An Exact Error Analysis for Low-Duty Pulsed DS-CDMA Systems in Flat Nakagami Fading

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***Abstract*— Based on the characteristic function method, an exact error analysis is presented for low-duty pulsed direct sequence code division multiple access (DS-CDMA) systems in flat Nakagami fading channel. The presented method is simple and good for any arbitrary pulse shape. The computational involvement of the method is simpler than the method based on improved Gaussian approximation (IGA), which is the only reliable method currently available for calculating error probabilities of such systems.**

***Index Terms*— Multiaccess communication, ultra-wideband, Nakagami fading.**

I. INTRODUCTION

RECENTLY, multiple access communications using ultra-wideband (UWB) signals have received considerable attention [1]. Several works have been presented on the topic [1]−[5]. The UWB technology usually transmits signal with low duty-cycle, whereas, a conventional communication system like direct sequence code division multiple access (DS-CDMA) transmits signal continuously i.e. with full duty [6]−[9]. Motivated by the emergence of UWB, new ideas regarding pulsed DS-CDMA with low chip-duty have evolved [2]−[3]. Envisioning the potential application of such signals in UWB, those have received much interest from the UWB community [1]−[3]. In [2], Chernoff bounds on bit error probabilities (BEP) were given for such systems. Later, in [3], we presented approximate BEP for such systems considering additive white Gaussian noise (AWGN) channel. The standard, improved and simplified improved Gaussian approximations: abbreviated as SGA, IGA and SIGA [6]−[7] respectively were considered in [3]. It was shown that the simple method SGA is not accurate for low-duty systems in general. The other simple method SIGA that was previously accurately used for conventional DS-CDMA systems [6], has restrictive region of validity for low-duty systems [3]. However, the IGA [7] provided accurate results at the expense of much higher computational complexity [3]. Presently, the method based on IGA presents the only-available reliable closed-form BEP expression for such systems [3].

The exact error analysis for such low-duty DS-CDMA systems is of interest, however, has not yet been addressed. The exact error analysis has been previously studied for impulse radio systems in AWGN [5] and for conventional DS-CDMA both in AWGN and fading channels [8], [9]. In this letter, we present a simple exact error analysis for low-duty DS-CDMA systems considering a general pulse shape and Nakagami fading channel. The BEP expression obtained from our method requires less computational complexity than that is required for the IGA method.

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II. SYSTEM MODEL

The pulsed DS-CDMA system uses binary phase shift keying (BPSK) modulation. A typical representation of the transmitted signal of an arbitrary user $k \in (1, 2, \ldots, K)$ has the form

$$s_k(t) = \sqrt{\frac{E_k}{N_s}} \sum_{n=-\infty}^{+\infty} b^{(k)}_{\lfloor n/N_s \rfloor} a^{(k)}_n \psi(t - nT_c) \cos(2\pi f t) \quad (1)$$

where t is time, $\psi(t)$ is the communication pulse with unit energy and f is the center frequency. The rest of the signal structure is described as follows:

- E_k is the bit energy of user k.
- N_s is the number of chips or pulses per information bit. T_c is the chip duration and hence, the bit duration is $T_b = N_s T_c$.
- The pulse $\psi(t)$ transmitted in each chip has a duration T_p ($\leq T_c$) and hence the chip-duty, $\delta = T_p/T_c$. Though it is possible that $0 < \delta \leq 1$, low-duty systems with $0 < \delta \leq 0.5$ is considered in this letter for simplicity. Note that the conventional DS-CDMA uses $\delta = 1$.
- $\{b^{(k)}_i\}$ is the i-th bit of user k which is a random variable (RV) uniform on $\{+1, -1\}$. Here, $i = \lfloor n/N_s \rfloor$ and $\lfloor . \rfloor$ represents floor function.
- $\{a^{(k)}_n\}$ is the random polarity code for user k which is also an RV uniform on $\{+1, -1\}$ and is periodic with period N_s.
- The processing gain of the system is $PG = N_s/\delta$.

Considering a time-invariant slow flat Nakagami faded channel, the received signal can be given by

$$r(t) = \sum_{k=1}^{K} \sqrt{\frac{E_k}{N_s}} \sum_{n=-\infty}^{+\infty} b^{(k)}_{\lfloor n/N_s \rfloor} a^{(k)}_n \beta_k \times \psi(t - nT_c - \tau_k) \cos(2\pi f t + \theta_k) + n(t) \quad (2)$$

where τ_k is the delay of the k-th user signal with respect to the signal of the desired user 1 ($\tau_1 = 0$) and $n(t)$ is AWGN noise with two-sided power spectral density of $N_o/2$. β_k is the fading amplitude having a Nakagami distribution given by $f_{\beta_k}(\nu) = \frac{2m_k^{m_k}\nu^{2m_k-1}}{\Omega_k^{m_k}\Gamma(m_k)} \exp\left(-\frac{m_k\nu^2}{\Omega_k}\right)$, $\nu \geq 0$, where m_k ($0.5 \leq m_k \leq \infty$) is the Nakagami fading parameter for user

k, $\Gamma(.)$ is gamma function and $\Omega_k = E[\beta_k^2]$, $E[.]$ representing mean value. θ_k models the phase of user k including the effect of delay and fading, and is uniformly distributed over $[0, 2\pi]$. Let $\tau_k = \Delta_k + \gamma_k T_c$, where γ_k is an RV uniform on $\{0, 1, \ldots, N_s - 1\}$ and Δ_k is uniform over $[0, T_c]$.

A coherent correlation receiver is considered. The received signal is correlated with a template of the form $s_{temp}^{(1)}(t) = \sum_{n=iN_s}^{(i+1)N_s-1} a_n^{(1)} \psi(t - nT_c)\cos(2\pi f t + \theta_1)$. Hence, the decision statistics, while detecting the the i-th bit of the desired user 1, can be given by

$$y^{(1)} = \sqrt{E_1 N_s} b_i^{(1)} \beta_1 + \textstyle\sum_{k=2}^{K} I_k + \eta. \tag{3}$$

The right hand side of (3) has three parts, of which the first part is the desired signal component, the second part is the multiple access interference (MAI) component and the third part η is the AWGN component having variance of $\sigma_\eta^2 = N_o N_s/2$. Here, I_k is the interference from user k, defined as $I_k = \sum_{n=iN_s}^{(i+1)N_s-1} \sqrt{E_k/N_s}\beta_k \cos\phi_k W_n^{(k)}$, where $\phi_k = \theta_k - \theta_1$. $W_n^{(k)}$ is the MAI component on the n-th chip of the desired user 1 from user k, which can be given for $0 < \delta \le 0.5$ by [3]

$$W_n^{(k)} = \begin{cases} L_n^{(k)} \hat{R}_\psi(\alpha_k), & 0 < \Delta_k \le T_p \\ 0, & T_p < \Delta_k \le T_c - T_p \\ M_n^{(k)} R_\psi(\alpha_k), & T_c - T_p < \Delta_k \le T_c \end{cases} \tag{4}$$

where $L_n^{(k)}$, $M_n^{(k)}$ are RVs uniform on $\{+1, -1\}$ and $\hat{R}_\psi(\alpha_k)$, $R_\psi(\alpha_k)$ are the partial autocorrelation functions of the baseband pulse $\psi(t)$, given by $\hat{R}_\psi(\alpha_k) = \int_{\alpha_k}^{T_p} \psi(t)\psi(t-\alpha_k)dt$ and $R_\psi(\alpha_k) = \hat{R}_\psi(T_p - \alpha_k)$. Here, α_k in relation to $\hat{R}_\psi(\alpha_k)$ and $R_\psi(\alpha_k)$ are given by $\alpha_k = \Delta_k, 0 < \Delta_k \le T_p$ and $\alpha_k = \Delta_k - (T_c - T_p), T_c - T_p < \Delta_k \le T_c$ respectively, while in both cases α_k is uniform in $[0, T_p]$ [3]. Note that $W_n^{(k)}$ assumes any of the three values of right side of (4) with probability δ, $1 - 2\delta$ and δ respectively from top to bottom.

III. The Exact Characteristic Function of MAI

The conditional characteristic function (CF) of MAI from an arbitrary user k can be given by

$$\Phi_{I_k|\alpha_k,\beta_k,\phi_k,L_n^{(k)},M_n^{(k)}}(\omega) = E\left[\exp(j\omega I_k) \mid \alpha_k, \beta_k, \phi_k, L_n^{(k)}, M_n^{(k)}\right] \tag{5}$$

where $j = \sqrt{-1}$. Note that (5) represents the CF conditioned on the independent RVs $\alpha_k, \beta_k, \phi_k$, $L_n^{(k)}$ and $M_n^{(k)}$. Putting the expression of I_k in (5) and taking help from (4), one can rewrite (5) for $0 < \delta \le 0.5$ as

$$\begin{aligned}\Phi_{I_k|\alpha_k,\beta_k,\phi_k,L_n^{(k)},M_n^{(k)}}(\omega) &= (1-2\delta) \\ &+\delta\left[\exp\left(j\omega\sqrt{\tfrac{E_k}{N_s}}\beta_k\cos\phi_k\hat{R}_\psi(\alpha_k)\textstyle\sum_{n=iN_s}^{(i+1)N_s-1} L_n^{(k)}\right)\right. \\ &\left.+\exp\left(j\omega\sqrt{\tfrac{E_k}{N_s}}\beta_k\cos\phi_k R_\psi(\alpha_k)\textstyle\sum_{n=iN_s}^{(i+1)N_s-1} M_n^{(k)}\right)\right].\end{aligned} \tag{6}$$

We now recall that both $L_n^{(k)}$ and $M_n^{(k)}$ are uniform on $\{+1, -1\}$. Utilizing this information and using the identity $\cos z = \{\exp(jz) + \exp(-jz)\}/2$, we obtain the following expression from (6):

$$\begin{aligned}\Phi_{I_k|\alpha_k,\beta_k,\phi_k}(\omega) &= (1-2\delta) \\ &+\delta\left[\cos^{N_s}\left(\omega\sqrt{\tfrac{E_k}{N_s}}\beta_k\cos\phi_k\hat{R}_\psi(\alpha_k)\right)\right. \\ &\left.+\cos^{N_s}\left(\omega\sqrt{\tfrac{E_k}{N_s}}\beta_k\cos\phi_k R_\psi(\alpha_k)\right)\right].\end{aligned} \tag{7}$$

Note that developing of (7) from (6) follows from the fact that random polarity sequences are considered in this paper. Hence, the chip polarities for a particular user are selected independently. Next, using the identities $\cos^{2n} z = 2^{-2n}\left\{\sum_{q=0}^{n-1} 2C_q^{2n}\cos[2(n-q)z] + C_n^{2n}\right\}$ and $\cos^{2n-1} z = 2^{-(2n-2)}\sum_{q=0}^{n-1}\left\{C_q^{2n-1}\cos[(2n-2q-1)z]\right\}$ from [10, appendix G], where $C_q^p = \frac{p!}{q!(p-q)!}$, cosines having power N_s in (7) can be rewritten in terms of cosines having unity power. After so doing, we take the following two actions in sequel. We first integrate the expressions obtained over the density of ϕ_k, which is uniform over $[0, 2\pi]$ and use the identity $J_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\cos(z\cos\phi)d\phi$. This provides $\Phi_{I_k|\alpha_k,\beta_k}(\omega)$ in terms of J_0, where J_0 is the zeroth order Bessel function of the first kind. Next, we integrate $\Phi_{I_k|\alpha_k,\beta_k}(\omega)$ over the Nakagami density of β_k to obtain $\Phi_{I_k|\alpha_k}(\omega)$. Again, by employing an identity given by

$$\int_0^\infty J_0(z\nu)f_{\beta_k}(\nu)d\nu = {}_1F_1\left(m_k; 1; -\frac{\Omega_k z^2}{4m_k}\right) \hat{=} \mathcal{F}_k(z^2) \tag{8}$$

from [11, (41), (42)], where $f_{\beta_k}(\nu)$ is the Nakagami density of β_k and ${}_1F_1(:;:;:)$ is the confluent hypergeometric function, $\Phi_{I_k|\alpha_k}(\omega)$ can now be given by

$$\begin{aligned}\Phi_{I_k|\alpha_k}(\omega) &= (1-2\delta) + \tfrac{\delta}{2^{N_s-1}} \\ &\times\left[\textstyle\sum_{q=0}^{\frac{N_s}{2}-1} C_q^{N_s}\left\{\mathcal{F}_k(\omega^2\hat{x}^2) + \mathcal{F}_k(\omega^2 x^2)\right\} + C_{N_s/2}^{N_s}\right]\end{aligned} \tag{9}$$

for N_s being even and by

$$\begin{aligned}\Phi_{I_k|\alpha_k}(\omega) &= (1-2\delta) \\ &+\tfrac{\delta}{2^{N_s-1}}\left[\textstyle\sum_{q=0}^{\frac{N_s-1}{2}} C_q^{N_s}\left\{\mathcal{F}_k(\omega^2\hat{x}^2) + \mathcal{F}_k(\omega^2 x^2)\right\}\right]\end{aligned} \tag{10}$$

for N_s being odd, where $\hat{x} = (N_s - 2q)\sqrt{E_k/N_s}\hat{R}_\psi(\alpha_k)$, $x = (N_s - 2q)\sqrt{E_k/N_s}R_\psi(\alpha_k)$ and $\mathcal{F}_k(.)$ is defined in (8). Finally, the unconditional exact CF is obtained by integrating (9) and (10) over the density of α_k. Since α_k is uniform over $[0, T_p]$, the unconditional CF of MAI from the k-th user is given by $\Phi_{I_k}(\omega) = \frac{1}{T_p}\int_0^{T_p}\Phi_{I_k|\alpha_k}(\omega)d\alpha_k$. Assuming that the interference from different users are independent, the exact CF of total MAI takes the form $\Phi_I(\omega) = \prod_{k=2}^{K}\Phi_{I_k}(\omega)$. Fig. 1 shows the exact CF of the MAI considering a Gaussian pulse shape given by $\psi(t + \frac{T_p}{2}) = e^{-\pi(\frac{t}{t_m})^2}/\sqrt{\mathcal{E}}$, $t_m = 0.4T_p$ for $K = 2$, implying one interfering user. Here, $\mathcal{E}$ is the energy of $e^{-\pi(\frac{t}{t_m})^2}$. The other parameters are $N_s = 7$ and $\delta = 0.1, 0.2, \ldots, 0.5$. Same Nakagami parameter ($m = m_1 = m_2 = 1, 5$) and same bit energy ($E_b = \Omega_1 E_1 = \Omega_2 E_2$) are considered for the two users, and E_b/N_o is set to 20 dB. Also, $\sigma_\eta^2 = N_o N_s/2 = 1$ is assumed without loss of generality, implying $E_b/N_o = E_b N_s/2$ [11]. As seen, the shape of the CF varies considerably with chip-duty.

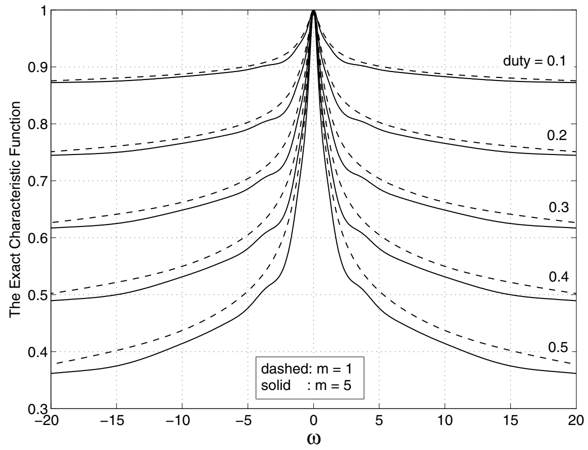

Fig. 1. The exact characteristic function of the pulsed DS-CDMA system with $N_s = 7$ and chip-duty $\delta = 0.1, 0.2, \ldots, 0.5$ for $K = 2$.

IV. Error Probabilities

1) The Exact: The unconditional exact BEP in flat Nakagami fading channel can now be given by [11]

$$P_e = \frac{1}{2} - \frac{1}{\pi}\frac{\Gamma(m_1+\frac{1}{2})}{\Gamma(m_1)}\sqrt{\frac{E_1 N_s \Omega_1}{m_1}}\int_0^\infty \Phi_I(\omega)\Phi_\eta(\omega) \times {}_1F_1\left(m_1+\frac{1}{2};\frac{3}{2};-\frac{E_1 N_s \Omega_1 \omega^2}{4m_1}\right)d\omega \quad (11)$$

where $\Phi_\eta(\omega) = \exp(-\sigma_\eta^2\omega^2/2)$ is the CF of AWGN. Note that the exact BEP of (11) is governed by the specific pulse shape and chip-duty only through $\Phi_I(\omega)$.

2) Based on the SGA: Under the SGA of MAI, the BEP conditioned on the fading amplitude of the desired user 1 can be given by $P_{e|\beta_1}^{SGA} = 0.5 \text{ erfc}\left(\beta_1\sqrt{0.5E_1N_s/\left(\sigma_\eta^2+\mu_I\right)}\right)$. Here, μ_I is the mean of total MAI variance given by $\mu_I = \sum_{k=2}^{K}\Omega_k E_k \rho_\psi \delta$, where $\rho_\psi = \frac{1}{T_p}\int_0^{T_p}\hat{R}_\psi^2(\alpha_k)d\alpha_k = \frac{1}{T_p}\int_0^{T_p}R_\psi^2(\alpha_k)d\alpha_k$. Now integrating $P_{e|\beta_1}^{SGA}$ over the density of β_1 and employing the identity from [12, p. 20, eqn(45)], the unconditional BEP under SGA can be given by

$$P_e^{SGA} = \frac{\Lambda^{m_1}}{2\sqrt{\pi}}\frac{\Gamma(m_1+\frac{1}{2})}{m_1\Gamma(m_1)}\ {}_2F_1\left(m_1, \frac{1}{2}; m_1+1; \Lambda\right) \quad (12)$$

where $\Lambda = \left(1+\frac{E_1N_s\Omega_1}{2m_1(\sigma_\eta^2+\mu_I)}\right)^{-1}$ and ${}_2F_1$ is Gauss hypergeometric function.

V. Numerical Examples and Conclusion

In this section, specific numerical examples are presented on error performance of the system. Gaussian pulse shape introduced in a previous section is used. Theoretical results are confirmed by Monte Carlo simulations. Fig. 2 shows the BEP versus total users K with $N_s = 7$ and $\delta = 0.05$, 0.5. Same Nakagami parameter and bit energy are considered for all users ($m = m_k, E_b = \Omega_k E_k, k = 1, 2, \ldots, K$) by setting $m = 1$, 5 and $E_b/N_o = 20$ dB. As before, without loss of generality, $\sigma_\eta^2 = 1$ is assumed for evaluating (11) [11]. An absolute match between the BEP from the exact method and simulations can be seen. The SGA, on the other hand, though provides reasonably accurate results for $m = 1$ (Rayleigh fading), becomes extremely optimistic for the relatively less-faded channel with $m = 5$, especially for systems with low duty. Lowering chip-duty improves performance, since processing gain is obtained. However, relatively less performance improvement is obtained than that is expected by the SGA in lightly faded channel. The exact method presented in this letter can be a powerful tool to investigate many such system trade-offs guaranteeing both credibility and simplicity.

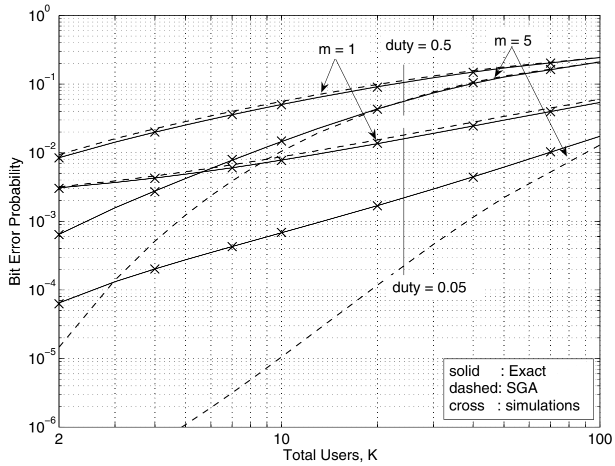

Fig. 2. BEP versus total users K for $N_s = 7$, $\delta = 0.05$, 0.5 with $m = 1$, 5 and $E_b/N_o = 20$ dB.

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