that diplanes (with different orientation angles) and helixes ($\left[\begin{smallmatrix}1 & i\\i & -1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1 & -i\\-i & -1\end{smallmatrix}\right]$) belong to this space. The general form of this space is $G = \{\left[\begin{smallmatrix}a & b\\b & -a\end{smallmatrix}\right]\}$. An interesting property of this is that $G_H(\theta) = G_H$, where θ is an arbitrary angle. According to Huynen's phenomenological theory [2], one knows that except for diplanes, all others in this space form the class of H-targets. From the definition of the co-null, it is easy to know that the echo from an H-target can be suppressed by using $(1/\sqrt{2})\left[\begin{smallmatrix}1\\i\end{smallmatrix}\right]$ and $(1/\sqrt{2})\left[\begin{smallmatrix}1\\-i\end{smallmatrix}\right]$ as the polarization states of the transmitting and receiving antennas. This result is identical with that in [10].

C. Relations Between the CO/X-POL Nulls and Co-Null of Targets

Now let us consider the relations between the CO/X-POL Nulls and the co-null of targets. Note that the CO/X-POL Nulls are defined as the polarization states of the transmitting antenna for which the radar receives zero power, whereas the co-null of two targets is defined as the special pair of the polarization states of the transmitting and receiving antennas. From these definitions, we conclude the following results.

R11) Let the CO-POL Nulls of $[S]$ be $\boldsymbol{a}$ and $\boldsymbol{b}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then two linearly independent rank-1 matrices $[S_1]$ and $[S_2]$ exist such that

1) The co-null of $[S]$ and $[S_1]$ is $(\boldsymbol{a}, \boldsymbol{a})$.
2) The co-null of $[S]$ and $[S_2]$ is $(\boldsymbol{b}, \boldsymbol{b})$.
3) The co-null of $[S_1]$ and $[S_2]$ is $(\boldsymbol{a}, \boldsymbol{b})$.

From the relation between the co-null and the rank-1 matrices [see R3)], one can easily obtain two linearly independent rank-1 matrices having the above properties.

R12) If a space has the co-null $(\boldsymbol{a}, \boldsymbol{a})$, then a rank-1 scattering matrix $[S_1]$ exists such that every singular scattering matrix in this space can be expressed as $c[S_1]$, where c is a complex constant.

Proof: According to R3, one easily obtains a rank-1 scattering matrix $[S_1]$ from the null $(\boldsymbol{a}, \boldsymbol{a})$. Now if there exists a rank-1 scattering matrix in the space that cannot be expressed as $c[S_1]$, it leads to two linearly independent rank-1 matrices in the co-null space. From R3 and R7, one deduces that the co-null of the space is $(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. This contradicts the condition of R11, which means that every singular scattering matrix in the space can be expressed as $c[S_1]$.#

From R3 and R8, we can obtain the following result:

R13) Let G_{aa} and G_{bb} denote two co-null spaces having the co-nulls $(\boldsymbol{a}, \boldsymbol{a})$ and $(\boldsymbol{b}, \boldsymbol{b})$, respectively. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then there exists a rank-1 scattering matrix $[S_1]$ in the space G_{aa} and a rank-1 scattering matrix $[S_2]$ in the space G_{bb} such that

1) $[S_1]$ and $[S_2]$ are linearly independent.
2) Every scattering matrix in the space G_{ab} can be expressed as the linear combination of $[S_1]$ and $[S_2]$, where G_{ab} denotes the co-null space which has the co-null $(\boldsymbol{a}, \boldsymbol{b})$.

Conversely, if the spaces G_{aa}, G_{bb}, and G_{ab} have the co-nulls $(\boldsymbol{a}, \boldsymbol{a})$, $(\boldsymbol{b}, \boldsymbol{b})$, and $(\boldsymbol{a}, \boldsymbol{b})$, respectively, and if $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then there exist two linearly independent rank-1 scattering matrices $[S_a]$ and $[S_b]$ in the space G_{ab} such that every singular scattering matrix in the space G_{aa} (or G_{bb}) can be expressed as $c[S_a]$ (or $c[S_b]$), where c is a complex constant.

Using R3, R12, and R13, one easily deduces the following result.

R14) G_{aa} and G_{bb} denote two spaces that have the co-nulls $(\boldsymbol{a}, \boldsymbol{a})$ and $(\boldsymbol{b}, \boldsymbol{b})$, respectively. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then the intersection of the space G_{aa} and the space G_{bb} is a subspace or one-dimensional (1-D) space. Except for $[S_0] = \left[\begin{smallmatrix}0 & 0\\0 & 0\end{smallmatrix}\right]$, the CO-POL Nulls of an arbitrary scattering matrix in this subspace are $\boldsymbol{a}$ and $\boldsymbol{b}$.

Next, let us consider the relations between the co-null and the X-POL Nulls. Using the definitions of the co-null and the CO-POL Nulls, we have the following results.

R15) Let the X-POL Nulls of $[S]$ be $\boldsymbol{a}$ and $\boldsymbol{a}^{\perp}$, where $\boldsymbol{a}^{\perp} = \left[\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right]\boldsymbol{a}^*$. Then two linearly independent rank-1 scattering matrices $[S_1]$ and $[S_2]$ exist such that

1) The co-null of $[S]$, $[S_1]$ and $[S_2]$ is $(\boldsymbol{a}, \boldsymbol{a}^{\perp})$.
2) $[S_1][S_2^*] = \left[\begin{smallmatrix}0 & 0\\0 & 0\end{smallmatrix}\right]$.

R16) $G_{a\,a^\perp}$ is a co-null space that has a co-null $(\boldsymbol{a}, \boldsymbol{a}^\perp)$. If $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \in G_{a\,a^\perp}$, then $\begin{bmatrix} s_3^* & -s_2^* \\ -s_2^* & s_1^* \end{bmatrix} \in G_{a\,a^\perp}$.

Proof: Let $[S] = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$, then

$$\boldsymbol{a}^t[S]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{a}^* = 0,$$

$$\boldsymbol{a}^H[S^*]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{a} = 0,$$

$$\boldsymbol{a}^H\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[S^*]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{a} = 0,$$

$$\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[S^*]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}\boldsymbol{a} \bullet \boldsymbol{a}^\perp = 0,$$

$$\begin{bmatrix} s_3^* & -s_2^* \\ -s_2^* & s_1^* \end{bmatrix}\boldsymbol{a} \bullet \boldsymbol{a}^\perp = 0.$$

It follows that $\begin{bmatrix} s_3^* & -s_2^* \\ -s_2^* & s_1^* \end{bmatrix} \in G_{a\,a^\perp}$.

R17) If $\begin{bmatrix} 1 & -\rho \\ -\rho & \rho^2 \end{bmatrix} \in G_{a\,a^\perp}$, then

1) The co-null of the space $G_{a\,a^\perp}$ is

$$\left(\frac{1}{\sqrt{1+|\rho|^2}}\begin{bmatrix} \rho \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{1+|\rho|^2}}\begin{bmatrix} 1 \\ -\rho^* \end{bmatrix}\right).$$

2) An arbitrary scattering matrix $[S]$ in $G_{a\,a^\perp}$ can be expressed as

$$[S] = c_1\begin{bmatrix} 1 & -\rho \\ -\rho & \rho^2 \end{bmatrix} + c^2\begin{bmatrix} (\rho^*)^2 & \rho^* \\ \rho^* & 1 \end{bmatrix}.$$

Proof:

1) Let $\boldsymbol{a} = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$, then we have

$$\begin{bmatrix} 1 & -\rho \\ -\rho & \rho^2 \end{bmatrix}\begin{bmatrix} \alpha \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -\alpha^* \end{bmatrix} = 0$$

that is, $(\alpha - \rho)(1 + \rho\alpha^*) = 0$, we therefore deduce the result in 1.

2) According to the relation between a co-null space and its co-null [see the proof of R7)], one easily concludes the result in 2. #

From the above result, one observes that the rank-1 scattering matrix plays an important role in the co-null space $G_{a\,a^\perp}$.

In the polarization basis $(\boldsymbol{a}, \boldsymbol{a}^\perp)$, all scattering matrices of the co-null space $G_{a\,a^\perp}$ are diagonal. Therefore, the results R15–R17 are very interesting and important.

Example 2: Using the results of R16 and R17, one easily obtains the following results.

1) If $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in G_{a\,a^\perp}$, then $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in G_{a\,a^\perp}$, and therefore, $G_{a\,a^\perp} = G_s = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\}$.

2) If $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \in G_{c\,c^\perp}$, then $\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \in G_{c\,c^\perp}$, and therefore, $G_{c\,c^\perp} = G_H = \{\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\}$. From R17, it is known that the co-null $(\boldsymbol{c}, \boldsymbol{c}^\perp)$ of $G_{c\,c^\perp}$ is

$$\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}\right)$$

The pair of the left- and right-circular polarization states. Using this pair as the polarization basis, we know that $G_{c\,c^\perp} = G_H = \{\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}\}$. This result is identical with that in [10].

D. *Target Decomposition and Co-Null of Targets*

R18) $[S_1]$, $[S_2]$, and $[S_3]$ are linearly independent. If $[S]$ is decomposed into

$$[S] = c_1[S_1] + c_2[S_2] + c_3[S_3]$$

then

$$c_1 = \frac{[S]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23}}{[S_1]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23}}, \qquad c_2 = \frac{[S]\boldsymbol{a}_{13} \bullet \boldsymbol{b}_{13}}{[S_2]\boldsymbol{a}_{13} \bullet \boldsymbol{b}_{13}},$$

and

$$c_3 = \frac{[S]\boldsymbol{a}_{12} \bullet \boldsymbol{b}_{12}}{[S_3]\boldsymbol{a}_{12} \bullet \boldsymbol{b}_{12}}$$

where $(\boldsymbol{a}_{ij}, \boldsymbol{b}_{ij})$ denotes the co-null of the scattering matrices $[S_i]$ and $[S_j]$, $i, j = 1, 2, 3, i \neq j$.

Proof: From $[S] = c_1[S_1] + c_2[S_2] + c_3[S_3]$, we have

$$[S]\boldsymbol{a}_{23}\bullet\boldsymbol{b}_{23} = c_1[S_1]\boldsymbol{a}_{23}\bullet\boldsymbol{b}_{23} + c_2[S_2]\boldsymbol{a}_{23}\bullet\boldsymbol{b}_{23} + c_3[S_3]\boldsymbol{a}_{23}\bullet\boldsymbol{b}_{23}.$$

Note that $(\boldsymbol{a}_{23}, \boldsymbol{b}_{23})$ is the co-null of the scattering matrices $[S_2]$ and $[S_3]$. It follows that $[S_2]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23} = 0$ and $[S_3]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23} = 0$. Therefore, we obtain that $c_1 = ([S]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23})/([S_1]\boldsymbol{a}_{23} \bullet \boldsymbol{b}_{23})$. Similarly, we can derive that

$$c_2 = \frac{[S]\boldsymbol{a}_{13} \bullet \boldsymbol{b}_{13}}{[S_2]\boldsymbol{a}_{13} \bullet \boldsymbol{b}_{13}} \quad \text{and} \quad c_3 = \frac{[S]\boldsymbol{a}_{12} \bullet \boldsymbol{b}_{12}}{[S_3]\boldsymbol{a}_{12} \bullet \boldsymbol{b}_{12}}.$$

#

Note that the co-nulls in Table I can be employed directly if the scattering matrices in Table I are selected for target decomposition.

R18 provides a method to find the coefficient numbers for the general target decomposition. The advantage of this method is that R18 affords us a concise form to express the coefficients. From this form, one easily observes the relation between target decomposition and the co-nulls.

IV. Co-Nulls of Targets for the General Scattering Matrix Case (Including Symmetric and Asymmetric Scattering Matrices)

In this section, we extend the above results to the general scattering matrix (asymmetric/symmetric scattering matrix) case. Note that for the asymmetric scattering matrix case, the reciprocity theorem does not hold, so $(\boldsymbol{a}, \boldsymbol{b}) \neq (\boldsymbol{b}, \boldsymbol{a})$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent.

R19) $[S_1]$ and $[S_2]$ are linearly independent. $(\boldsymbol{a}, \boldsymbol{b})$ is a co-null (pair) of $[S_1]$ and $[S_2]$ if and only if there exists a constant λ such that $[S_1]\boldsymbol{a} = \lambda[S_2]\boldsymbol{a}$ and $[S_1]^t\boldsymbol{b} = \lambda[S_2]^t\boldsymbol{b}$.

Referring to the proof of R1, one can easily prove the above conclusion. From this result, we conclude that there exist two co-nulls for two independent asymmetric scattering matrices.

R20) If $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ are the co-nulls of $[S_1]$ and $[S_2]$, then $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ are the co-nulls of $[S_1]$, $[S_2]$ and $\gamma_1[S_1] + \gamma_2[S_2]$, too, where γ_1 and γ_2 are two arbitrary complex numbers.

R21) The co-nulls of two different rank-1 scattering matrices $\begin{bmatrix} 1 & \rho_1 \\ \alpha_1 & \alpha_1\rho_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2\rho_2 & \alpha_2 \\ \rho_2 & 1 \end{bmatrix}$ are

$$\left(\frac{1}{\sqrt{1+|\rho_1|^2}}\begin{bmatrix} -\rho_1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{1+|\alpha_2|^2}}\begin{bmatrix} 1 \\ -\alpha_2 \end{bmatrix}\right)$$

and

$$\left(\frac{1}{\sqrt{1+|\rho_2|^2}}\begin{bmatrix} 1 \\ -\rho_2 \end{bmatrix}, \frac{1}{\sqrt{1+|\alpha_2|^2}}\begin{bmatrix} -\alpha_1 \\ 1 \end{bmatrix}\right).$$

This conclusion shows that there also exists a simple relation between the co-nulls and two different rank-1 scattering matrices.

R22) If we regard $[S_0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as a special scattering matrix, then all scattering matrices (including symmetric and asymmetric scattering matrices) that possess the null $(\boldsymbol{a}, \boldsymbol{b})$ form a linear space, called the mono-co-null space. In this case, the null $(\boldsymbol{a}, \boldsymbol{b})$ is also called the co-null of the (mono-co-null) space.

R23) If we regard $[S_0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as a special scattering matrix, then all scattering matrices (including symmetric and asymmetric scattering matrices) that possess two different nulls $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ form a linear space, called the bi-co-null space. In this case, the two nulls $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ are also called the co-nulls of the (bi-co-null) space.

Note: In R23, "two different nulls $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ and $(\boldsymbol{a}_2, \boldsymbol{b}_2)$" means that there exists no complex number c such that $(\boldsymbol{a}_1, \boldsymbol{b}_1) = (c\boldsymbol{a}_2, c\boldsymbol{b}_2)$.

R24) The intersection of two different mono-co-null spaces is a bi-co-null space, and the co-nulls of the bi-co-null space consists of the co-nulls of the two mono-co-null spaces.

This conclusion demonstrates the relation between the mono-co-null space and the bi-co-null space.

R25) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. If $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{b}, \boldsymbol{a})$ are the co-nulls of a bi-co-null space G, then all scattering matrices in G are symmetric.

Proof: Let $[S] = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$ be an arbitrary scattering matrix in the bi-co-null space G. Setting $\boldsymbol{a} = (a_1, a_2)^t$ and $\boldsymbol{b} = (b_1, b_2)^t$, then we know from the given conditions that

$$s_2a_1b_2 + s_3a_2b_1 = s_2a_2b_1 + s_3a_1b_2$$

or

$$s_2(a_1b_2 - a_2b_1) = s_3(a_1b_2 - a_2b_1).$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, $a_1b_2 - a_2b_1 \neq 0$. Therefore $s_2 = s_3$. #

R26) For an arbitrary mono-co-null space, there exist three linearly independent nonsingular scattering matrices such that every scattering matrix in the space can be expressed as the linear combination of them.

From this conclusion, one observes that a mono-co-null space is 3-D.

R26′) For an arbitrary mono-co-null space, there exist three linearly independent rank-1 scattering matrices such that every scattering matrix in the space can be expressed as the linear combination of them.

R27) For an arbitrary bi-co-null space, there exist two linearly independent nonsingular scattering matrices such that every scattering matrix in the space can be expressed as the linear combination of them.

From this conclusion, one can find that a bi-co-null space is 2-D.

R27′) For an arbitrary bi-co-null space, there exist two linearly independent rank-1 scattering matrices such that every scattering matrix in the space can be expressed as the linear combination of them.

R28) For an arbitrary mono-co-null space G, there exists a 2-D space consisting of the symmetric scattering matrices as a subspace of G.

R29) For an arbitrary bi-co-null space G, there exists one symmetric scattering matrix $[S]$ in G such that all the symmetric scattering matrices of G can be expressed as $c[S]$.

R30) If $[S_1]$, $[S_2]$, $[S_3]$ and $[S_4]$ satisfy the following conditions.

1) $[S_1]$, $[S_2]$, $[S_3]$ and $[S_4]$ are linearly independent.
2) $[S_2]$, $[S_3]$ and $[S_4]$ have the co-null $(\boldsymbol{a}_1, \boldsymbol{b}_1)$; $[S_1]$, $[S_3]$ and $[S_4]$ have the co-null $(\boldsymbol{a}_2, \boldsymbol{b}_2)$; $[S_1]$, $[S_2]$ and $[S_4]$ have the co-null $(\boldsymbol{a}_3, \boldsymbol{b}_3)$; and $[S_1]$, $[S_2]$ and $[S_3]$ have the co-null $(\boldsymbol{a}_4, \boldsymbol{b}_4)$. An arbitrary scattering matrix can be decomposed into

$$[S] = c_1[S_1] + c_2[S_2] + c_3[S_3] + c_4[S_4]$$

where

$$c_i = \frac{[S]\boldsymbol{a}_i \bullet \boldsymbol{b}_i}{[S_i]\boldsymbol{a}_i \bullet \boldsymbol{b}_i}, \qquad i = 1, 2, 3, 4.$$

The result R30 demonstrates the relation between target decomposition and co-nulls of targets for the general scattering matrix case.

V. Summary

The classical target null theory was founded for the cases of the co-polarized and cross-polarized channels. In these two channels, the polarization state of the receiving antenna depends on that of the transmitting antenna. For fully utilizing the advantage of polarization, we need to extend the classical target null theory to the case of two independent antenna polarization states.

This paper has employed the concepts of the co-null and the co-null space for developing the target null theory. These concepts are very important and useful. For example, let us assume that there exist two different undesired targets (clutter) and assume that their co-null is $(\boldsymbol{a}, \boldsymbol{b})$. Then it is easy to suppress all echoes from the two undesired targets by utilizing $\boldsymbol{a}$ and $\boldsymbol{b}$ as the polarization states of the transmitting and receiving antennas, respectively. Theoretically, this method can be applied to the suppression of all echoes from the targets in the space that possesses the co-null $(\boldsymbol{a}, \boldsymbol{b})$. However, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are neither the same nor orthogonal, it is impossible to suppress all echoes from the two undesired targets for the cases of the co- and cross-polarized channels.

In this paper, we have proposed a simple method to obtain the co-null/nulls of two targets and presented a new physical interpretation of an eigenvalue equation for the symmetric/asymmetric scattering matrix case. From the proposed method, we have proved that there exists only one co-null for two symmetric scattering matrices and that there exist two co-nulls for two asymmetric scattering matrices. Then using the concepts of the co-null and the co-null space, we have obtained many interesting and important results, e.g., the relations between the co-null and the CO-POL/X-POL Nulls, the structure of the co-null space, the relation between two different co-null spaces, and the relation between target decomposition and co-nulls. In addition, we have obtained two important classes of targets: the classes of the symmetric targets and H-targets. From the presented results, one observes that the rank-1 targets play an important role in the co-null space. Finally, the concepts of the co-null and the co-null space were extended to the asymmetric scattering matrix case. We have defined the mono-co-null space and the bi-co-null space and have presented some properties of the two spaces, e.g., the relations between the mono-co-null space and the bi-co-null space.

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References

[1] E. M. Kennaugh, "Polarization properties of radar reflections," M.Sc. thesis, Ohio State Univ., Columbus, 1952.

[2] J. R. Huynen, "Phenomenological theory of radar target," Ph.D. dissertation, Tech. Univ., Delft, The Netherlands, 1970.

[3] W.-M. Boerner and A.-Q. Xi, "The characteristic radar target polarization state theory for the coherent monostatic and reciprocal case using the generalized polarization transformation ratio formulation," *AEU*, vol. 44, no. 4, pp. 273–281, 1990.

[4] W.-M. Boerner, W.-L. Yan, A.-Q. Xi, and Y. Yamaguchi, "On the principles of radar polarimetry: The target characteristic polarization state theory of Kennaugh, Huynen's polarization fork concept, and its extension to the partially polarized case," *Proc. IEEE*, vol. 79, pp. 1538–1550, Oct. 1991.

[5] A. P. Agrawal and W.-M. Boerner, "Redevelopment of Kennaugh's target characteristic polarization state theory using the polarization transformation ratio formalist for the coherent case," *IEEE Trans. Geosci. Remote Sensing*, vol. 27, pp. 2–14, Jan. 1989.

[6] M. Davidovitz and W.-M. Boerner, "Extension of Kennaugh's optimal polarization concept to the asymmetric matrix case," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 569–574, Apr. 1986.

[7] C.-L. Liu, X. Zhang, Y. Yamaguchi, H.-J. Eom, and W.-M. Boerner, "Comparison of optimization procedures for 2×2 Sincair, 2×2 Graves, 3×3 covariance, and 4×4 Mueller (symmetric) matrix in coherent radar polarimetry and its application to target versus background discrimination in microwave remote sensing and imaging," *Proc. SPIE*, vol. 1748, pp. 144–173, 1992.

[8] A. B. Kostinski and W.-M. Boerner, "On the foundations of radar polarimetry," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 1395–1404, Dec. 1986.

[9] A.-Q. Xi and W.-M. Boerner, "Determination of the characteristic polarization states of the target scattering matrix $[S(AB)]$ for the coherent, monostatic, and reciprocal propagation space," *J. Opt. Soc. Amer. A*, vol. 9, no. 3, pp. 437–455, 1992.

[10] H. Mott, *Antennas for Radar and Communications—A Polarimetric Approach*. New York: Wiley, 1992.

[11] J. J. Van Zyl, C. H. Papas, and C. Elachi, "On the optimum polarizations of incoherently reflected waves," *IEEE Trans. Antennas Propagat.*, vol. 35, pp. 818–825, July 1987.

[12] J. J. Van Zyl, "On the importance of polarization in radar scattering problems," Ph.D. dissertation, California Inst. Technol., Pasadena, 1986.

[13] Y. Yamaguchi, M. Mitsumoto, M. Sengoku, and T. Abe, "Synthetic aperture FM-CW radar applied to the detection of objects buried in snowpack," *IEEE Trans. Geosci. Remote Sensing*, vol. 32, pp. 11–18, Jan. 1994.

[14] Y. Yamaguchi and T. Moriyama, "Polarimetric detection of objections buried in snowpack by a synthetic aperture FM-CW radar," *IEEE Trans. Geosci. Remote Sensing*, vol. 34, pp. 45–51, Jan. 1996.

[15] T. Moriyama, Y. Yamaguchi, H. Yamada, and M. Sengoku, "Reduction of surface clutter by a polarimetric FM-CW radar in underground target detection," *IEICE Trans. Commun.*, vol. E78-B, pp. 625–629, Apr. 1995.

[16] T. Moriyama, H. Kasahara, Y. Yamaguchi, and H. Yamada, "Advanced polarimetric subsurface FM-CW radar," *IEEE Trans. Geosci. Remote Sensing*, vol. 36, pp. 725–731, May 1998.

[17] Y. Yamaguchi, W.-M. Boerner, H. J. Eom, M. Sengoku, S. Motooka, and T. Abe, "On characteristic polarization states in the cross-polarized radar channel," *IEEE Trans. Geosci. Remote Sensing*, vol. 30, pp. 1078–1081, Sept. 1992.

[18] J. Yang, Y. Yamaguchi, H. Yamada, M. Sengoku, and S. M. Lin, "Simple method for obtaining characteristic polarization states," *Electron. Lett.*, vol. 34, pp. 441–443, Mar. 1998.

[19] J. Yang, Y. Yamaguchi, H. Yamada, and S. M. Lin, "The formulae of the characteristic polarization states in the co-POL channel and the optimal polarization state for contrast enhancement," *IEICE Trans. Commun.*, vol. E80-B, pp. 1570–1575, Oct. 1997.

[20] J. Yang and S. M. Lin, "On the problem of the polarimetric contrast optimization," *IEEE Antennas Propagat.*, vol. 1, pp. 558–560, 1990.

[21] J. Yang, "The optimum antenna polarizations for target discrimination in clutter" (in Chinese), *J. Northwestern Polytech. Univ.*, vol. 12, pp. 316–320, Apr. 1994.

[22] J. Yang, Y. Yamaguchi, and H. Yamada, "Co-null of targets and co-null Abelian group," *Electron. Lett.*, vol. 35, pp. 1017–1019, June 1999.

[23] J. Yang, "On theoretical problems in radar polarimetry," Ph.D. dissertation, Niigata Univ., Niigata, Japan, Sept. 1999.

[24] S. R. Cloude, "Lie groups in electromagnetic wave propagation and scattering," *J. Electromagn. Wave Applicat.*, vol. 6, no. 8, pp. 947–974, 1992.

[25] ——, "Recent advances in the application of group theory to polarimetry," in *Proc. PIERS*. Pasadena, CA, July 1993, p. 194.
[26] ——, "Group theory and polarization algebra," *OPTIK*, vol. 75, no. 1, pp. 26–36, 1986.
[27] D. J. Lewis, *Introduction to Algebra*. New York: Harper Int., 1965.
[28] W.-M. Boerner, H. Mott, E. Lüneburg, C. Livingstone, B. Brown, J. S. Paterson, S. R. Cloude, E. Krogager, J. S. Lee, D. L. Schuler, J. J. van Zyl, D. Randall, and P. Budkewitsch, *Polarimetry in Radar Remote Sensing—Basic and Applied Concepts*, F. M. Henderson and A. J. Lewis, Eds. New York: Wiley, 1998.
[29] E. Lüneburg and S. R. Cloude, "Bistatic scattering," in *Wideband Interferometric Sensing and Imaging Polarimetry*, H. Mott and W.-M. Boerner, Eds. San Diego, CA: SPIE, July 1997, vol. 3120, pp. 56–68.
[30] E. Lüneburg, "Radar polarimetry: A revision of basic concepts," in *Direct and Inverse EM Scattering*, A. H. Serbest and S. R. Cloude, Eds. White Plains, NY: Longman, 1996, vol. 361, pp. 257–275.

Jian Yang was born in Hubei Province, China, on February 1, 1965. He received the B.S. and M.S. degrees from Northwestern Polytechnical University, Xian, China, in 1985 and 1990, and the Ph.D. degree from Niigata University, Niigata, Japan, in 1999.

He joined the Department of Applied Mathematics, Northwestern Polytechnical University, Xian, China, in 1985 as an Assistant, and was promoted to the ranks of Lecture and Associate Professor in 1990 and 1994, respectively. From 1999 to 2000, he was employed as an Assistant Professor, Department of Information Engineering, Niigata University. In April 2000, he joined the Department of Electronic Engineering, Tsinghua University, where he is now an Associate Professor.

Dr. Yang is a Member of the Chinese Institute of Electronics, Chinese Society of Mathematics, Chinese Society of Industrial and Applied Mathematics, and a member of the Japanese Institute of Electronics, Information and Communication Engineers.

Yoshio Yamaguchi (M'83–SM'94) received the B.E. degree in electronics engineering from Niigata University, Niigata, Japan, in 1976, and the M.E. and Dr.Eng. degrees from Tokyo Institute of Technology, Tokyo, Japan, in 1978 and 1983, respectively.

In 1978, he joined the Faculty of Engineering, Niigata University, where he is now a Professor. From 1988 to 1989, he was a Research Associate with the University of Illinois, Chicago. His interests are in the field of propagation characteristics of electromagnetic waves in lossy medium, radar polarimetry, and microwave remote sensing and imaging.

Dr. Yamaguchi is a member of the Japan Society for Snow Engineering and a Member of the Japanese Institute of Electronics, Information and Communication Engineers.

Hiroyoshi Yamada (M'91) was born in Hokkaido, Japan, on November 2, 1965. He received the B.S., M.S., and Ph.D. degrees from Hokkaido University, Sapporo, Japan, all in electronic engineering, in 1988, 1990, and 1993, respectively.

In 1993, he joined the Faculty of Engineering, Niigata University, where he is now an Associate Professor. His current research involves superresolution techniques, time-frequency analysis, electromagnetic wave measurements, and radar signal processing.

Dr. Yamada is a Member of the Japanese Institute of Electronics, Information and Communication Engineers.

Wolfgang-Martin Boerner (SM'75–F'84) received the B.S. degree from the August von Platen Gymnasium, Ansbach, Germany, the M.S. degree from the Technical University of Munich, Germany, and the Ph.D. degree from the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, in 1958, 1963, and 1967, respectively.

From 1967 to 1968, he was a Research Assistant Engineer with the Department of Electrical and Computer Engineering, Radiation Laboratory, University of Michigan, Ann Arbor. From 1968 to 1978, he was with the Electrical Engineering Department, University of Manitoba, Winnipeg, MB, Canada. In 1978, he joined the Department of Electrical Engineering and Computer Science, University of Illinois, Chicago (UIC), where he is a Professor and Director of the Communications, Sensing and Navigation Laboratory.

Dr. Boerner has been awarded the Alexander von Humboldt Senior U.S. Scientist, the Japan Society for the Promotion of Science Senior U.S. Scientist, and the U.S. Navy Distinguished Senior Professor awards. He is a University of Illinois Distinguished Senior Professor, all in recognition for his contributions to the advancements of electromagnetic inverse scattering, radar polarimetry and interferometry. He is a Senior Member of the Canadian Association of Physicists, the American Society for Engineering Education, the American and the International Societies for Remote Sensing and Photogrammetry; he is Fellow of OSA, SPIE, and AAAS. He is a Registered Engineer with Verein Deutscher Ingenieure, Association of Professional Engineers of the Province of Manitoba, Illinois, and U.S. National Societies of Professional Engineers. Among others, he is a Member of the honor societies Sigma Xi, the American and the German Fulbright Associations, and the Alexander von Humboldt Association.

Harold Mott (M'54–SM'60) was born in Harris, NC. He received the Ph.D. degree from North Carolina State University, Raleigh, in 1960.

He was Associate Professor and Professor of Electrical Engineering with the University of Alabama, Huntsville, from 1960 to 1993, and is presently Professor Emeritus. He is the author of *Polarization in Antennas and Radar* and *Antennas for Radar and Communications: A Polarimetric Approach*, published by Wiley-Interscience in 1986 and 1992, respectively.

Yingning Peng (M'93–SM'96) received the B.S. and M.S. degrees from Tsinghua University, Beijing, China, in 1962 and 1965, respectively.

Since 1993, he has been with the Department of Electronic Engineering, Tsinghua University, where he is now a Professor and Director of the Institute of Signal Detection and Processing. He has worked with real-time signal processing for many years and has published more than 200 papers. His recent research interests include spectral estimation, adaptive filtering, adaptive array signal processing, parallel signal processing, and radar polarimetry.

Prof. Peng is a Member of the Chinese Institute of Electronics. He has received many awards for his contributions to research and education in China.