Accurate Estimation of Minimum Filter Length for Optimum FIR Digital Filters

Koichi Ichige, *Member, IEEE*, Mamoru Iwaki, *Member, IEEE*, and Rokuya Ishii, *Fellow, IEEE*

***Abstract*—This paper presents an accurate estimation formula of minimum filter length for optimum (minimax criterion based) linear-phase finite impulse response (FIR) low-pass digital filters. Two estimation formulae have been already proposed; however, they have some problems which lead to wrong estimation. Discussing the problems based on our experimental results, a new estimation formula is newly proposed. The accuracy of the proposed formula is evaluated in comparison with that of the conventional formulae. Furthermore, the proposed formula is applied to the design of high-pass, band-pass, and band-stop filters. The estimation accuracy is discussed also in this case.**

***Index Terms*—Estimation, filter length, FIR digital filters, Parks–McClellan algorithm.**

I. INTRODUCTION

SHORTER filters (filters with shorter filter length) have advantages over longer filters in that they have fewer circuit elements in a hardware implementation or less computational cost in a software implementation. Therefore, any filter design problem can be considered to be an optimization problem to find a filter satisfying the given specifications with a minimum filter length. Especially for designing optimum (minimax criterion based) linear-phase finite impulse response (FIR) low-pass filters, an iterative optimization algorithm using Remez exchange method [1] (called Parks–McClellan algorithm) has been established for those with odd filter length [2], and for those with even filter length [3]. This algorithm is most widely used for the design of linear-phase FIRs because of its flexible and efficient performance. However, the algorithm requires the filter length of the designed filter to be known in advance, and optimizes the amplitude characteristics in the minimax sense for a specified filter length.

Suppose the case to design an FIR digital filter of low-pass type. Specifications of a target filter are generally given by four parameters: passband edge frequency f_p, stopband edge frequency f_s ($f_p < f_s$), passband ripple δ_p and stopband ripple δ_s (usually $\delta_p \geq \delta_s$). In many practical cases, the above four parameters of a target filter is first specified, and then many filters are designed so as to see how long the minimum filter length N must be. It is hard to know the exact value of the minimum filter length N which satisfies the given specifications.

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To conjecture an appropriate filter length from given specifications in advance, two estimation formulae have been proposed by Herrmann *et al.* [4], [5] and by Kaiser [6] for designing FIR low-pass filters. Such estimation formulae would be helpful for the automated design tools of digital filters such as MATLAB signal processing toolbox [7]. Those formulae estimate the filter length moderately well, however, they cannot achieve enough accuracy because of lack of some considerations especially for longer filters. It is mentioned here that those formulae have been proposed in 1970s, at the beginning of digital filters when the longer filters can hardly be implemented. Since it takes so much time and trouble to establish a more accurate formula, no other formula has been proposed, and the conventional formulae are still used in practice for the estimation. Such a situation must be improved.

This paper presents an accurate estimation formula of minimum filter length for optimum (minimax criterion based) linear-phase FIR digital filters. This paper consists of the following sections. Drawbacks of the conventional formulae are discussed in Section II, and then a new estimation formula for the case $\delta_p > \delta_s$ is developed in Section III based on our experimental results. The accuracy of the proposed formula is evaluated in comparison with that of the conventional formulae in Section IV. Furthermore, the application of the proposed formula to the design of high-pass, band-pass, and band-stop filters is discussed in Section V. The accuracy is evaluated for some example filters. Section VI makes some concluding remarks.

II. CONVENTIONAL APPROACHES AND THEIR PROBLEMS

A. *Conventional Estimation Formulae*

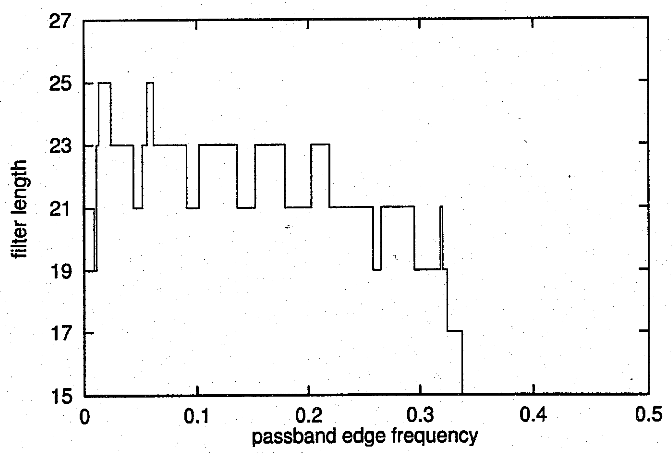
In this subsection, $\langle a \rangle$ and $\lceil a \rceil$ denote the nearest odd integer from a and the minimum odd integer not less than a, respectively.

Herrmann *et al.* [4], [5] proposed the following estimation formula:

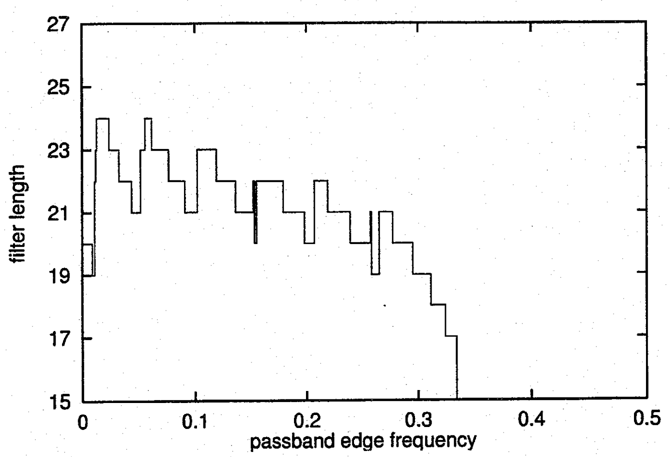
$$\hat{N}_1(\Delta F, \delta_p, \delta_s) = \left\langle \frac{D_\infty(\delta_p, \delta_s)}{\Delta F} - f(\delta_p, \delta_s) \cdot \Delta F + 1 \right\rangle \tag{1}$$

where

$$\begin{aligned} D_\infty(\delta_p, \delta_s) &= \{a_1(\log_{10}\delta_p)^2 + a_2\log_{10}\delta_p + a_3\}\log_{10}\delta_s \\ &\quad + \{a_4(\log_{10}\delta_p)^2 + a_5\log_{10}\delta_p + a_6\} \\ f(\delta_p, \delta_s) &= b_1 + b_2(\log_{10}\delta_p - \log_{10}\delta_s) \\ a_1 &= 5.309 \times 10^{-3}, \quad a_2 = 7.114 \times 10^{-2} \end{aligned}$$



(a)



(b)

Fig. 1. Behavior of (a) minimum odd filter length N_{odd} and (b) minimum (integer) filter length N in the case $\delta_p = 0.01, \delta_s = 0.0001$ and $\Delta F = 0.158$.

$$\begin{aligned} a_3 &= -4.761 \times 10^{-1}, \quad a_4 = -2.660 \times 10^{-3} \\ a_5 &= -5.941 \times 10^{-1}, \quad a_6 = -4.278 \times 10^{-1} \\ b_1 &= 11.012\,17, \quad b_2 = 0.512\,44. \end{aligned}$$

In (1), ΔF denotes the transition width $(f_s - f_p)$. In addition, Kaiser [6] also proposed the following formula independently:

$$\hat{N}_2(\Delta F, \delta_p, \delta_s) = \left\lceil \frac{-20\log_{10}\sqrt{\delta_p\delta_s} - 13}{14.6\Delta F} + 1 \right\rceil. \quad (2)$$

In the case $\delta_p = \delta_s (= \delta)$, (1) and (2) are rearranged as functions of two parameters ΔF and δ

$$\hat{N}_1(\Delta F, \delta) = \left\langle \frac{D_\infty(\delta, \delta)}{\Delta F} - b_1 \cdot \Delta F + 1 \right\rangle \quad (3)$$

$$\hat{N}_2(\Delta F, \delta) = \left\lceil \frac{-20\log_{10}\delta - 13}{14.6\Delta F} + 1 \right\rceil. \quad (4)$$

B. Problems

1) Historical Problems: As the estimated value $\hat{N}_1$ in (1) is rounded to the nearest odd integer, it is very particular to the FIRs of odd filter length, and does not consider those of even filter length. Similarly, $\hat{N}_2$ in (2) is rounded up to the nearest odd integer. This means that the estimation formula (2) also does not include the FIRs of even filter length. Fig. 1(a) shows the behavior of the minimum odd filter length N_{odd} as a function of f_p, which is obtained by actually designing the example filters used in [4, Fig. 14] for the case $\delta_p = 0.01$, $\delta_s = 0.0001$ and $\Delta F = 0.158$. The estimation formulae (1) and (2) have been established based on the odd data of Fig. 1 in 1973 and 1974, respectively. However, a design algorithm for FIRs of even filter length [3] was proposed in 1973. Fig. 1(b) shows the behavior of the minimum filter length N as a function of f_p for the same specifications of Fig. 1(a), however N can be even by the algorithm [3]. From Fig. 1, we can see that

$$N \le N_{\mathrm{odd}}, \quad \forall f_p \in [0, 0.5 - \Delta F].$$

Since the design algorithm has already been established for FIRs of both odd and even filter length, the estimation formulae for every integer filter length should be newly considered based on the data of Fig. 1(b) to make the algorithm more useful.

Furthermore, when (1) and (2) were established, FIRs of longer filter length (approximately more than 150) were not feasible, hence the formulae must have been formulated only for the FIRs of short filter length. Actually, as shown later, the estimation accuracies of (1) and (2) become worse as the filter length becomes long. Now that FIRs of longer filter length can be designed, the accuracy for longer filter length should be improved.

From the above discussions, we can summarize the historical problems of the conventional formulations (1) and (2) as follows.

Problem 1: Formulas (1) and (2) are made based on the data of FIRs of odd filter length only, and do not mention those of even filter length.

Problem 2: Formulas (1) and (2) do not correspond to the FIRs of long filter length.

2) Problems on Formulation: Since specifications of a filter are given by four parameters: f_p, f_s, δ_p, and δ_s, the minimum filter length N can be a function of those four variables. In the case using $\Delta F (= f_s - f_p)$ instead of f_s, N can be rewritten as a function of f_p, ΔF, δ_p and δ_s. However, in (1) and (2), the estimated minimum filter length $\hat{N}_1$ and $\hat{N}_2$ are written as functions of three-variables: ΔF, δ_p, and δ_s. This means that they are constant irrespective of f_p.

If N is really independent of f_p, the graph of N versus f_p must be drawn as a horizontal straight line. However, as shown in Fig. 1, N becomes small as f_p increases. This result leads to the fact that the minimum filter length N depends on all of the four variables including f_p.

Hereafter, $\langle a \rangle$ and $\lceil a \rceil$ newly denote the nearest integer from a and the minimum integer not less than a respectively, to deal with all the integers. The case $\delta_p = \delta_s$ $(= \delta)$ is studied first. Fig. 2 shows the behavior of the minimum filter length N as a function of f_p. Fig. 2(a) shows the behavior for the case $\delta = 0.001$ and $\Delta F = 0.05$, and Fig. 2(b) for the case $\delta = 0.0001$ and $\Delta F = 0.1$, where the solid, broken and dotted lines denote the required minimum filter length N by trial and error experiments, the estimated filter length $\hat{N}_1$ by (3) and the estimated filter length $\hat{N}_2$ by (4), respectively. From Fig. 2, we can see that

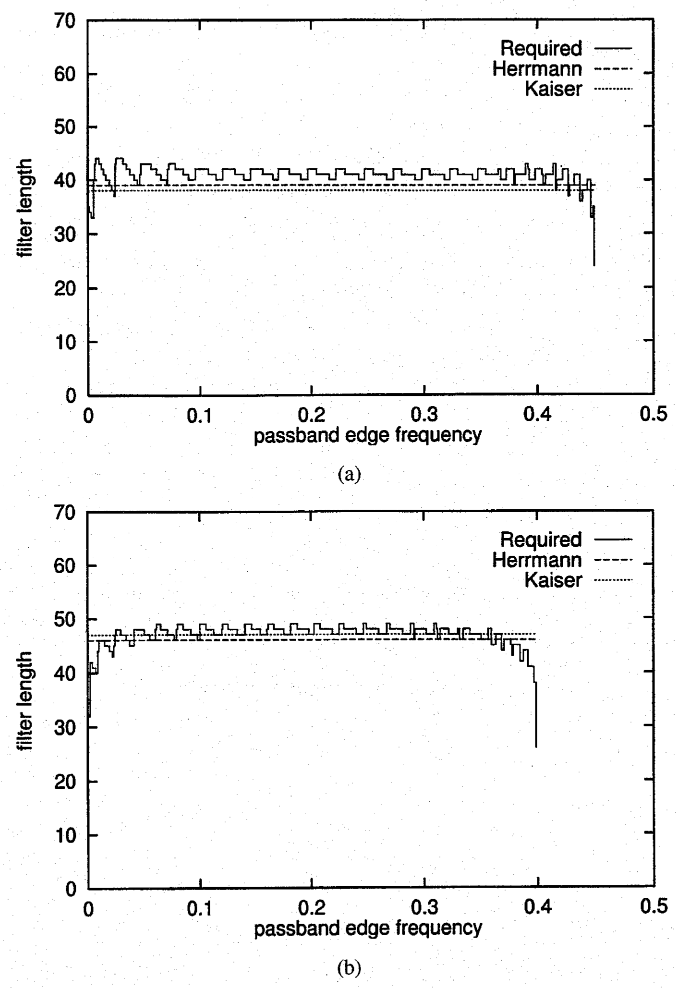


Fig. 2. Behavior of the required filter length N and the estimated filter length $\hat{N}_1$ and $\hat{N}_2$ as a function of f_p. (a) In the case $\delta_p = 0.1$, $\delta_s = 0.001$ and $\Delta F = 0.03$. (b) In the case $\delta = 0.0001, \Delta F = 0.1$.

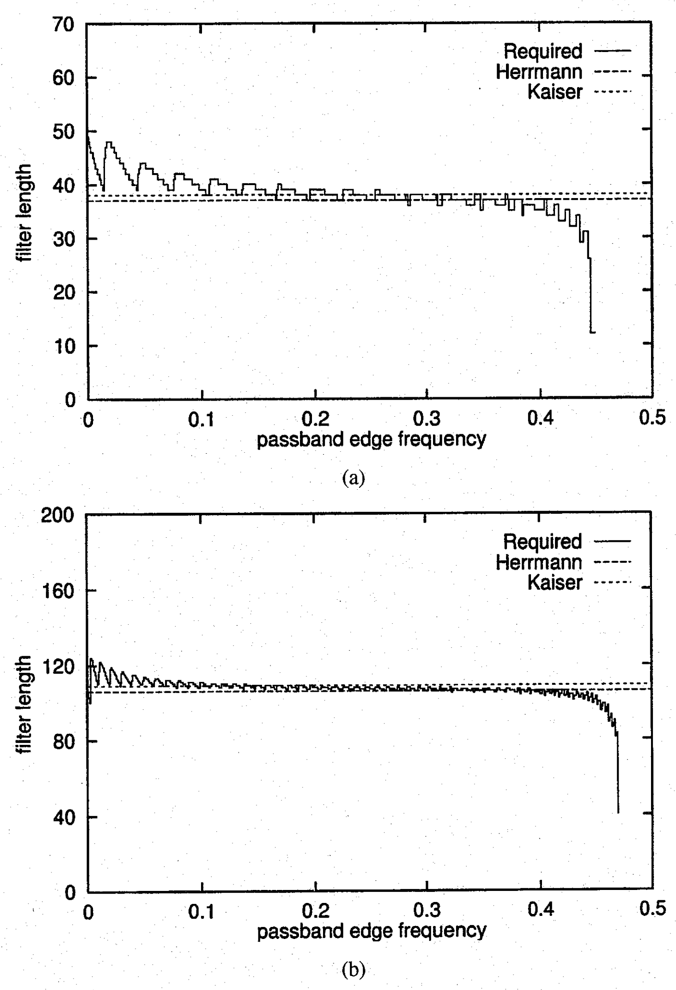


Fig. 3. Behavior of the required minimum filter length N (real line), the estimated filter length $\hat{N}_1$ (broken line) and $\hat{N}_2$ (dotted line) as a function of f_p. (a) In the case $\delta_p = 0.1$, $\delta_s = 0.001$ and $\Delta F = 0.03$. (b) In the case $\delta_p = 0.01, \delta_s = 0.0001$ and $\Delta F = 0.05$.

the conventional formulas (3) and (4) in the case $\delta_p = \delta_s$ have the following problems.

Problem 3: For any ΔF and δ, the minimum filter length N becomes shorter as f_p gets close to zero or $0.5 - \Delta F$.

Problem 4: Let f_c denote the center frequency in the interval $[0, 0.5 - \Delta F]$, i.e., $f_c = (0.5 - \Delta F)/2$. For any ΔF and δ, (3) and (4) tend to give the shorter filter length than actually required when f_p is placed around f_c.

The above problems of (1) and (2) are due to the lack of consideration of some significant theoretical properties, which are mentioned later.

Next, the case $\delta_p \neq \delta_s$ is studied. Fig. 3 shows the behavior of the minimum filter length N as a function of f_p. From Fig. 3, we can see that the conventional formulas (1) and (2) have the following problem.

Problem 5: For any ΔF and any $\delta_p > \delta_s$, the minimum filter length N becomes longer as f_p approaches zero, and becomes shorter as f_p approaches $0.5 - \Delta F$, however (1) and (2) are constant, irrespective of f_p.

Since specifications of low-pass filters are given by four-parameters, estimation formulae must be written as four-variable functions. However, both (1) and (2) are three-variable functions of ΔF, δ_p, and δ_s. As seen in Fig. 3, the behavior of N should be respective to f_p.

III. Proposed Approach

A. For Low-Pass Filters with Identical Passband and Stopband Ripples

An accurate estimation formula $\hat{N}_3(f_p, \Delta F, \delta)$ is proposed in this section. First, the relations between filter parameters are studied based on experimental results and theoretical considerations. Then the formula $\hat{N}_3$ is formulated.

1) Relations Between Filter Parameters: In the case $\delta_p = \delta_s = \delta$, the required minimum filter length N_3 is determined by three parameters: f_p, f_s and δ. Hence, we express N by

$$N_3 = F_1(f_p, f_s, \delta).$$

Since f_s can be represented by f_p and ΔF, N can also be expressed by

$$N_3 = F_2(f_p, \Delta F, \delta) = F_1(f_p, f_p + \Delta F, \delta). \qquad (5)$$

a) Relation Between N_c and δ: Let N_c denote the minimum filter length N for fixed $f_p = f_c$. First, the relation between N_c and δ is studied for various ΔF.

Fig. 4 shows a behavior of the minimum filter length N_c as a function of ripple δ for some ΔF. Here, the following proposition 1 theoretically holds.

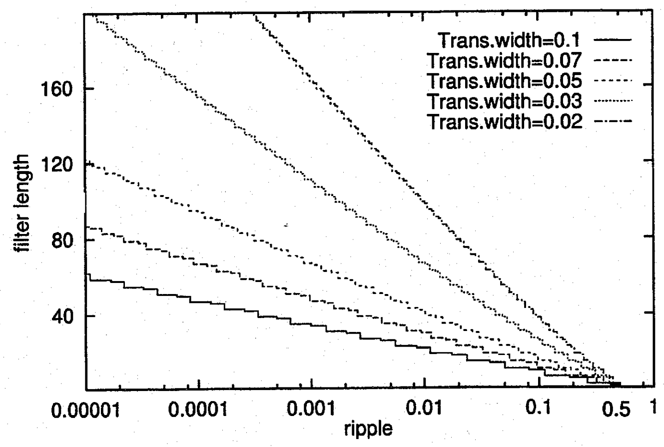


Fig. 4. Behavior of the minimum filter length N_c as a function of δ (log-to-linear scale).

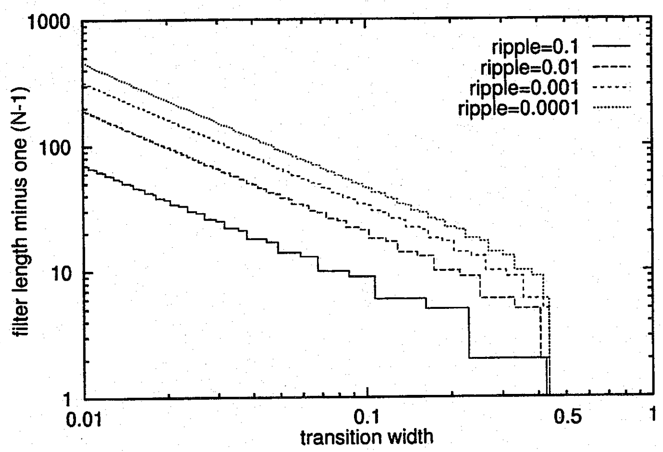


Fig. 5. Behavior of the minimum filter length N_c as a function of ΔF (log-to-log scale).

Proposition 1: For any ΔF, the minimum filter length N_c is one if $\delta = 0.5$.

Proof: Consider a filter $H(z) = 0.5$ of the minimm filter length one. For any ΔF, the amplitude of the ripples caused by the filter in both the passband and the stopband is 0.5. This means that $H(z)$ is a filter satisfying $\delta = 0.5$. Therefore, the minimum filter length is one for any ΔF when $\delta = 0.5$. ■

The fact of Proposition 1 is also observed experimentally in Fig. 4. The conventional formulas (3) and (4) lacked consideration to this fact. From Fig. 4, we also have the following observation.

Observation 1: For any ΔF, $(N_c - 1)$ is almost a linear function of $\log_{10} \delta$.

This observation seems to be noticed in Kaiser's formula (4), however it is missed in Herrmann's formula (3).

b) Relation Between N_c and ΔF: Next, the relation between N_c and ΔF is studied for various δ.

Fig. 5 illustrates the behavior of $(N_c - 1)$ as a function of ΔF for some values of δ. From Fig. 5, the following observation seems to hold.

Observation 2: For any δ, it seems that $\log_{10}(N_c - 1)$ is a linear function of $\log_{10} \Delta F$ with the gradient -1 when $N_c \geq 10$.

This is equivalent to the following observation.

Observation 3: For any δ, $(N_c - 1)$ is nearly in inverse proportion to ΔF when $N_c \geq 10$.

These are also mentioned in Kaiser's formula (4); however, the consideration to this observation in the Herrmann's formula (3) is not in enough detail.

c) Relation Between N and f_p: Next, the relation between N and f_p are studied for some values of ΔF and δ. As seen later in Fig. 2, the following seems to hold.

Observation 4: For any ΔF and δ, the graph of N versus f_p resembles the shape of an arch. In other words, the minimum filter length N is shorter than N_c when $f_p \simeq 0$ or $f_p \simeq 0.5 - \Delta F$.

2) Formulation:

a) Estimation of N_c for $f_p = f_c$: First, the estimated value $\hat{N}_c$ of the minimum filter length N_c for $f_p = f_c$ is formulated. From Proposition 1 and Observation 1, $\hat{N}_c$ can be written as a function of δ as follows:

$$\begin{aligned}\hat{N}_c(\delta) &= \lceil a\{-\log_{10}\delta + \log_{10} 0.5\}^b + 1\rceil \\ &= \lceil a\{-\log_{10}(2\delta)\}^b + 1\rceil, \quad b \simeq 1. \end{aligned} \tag{6}$$

From Observations 2 and 3, $\hat{N}_c$ should also satisfy, as a function of ΔF, that

$$\log_{10}\left(\hat{N}_c(\Delta F) - 1\right) = c\{\log_{10}(\Delta F)\} + d, \quad c \simeq -1$$

namely

$$\hat{N}_c(\Delta F) = \lceil 10^d \cdot (\Delta F)^c + 1\rceil. \tag{7}$$

Equations (6) and (7) lead to one possible expression of $\hat{N}_c$

$$\hat{N}_c(\Delta F, \delta) = \lceil p\{-\log_{10}(2\delta)\}^q (\Delta F)^r + 1\rceil. \tag{8}$$

Here, the parameters p, q and r in (8) are determined based on the following least mean square (LMS) approximation criterion for thousands of combinations of $\{\Delta F, \delta, N_c\}$

$$\sum |\hat{N}_c - N_c|^2 \to \min.$$

Consequently, the estimation formula $\hat{N}_c$ is obtained as

$$\hat{N}_c(\Delta F, \delta) = \left\lceil \frac{1.101\{-\log_{10}(2\delta)\}^{1.1}}{\Delta F} + 1 \right\rceil.$$

b) The Proposed Estimation Formula $\hat{N}_3$ for $\delta_p = \delta_s = \delta$: Finally, the approximation function for the arch-like curves is found so as to determine a new estimation formula $\hat{N}_3$.

From the experiments of fitting various elementary functions to the arch-like curves, we found that the following modified arctangent function (9) is most suitable for approximating them:

$$g(f_p, \Delta F, \delta) := \frac{2}{\pi} \arctan\left\{v(\Delta F, \delta) \cdot \left(\frac{1}{f_p} - \frac{1}{(0.5 - \Delta F)}\right)\right\} \tag{9}$$

$$v(\Delta F, \delta) := 2.325 \cdot (-\log_{10}\delta)^{-0.445} \cdot (\Delta F)^{-1.39}. \tag{10}$$

The coefficients in (10) are also determined based on LMS approximation criterion. In this case, several hundred thousands of combinations of $\{f_p, \Delta F, \delta, N\}$ were used to determine the coefficients. This approximation problem is formulated as

$$\sum \left|\hat{N}_3 - N\right|^2 \to \min.$$

Finally we obtain the estimation formula $\hat{N}_3$ in (11), shown at the bottom of the page.The formula (11) is a function of three-variables: f_p, ΔF and δ.

B. For any Low-Pass Filter

Using (11), we expand a new estimation formula $\hat{N}_4$ which corresponds to the case $\delta_p \neq \delta_s$.

1) Relations Between Filter Parameters: In the case $\delta_p \neq \delta_s$, the minimum filter length N_4 is determined by four parameters: f_p, f_s, δ_p, and δ_s. Hence, we express N_4 by

$$N_4 = G_1(f_p,\, f_s,\, \delta_p,\, \delta_s).$$

Since f_s can be represented by f_p and ΔF, N can also be expressed by

$$N_4 = G_2(f_p,\, \Delta F,\, \delta_p,\, \delta_s) = G_1(f_p,\, f_p + \Delta F,\, \delta_p,\, \delta_s).$$

We assume that the minimum filter length N_4 for $\delta_p \neq \delta_s$ is represented in a form of an addition of the formula N_3 for $\delta_p = \delta_s$ in (5) and the distance DN, i.e.,

$$\begin{aligned} N_4(f_p,\, \Delta F,\, \delta_p,\, \delta_s) \\ = N_3(f_p,\, \Delta F,\, \delta_p) + DN(f_p,\, \Delta F,\, \delta_p,\, \delta_s). \end{aligned}$$

First we aim to formulate the approximation $D\hat{N}$ for the distance DN.

2) Formulation of $D\hat{N}$: We do not optimize the approximate distance $D\hat{N}$ for DN, however optimize the whole filter length $\hat{N}_4 = \hat{N}_3 + D\hat{N}$ for the required filter length N_4. This approximation problem can be formulated as

$$\sum |\hat{N}_4 - N|^2 \rightarrow \min. \qquad (12)$$

Furthermore, the approximation $D\hat{N}$ must satisfy

$$D\hat{N} = 0, \qquad \delta_p = \delta_s$$

so as to be $N_4 = N_3$.

Now we study the behavior of the distance DN. For example, Fig. 6 shows the behavior of the required filter length N_4 for the case $\Delta F = 0.05$, $\delta_p = 0.01$, $\delta_s = 0.000\,01$, the required filter length N_3 for the case $\Delta F = 0.05$, $\delta_p = \delta_s = 0.01$, and their distance DN as a function of f_p. From Fig. 6, we have the following observation.

Observation 5: The distance DN gets smaller as f_p become larger. Especially when f_p get close to $0.5 - \Delta F$, the distance DN decreases rapidly. This behavior of DN resembles the behavior of N_3 if f_p is larger than f_c.

It would be mentioned here that the similar behavior were observed for other specifications. This observation suggests that the modified arctangent function like g of (9) may be suited for approximating DN. From the experiments of fitting various elementary functions to the curves of DN, we found that the following modified arctangent function h is most suitable. These approximate functions can be written as

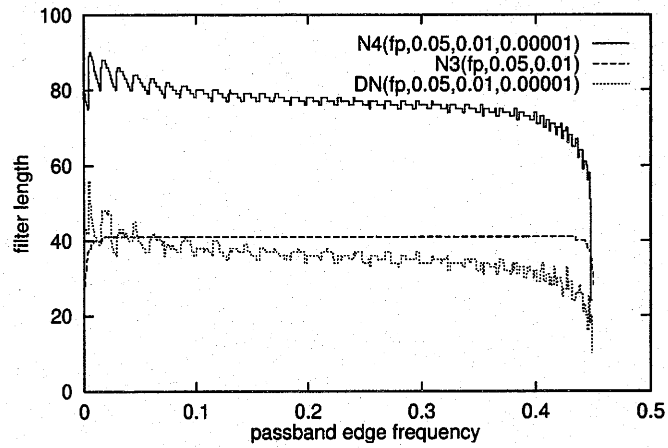


Fig. 6. Behavior of the minimum filter length N_4 for the case $\Delta F = 0.05$, $\delta_p = 0.01$, $\delta_s = 0.000\,01$, the minimum filter length N_3 for the case $\Delta F = 0.05$, $\delta_p = \delta_s = 0.01$, and their distance DN as a function of f_p.

$$\begin{aligned} &D\hat{N}(f_p,\, \Delta F,\, \delta_p,\, \delta_s) \\ &:= \left\lceil \hat{N}_m(\Delta F,\, \delta_p,\, \delta_s) \cdot \left\{ h(f_p,\, \Delta F,\, 1.1) - \frac{h(0.5 - \Delta F - f_p, \Delta F, 0.29) - 1}{2} \right\} \right\rceil \qquad (13) \\ &h(f_p,\, \Delta F,\, c) \\ &:= \frac{2}{\pi} \cdot \arctan\left\{ \frac{c}{\Delta F} \cdot \left(\frac{1}{f_p} - \frac{1}{0.5 - \Delta F} \right) \right\}. \end{aligned}$$

3) Formulation of N_m: The function $\hat{N}_m(\Delta F,\, \delta_p,\, \delta_s)$ in (13) is an approximation for N_m which is nearly the same as $DN(f_c,\, \Delta F,\, \delta_p,\, \delta_s)$. This approximation $\hat{N}_m$ works like $\hat{N}_c$ for the formula $\hat{N}_3$ in (11). Now the behavior of N_m is studied, and its approximation $\hat{N}_m$ is formulated.

a) Relation Between N_m and ΔF: First, the relation between N_m and ΔF is studied for various δ_p and δ_s. Fig. 7 illustrates the behavior of N_m as a function of ΔF for some values of δ_p and δ_s. From Fig. 7, the following observation seems to hold.

Observation 6: For any δ_p and δ_s, it seems that $\log_{10} N_m$ is a linear function of $\log_{10} \Delta F$ with the gradient -1. This is equivalent to the following observation.

Observation 7: For any δ_p and δ_s, N_m is nearly in inverse proportion to ΔF.

b) Relation Between N_m and $\log(\delta_p/\delta_s)$: Next, the relation between N_m and a function $\log(\delta_p/\delta_s)$ is studied for various ΔF, δ_p and δ_s. Fig. 8 illustrates the behavior of N_m as a function of $\log(\delta_p/\delta_s)$ for some values of δ_p and ΔF. From Fig. 8, we have the following observation.

Observation 8: For any δ_p, δ_s and ΔF, N_m is almost a linear function of $\log(\delta_p/\delta_s)$.

$$\hat{N}_3(f_p,\, \Delta F,\, \delta) := \left\lceil \hat{N}_c(\Delta F,\, \delta) \cdot \frac{g(f_p, \Delta F, \delta) + g(0.5 - \Delta F - f_p, \Delta F, \delta) + 1}{3} \right\rceil. \qquad (11)$$

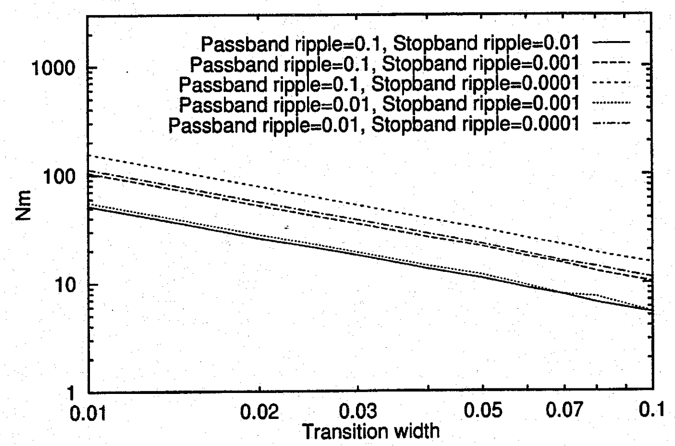


Fig. 7. Behavior of the function N_m for some δ_p and δ_s as a function of $1/\Delta F$ (log-to-log scale).

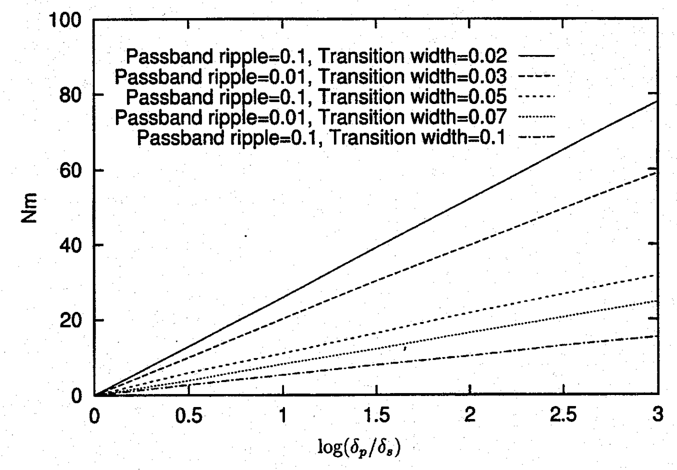


Fig. 8. Behavior of function N_m as a function of $\log(\delta_p/\delta_s)$.

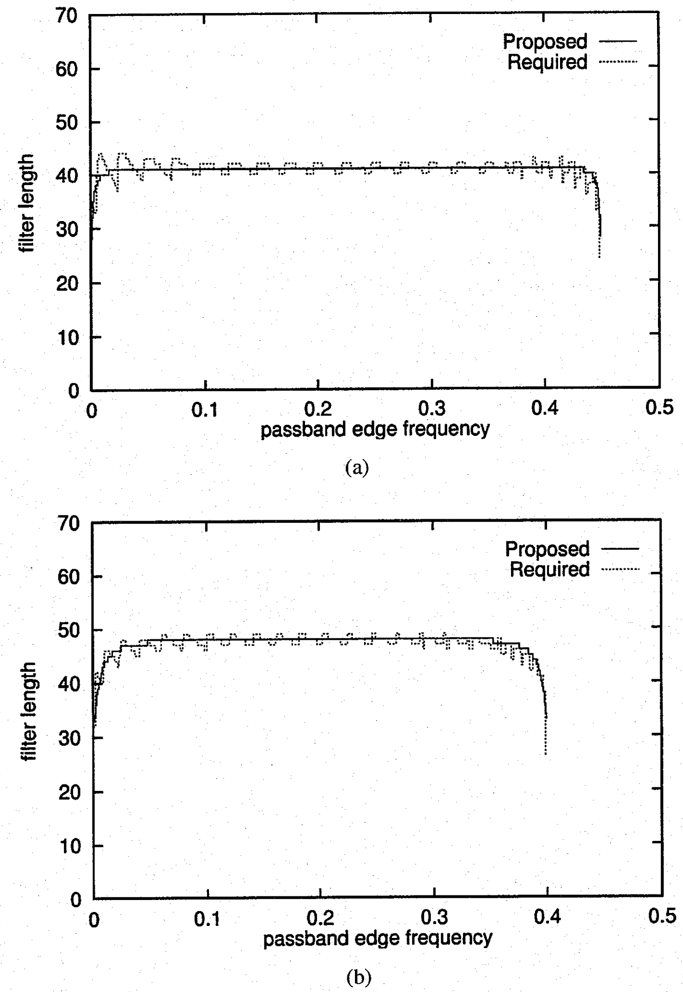


Fig. 9. Behavior of the minimum filter length N and the estimated filter length $\hat{N}_3$ as a function of f_p. (a) In the case $\delta = 0.01$ and $\Delta F = 0.05$. (b) In the case $\delta = 0.0001$ and $\Delta F = 0.1$.

Based on Observations 7, 8, and some more experiments, we formulate the approximation $\hat{N}_m$ as

$$\hat{N}_m(\Delta F, \delta_p, \delta_s) = p \cdot \frac{\log_{10}(\delta_p/\delta_s)}{\Delta F} \cdot (-\log_{10}\delta_p)^q. \quad (14)$$

The coefficients p and q in (14) are also determined based on LMS approximation criterion in (12). In this case, millions of combinations of $\{f_p, \Delta F, \delta_p, \delta_s, N\}$ are used to determine the coefficients. Finally, we have

$$p = 0.52, \qquad q = 0.17.$$

4) The Proposed Estimation Formula $\hat{N}_4$ for $\delta_p \neq \delta_s$: As a result, the proposed estimation formula $\hat{N}_4$ can be summarized as follows:

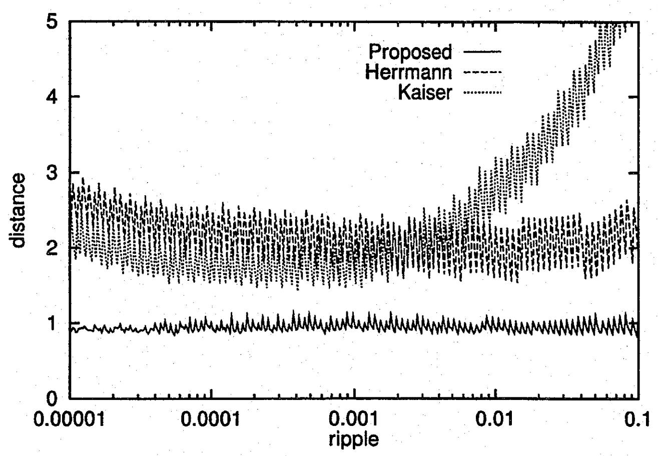
$$\begin{aligned} &\hat{N}_4(f_p, \Delta F, \delta_p, \delta_s) \\ &\quad := \hat{N}_3(f_p, \Delta F, \delta_p) + D\hat{N}(f_p, \Delta F, \delta_p, \delta_s) \quad (15) \\ &D\hat{N}(f_p, \Delta F, \delta_p, \delta_s) \\ &\quad := \left\lceil \hat{N}_m(\Delta F, \delta_p, \delta_s) \cdot \left\{ h(f_p, \Delta F, 1.1) - \frac{h(0.5 - \Delta F - f_p, \Delta F, 0.29) - 1}{2} \right\} \right\rceil \\ &\hat{N}_m(\Delta F, \delta_p, \delta_s) \\ &\quad := 0.52 \cdot \frac{\log_{10}(\delta_p/\delta_s)}{\Delta F} \cdot (-\log_{10}\delta_p)^{0.17}. \\ &h(f_p, \Delta F, c) \\ &\quad := \frac{2}{\pi} \cdot \arctan\left\{ \frac{c}{\Delta F} \cdot \left(\frac{1}{f_p} - \frac{1}{0.5 - \Delta F} \right) \right\}. \end{aligned}$$

The formula $\hat{N}_4$ is a four-variable function of f_p, ΔF, δ_p, and δ_s.

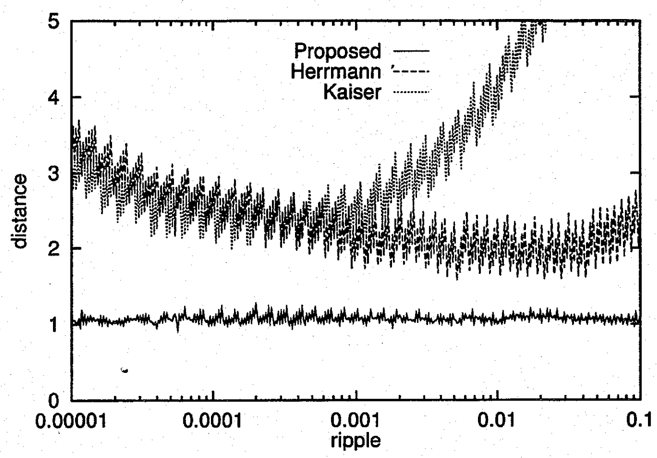
IV. Evaluation

A. For Low-Pass Filters with Identical Passband and Stopband Ripples

The proposed estimation formula (11) is evaluated in comparison with the conventional formulas (3) and (4). Fig. 9 shows the behavior of the minimum filter length N and the estimated values $\hat{N}_i$ $(i = 1, 2, 3)$ as a function of f_p. Fig. 9(a) shows the behavior for the case $\delta = 0.001$ and $\Delta F = 0.05$, and Fig. 9(b) for the case $\delta = 0.0001$ and $\Delta F = 0.1$. Fig. 9 demonstrates that the estimated filter length $\hat{N}_3$ is better than the conventional estimations by (3) and (4), which are irrespective of f_p.

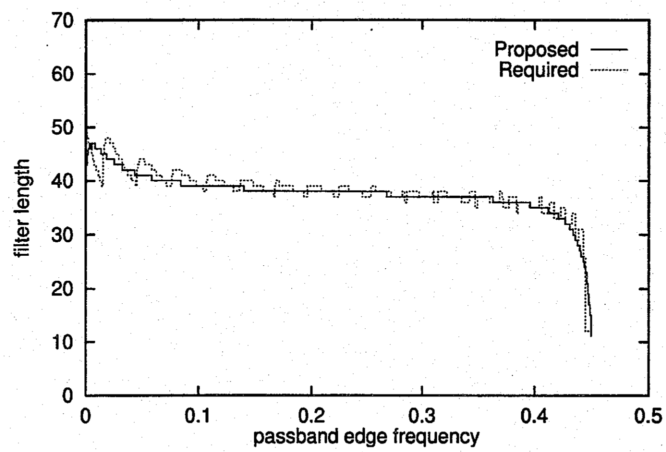


(a)

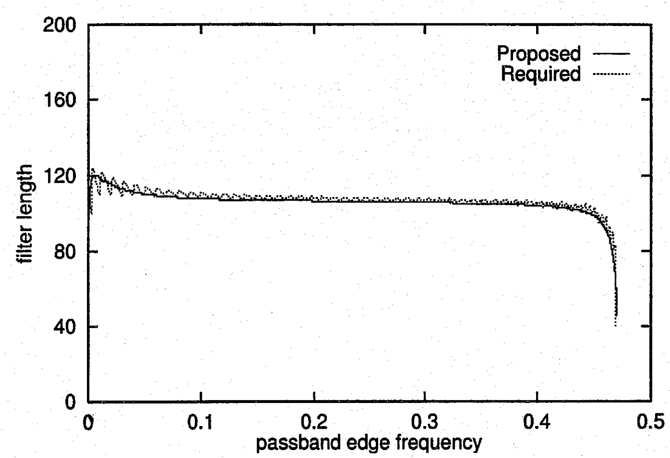


(b)

Fig. 10. Behavior of the distances ΔN_i ($i = 1, 2, 3$) as a function of δ (log-to-linear scale). (a) In the case $\Delta F = 0.05$. (b) In the case $\Delta F = 0.03$.



(a)



(b)

Fig. 11. Behavior of the estimated filter length $\hat{N}_4$ (real line) and the required minimum filter length N (broken line) as a function of f_p. (a) In the case $\delta_p = 0.1$, $\delta_s = 0.001$ and $\Delta F = 0.03$. (b) In the case $\delta_p = 0.01$, $\delta_s = 0.0001$ and $\Delta F = 0.05$.

The estimation performances of (3) and (4) are evaluated by the following distance $\Delta N_i (i = 1, 2, 3)$, defined by the normalized L^1 norm:

$$\Delta N_i(\Delta F, \delta) := \frac{1}{0.5 - \Delta F} \int_0^{0.5-\Delta F} |\hat{N}_i - N| \, df_p, \quad (i = 1, 2, 3). \tag{16}$$

In (16), ΔN_i means the average distance with respect to $f_p \in [0, 0.5 - \Delta F]$, which is a function of ΔF and δ.

Fig. 10 shows the behavior of the distances ΔN_i ($i = 1, 2, 3$) as a function of δ for some values of ΔF. In Fig. 10, we can see that the distance ΔN_3 stays around one and is smaller than both ΔN_1 and ΔN_2 for any ΔF and δ.

Especially in the case where both ΔF and δ get smaller, i.e., when the minimum filter length N gets longer, the distances ΔN_1 and ΔN_2 become worse. This is because the conventional formulas (3) and (4) do not correspond to the FIRs of long filter length and some significant theoretical considerations were missed. Since the distance ΔN_3 stays around one for any ΔF and δ, it is obvious that the proposed estimation formula (11) makes a good estimation for any case.

B. For Any Low-Pass Filter

The proposed estimation formula (15) is evaluated in comparison with the conventional formulae (1) and (2).

Fig. 11 shows the behavior of the estimated filter length $\hat{N}_4$ by (11) and the required minimum filter length N as a function of f_p. Specifications of Fig. 11(a) and (b) are the same as those of Fig. 3(a) and (b), respectively. Figs. 3 and 11 demonstrate that the proposed estimation formula $\hat{N}_4$ is better than the conventional estimations by (1) and (2), which are constant irrespective to f_p.

The estimation performance of (1) and (2) is evaluated by the following distance $\Delta N_i (i = 1, 2, 4)$ defined by the normalized L^1 norm

$$\Delta N_i(\Delta F, \delta_p, \delta_s) := \frac{1}{0.5 - \Delta F} \int_0^{0.5-\Delta F} |\hat{N}_i - N| \, df_p, \quad (i = 1, 2, 4). \tag{17}$$

In (17), ΔN_i means the average distance with respect to $f_p \in [0, 0.5 - \Delta F]$. Fig. 12 shows the behavior of the distances ΔN_i ($i = 1, 2, 4$) as a function of ΔF for some values of δ_p and

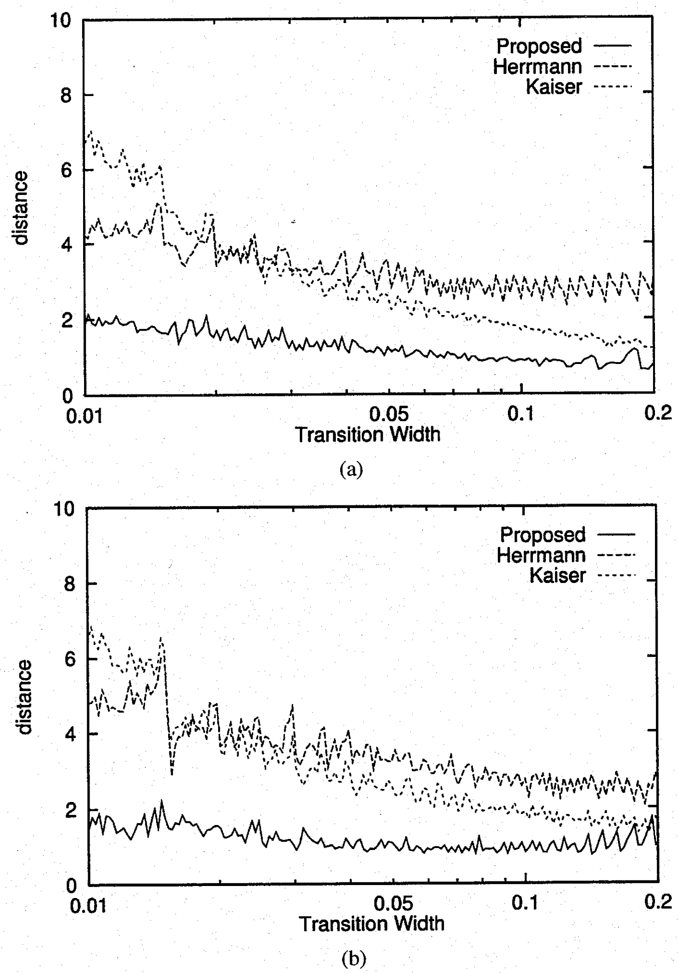


Fig. 12. Behavior of the distance ΔN_4 of the proposed estimation formula (real line), the distance ΔN_1 of the Herrmann's formula (broken line) and the distance ΔN_2 of the Kaiser's formula (dotted line) as a function of ΔF (log-to-linear scale). (a) In the case $\delta_p = 0.1$ and $\delta_s = 0.001$. (b) In the case $\delta_p = 0.01$ and $\delta_s = 0.0001$.

δ_s. In Fig. 12, we can see that the distance ΔN_4 stays less than two and is smaller than both ΔN_1 and ΔN_2. This means that the proposed formula $\hat{N}_4$ makes better estimation than both $\hat{N}_1$ and $\hat{N}_2$.

V. Minimum Filter Length for High-Pass, Band-Pass, and Band-Stop Filters

The proposed formula (15) is applied to the design of high-pass, band-pass, and band-stop filters. Since the conventional estimation formulae correspond only to the estimation for low-pass filters design, they are not applied to the design of other types of filters. Thus, the proposed formulae for high-pass, band-pass, and band-stop filters are evaluated by comparing the estimated minimum filter length with the required one for some design examples. It is mentioned that the estimation of the minimum filter length for FIR differentiators and FIR Hilbert transformers has been studied by Rabiner and Schafer [8], [9] and it realizes good accuracy.

A. High-Pass Filters

In designing FIR high-pass filters, the proposed estimation formula (11) can be applied.

TABLE I
General Specifications of High-Pass, Band-Pass, and Band-Stop Filters

	High-pass	Band-pass	Band-stop
passband	$[F_p, 0.5]$	$[f_{p1}, f_{p2}]$	$[0, F_{p1}]$ $[F_{p2}, 0.5]$
stopband	$[0, F_s]$	$[0, f_{s1}]$ $[f_{s2}, 0.5]$	$[F_{s1}, F_{s2}]$
passband ripple	δ_p	δ_p	δ_p
stopband ripple	δ_s	δ_s	δ_s

TABLE II
Example Specifications of High-Pass, Band-Pass, and Band-Stop Filters

	High-pass	Band-pass	Band-stop
passband	[0.2, 0.5]	[0.10, 0.15]	[0, 0.17] [0.25, 0.5]
stopband	[0, 0.1]	[0, 0.08] [0.2, 0.5]	[0.2, 0.22]
passband ripple	0.01	0.1	0.01
stopband ripple	0.0001	0.001	0.0001

Let F_s, F_p, δ_s and δ_p respectively denote the edge frequency of stopband, that of passband ($0 \le F_s < F_p \le 0.5$), the stopband ripple, and the passband ripple, as shown in Table I. In this case, the estimation of minimum filter length $\hat{N}_H$ for the high-pass filter is given by

$$\hat{N}_H(F_p, \Delta F, \delta_p, \delta_s) = \hat{N}_4(0.5 - F_p, \Delta F_H, \delta_p, \delta_s) \quad (18)$$

where $\Delta F_H = F_p - F_s$.

Consider designing the high-pass filter for which specifications are given in Table II. The estimated filter length $\hat{N}_H$ by (18) is 33, and the required filter length N_H is 35. The distance is only two in this case.

B. Band-Pass Filters

In designing FIR band-pass filters, the proposed estimation formula (11) can be also applied as follows.

Suppose that the specifications of a band-pass filter is given as Table I, where $0 \le f_{s_1} < f_{p_1} \le f_{p_2} < f_{s_2} \le 0.5$ and $\delta_s < \delta_p$. In this case, the estimation of minimum filter length $\hat{N}_{BP}$ for the band-pass filter is given by

$$\begin{aligned}\hat{N}_{BP}(f_{p_1}, f_{p_2}, \Delta f_1, \Delta f_2, \delta_p, \delta_s) \\ = \max\{\hat{N}_4(f_{p_2}, \Delta f_2, \delta_p, \delta_s) \\ \hat{N}_4(0.5 - f_{p_1}, \Delta f_1, \delta_p, \delta_s)\}\end{aligned} \quad (19)$$

where $\Delta f_1 = f_{p_1} - f_{s_1}$ and $\Delta f_2 = f_{s_2} - f_{p_2}$.

For example, We design a band-pass filter which satisfy the specifications in Table II. The estimated filter length $\hat{N}_{BP}$ by (19) is $\max\{38, 91\} = 91$, and the required filter length N_{BP} is 92. The distance is only one in this case.

C. Band-Stop Filters

Specifications of a band-stop filter is given as Table I, where $0 \le F_{p_1} < F_{s_1} \le F_{s_2} < F_{p_2} \le 0.5$ and $\delta_s < \delta_p$. In this case,

TABLE III
EXAMPLE SPECIFICATIONS OF THE PARTICULAR BAND-PASS AND BAND-STOP FILTERS

	Band-pass	Band-stop
passband	[0.20, 0.21]	[0, 0.2] [0.3, 0.5]
stopband	[0, 0.10] [0.25, 0.5]	[0.25, 0.26]
passband ripple	0.01	0.01
stopband ripple	0.0001	0.0001

the estimation of minimum filter length $\hat{N}_{BS}$ for the band-stop filter is given by:

$$\begin{aligned}\hat{N}_{BS}(F_{p_1}, F_{p_2}, \Delta F_1, \Delta F_2, \delta_p, \delta_s) \\ = \max\{\hat{N}_4(F_{p_1}, \Delta F_1, \delta_p, \delta_s), \\ \hat{N}_4(0.5 - F_{p_2}, \Delta F_2, \delta_p, \delta_s)\}\end{aligned} \tag{20}$$

where $\Delta F_1 = F_{s_1} - F_{p_1}$ and $\Delta F_2 = F_{p_2} - F_{s_2}$.

Here we design the band-stop filter of which specifications are given in Table II. The estimated filter length $\hat{N}_{BS}$ by (20) is $\max\{106, 107\} = 107$, and the required filter length N_{BS} is 105. The distance is two in this case.

VI. CONCLUDING REMARKS

This paper proposed the accurate estimation formula (15) of the minimum filter length for optimum FIR low-pass digital filters. In comparison with the conventional estimation formulae (1) and (2), the proposed formula (15) realizes much better accuracy. Furthermore, the proposed formula (15) was applied to the design of high-pass, band-pass and band-stop filters.

The estimation formulae (19) and (20) for band-pass and band-stop filters still have problems which may lead to wrong estimation. If we estimate the minimum filter length of a particular band-pass filter with the narrow passband (or, a band-stop filter with the narrow stopband), the proposed estimation formulae (19) and (20) tend to give longer filter length than required.

For example, we design the band-pass filter with the narrow passband for which specifications are given in Table III. The estimation $\hat{N}_{BP}$ by the proposed formula (19) is $\max\{33, 82\} = 82$, and the required filter length N is 69. The estimation is 13 longer than the required. The band-stop filter with the narrow stopband (called notch filter) is also designed. Specifications of the band-stop filter are given in Table III. The estimation $\hat{N}_{BS}$ by the proposed formula (20) is $\max\{66, 83\} = 83$, and the required N is 65. The distance is 18 in this case. Behavior of the minimum filter length of those filters would be studied further, and the accuracy of the estimation must be improved.

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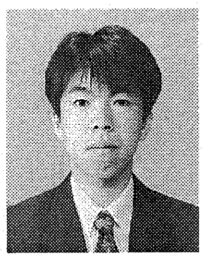

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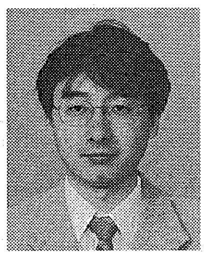

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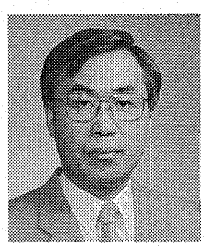

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