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Existence Theorems for Some Initial Boundary Value Problems in Isentropic Gas Dynamics

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Chapter 1

Introduction

The motion of ideal gas is governed by a system of nonlinear partial differential equations, which is derived from the conservation laws of mass, momentum and energy, and which is call a system of conservation laws.

In general, it happens that the solutions of some nonlinear differential equations blow up or the regularity of the solutions is lost in finite time. It is well known that the system of conservation laws has discontinuous solutions even for smooth initial data. Such a discontinuous solution corresponds to a shock wave in physical phenomena. We introduce the concept of weak solutions in order to regard the discontinuous functions as solutions of the system of the differential equations. But, even if we consider the solutions of the system in the class of weak solutions, we still have many basic open problems of whether or not there exists at least one solution, and whether or not the solution is unique. Especially, although few results for 2-dimensional and 3-dimensional motion of the gas are known, there are several results for 1-dimensional motion which concern with the existence of weak solutions.

In particular, P.D.Lax ([13]) studied the typical initial value problem, called the Riemann problem, for general nonlinear systems of hyperbolic conservation laws, O.A.Oleĭnik ([22]) gave the existence of the weak solutions of a scalar conservation law for the general data by showing the convergence of the Lax-Friedrichs difference approximation, and J.Glimm ([12]) established the existence of the weak solutions of genuinely nonlinear system of conservation laws for small data by using his original difference scheme. Subsequently, many authors studied the system for the ideal gas motion by using the Glimm's method and results, ([15],[16],[19],[20],[21]).

In those researches, it was proved that there exist weak solutions of initial or initial boundary value problems of isothermal gas motion for large data and of isentropic gas motion for data with sufficiently small total variations. But, the existence problem for large data remained unsolved. In this case, it is difficult to analyze vacuum state which occurs by the gas moving rapidly.

Moreover, it was not known well that the classical approximations, for example, the Lax-

Friedrichs scheme, or the approximation by artificial viscosity method, converge to a certain solution for any data.

In 1983, R.J.DiPerna ([9]) showed the convergence of these approximations for initial value problems by using the weak compactness theory, called the compensated compactness method, which were studied in the latter part of '70th. It followed from his result that the weak solutions including the vacuum states exist for large data. These result were extended by Ding, Chen, Luo ([1],[5]).

In this paper, we show the existence of weak solutions for several initial boundary value problems by DiPerna's method. In Glimm's method, the motion of the gas is considered on coordinate systems moving with the gas, called Lagrange coordinate system, but, in DiPerna's method, we study the motion of gas on the fixed coordinate system, called Euler coordinate system, and therefore the difficulty arises on the analysis at the boundary. In this paper, we consider the following four initial boundary value problems: one-piston problem such that the isentropic gas fills a narrow and long cylinder and the moving piston is given to the left of the gas, two-piston problem such that the gas fills in the cylinder and two moving pistons are given to the both of right and left of the gas, free piston problem such that a piston in a cylinder can move without friction and the gas fills to the left and right of the piston, and spherically symmetric motion problem such that the gas is outside a ball (i.e. a star) and the motion of the gas is symmetric spherically. The research on the last problem is a joint work with Professor Tetu Makino at Osaka Sangyo University.

There exist few researches for 3-dimensional motion of gas, but in a spherically symmetric case, we can treat the motion by the methods on 1-dimensional motion. However, our result for this spherically symmetric problem does not satisfy us because the solutions obtained are not global. It is still an open problem whether or not the global solution exists for any large data.

In general, it is hard in practical computation to calculate the approximate solutions near the boundary for initial boundary value problems. We observe simple approximations of the initial boundary value problems for this system, by which we can carry out the simulation easily, and we show the convergence of the approximations.

The structure of this paper is the following.

In Chapter 2, we explain the physical means of the system of gas dynamics, in Chapter 3, we mention the initial boundary value problems treating in this paper and state our results for the problems in Chapter 4. Notations are in Chapter 5. The notion of weak solutions is defined in Chapter 6. A weak solution satisfies the considering equations in the sense of distribution and satisfy the initial and boundary conditions as a weak sense. However, the definition of weak solutions is given by integration by parts in general. We investigate the relation between the above two statements. In Chapter 7, we introduce the approximate solutions which are using in this paper. We use mainly the Lax-Friedrichs difference scheme approximation, but this approximation values discretely. Therefore, we must explain how this approximation is extended to one which values continuously. This extension is done by using the solutions of the Riemann problem, a basic and important initial value problem for the system, and by using the solutions of the some initial boundary value problems corresponding to the Riemann problem. We explain these problems and the solutions in Chapter 8. As mentioned before, R.J.DiPerna established a theorem of existence of the convergent subsequence of approximate solutions by the compensated compactness theory. In order to apply the theorem to our initial boundary value problems, it is necessary to show the uniform boundedness and weak compactness of the approximate solutions. The boundedness is led by the existence of invariant regions for the system. The concept of the invariant region of the solution is introduced in Chapter 9, and we construct the approximation to prove the boundedness in Chapter 10. The weak compactness mentioned above is not the compactness of the certain approximate solutions but the compactness of special functionals of the approximation. The functionals are called generalized entropies. We define the notion of entropies in Chapter 11. The entropies are not only used in the compensated compactness theory, but used to choose the admissible weak solutions. This admissibility is called an entropy condition. But, the entropies do not always exist for any system of conservation laws. For example, the system consisting of 3-conservation laws, mass conservation, momentum conservation, energy conservation, has only one kind of the entropy almost equivalent the classical physical entropy. We prove this fact in this Chapter. There are known many kinds of the approximation for the system. The convergence of another kind of approximation and the convergence of more simple approximation for computer calculation mentioned before are proved in Chapter 12. We show the annoying calculate part of the proof of the existence theorems in Chapter 13.

Chapter 2

Systems of conservation laws

In physics, a gas is considered as continuum, called fluid. The basic system of equations of motion of fluid dynamics is led by the conservation laws of physical quantities with the fluid. We will derive the system.

First, we consider the conservation of mass. Let the gas be in a long and narrow cylinder, and let the gas move only left or right. We suppose that the state of the gas is constant on any cross section of the cylinder, and the gas do not have viscosity and heat conduction.

Let x-axis be the direction along the cylinder, let S be the area of cross section of the cylinder, and let m be the molecular weight with the gas. And we denote the mole number of the gas in a interval I on x-axis by $n(I)$ or $n(t; I)$, where t means the time.

The density of the gas $\rho = \rho(t, x)$ is defined by

$$\rho(t,x) = \lim_{a\uparrow x, b\downarrow x} \frac{mn(t; [a,b])}{(b-a)S} \tag{2.1}$$

for the gas is constant on any cross section. By this, the mass of the gas for a interval having minute length Δx is

$$m\, n(t; [x, x+\Delta x]) = \Delta x S \rho(t,x). \tag{2.2}$$

Thus, the total mass of the gas in a interval (a, b) is

$$S \int_a^b \rho(t,x)dx. \tag{2.3}$$

Let $u(t,x)$ be the velocity of the gas as the fluid at (t,x), then the gas which is on the position x at a time t will be on the position

$$x + u(t,x)\Delta t \tag{2.4}$$

after the minute time Δt. Thus, the increment of the mass flowed into the interval (a, b) is

$$-\{u(t,b)\Delta t\}S\rho(t,b) + \{u(t,a)\Delta t\}S\rho(t,a). \tag{2.5}$$

Hence, we have

$$\begin{aligned} & \frac{d}{dt}\left(S\int_a^b \rho(t,x)dx\right) \\ = & -\rho(t,b)u(t,b)S + \rho(t,a)u(t,a)S \\ = & -S\int_a^b \{\rho(t,x)u(t,x)\}_x dx \end{aligned}$$

where $()_x = \frac{\partial}{\partial x}()$. Thus, we obtain that

$$\int_a^b \{\rho_t + (\rho u)_x\}dx = 0 \tag{2.6}$$

for any interval (a,b). Therefore, it follows that

$$\rho_t + (\rho u)_x = 0 \tag{2.7}$$

for any point (t,x). This equation means the conservation of mass and is called the equation of continuity.

Next, we consider the conservation of momentum. Since the momentum is equal to the mass times the speed, the momentum of a interval $[x, x+\Delta x]$ with minute length Δx becomes

$$mn(t;[x,x+\Delta x])u(t,x) = \Delta x S\rho(t,x)u(t,x). \tag{2.8}$$

Thus, the total momentum of a interval (a,b) is given by

$$S\int_a^b \rho u\, dx. \tag{2.9}$$

Now, the increment of the momentum flowed into the interval while the minute time Δt is

$$-\{u(t,b)\Delta t\}S\rho(t,b)u(t,b) + \{u(t,a)\Delta t\}S\rho(t,a)u(t,a). \tag{2.10}$$

Since the second law of the Newtonian mechanics, the momentum change is equal to the impulse

$$-SP(t,b)\Delta t + SP(t,a)\Delta t, \tag{2.11}$$

where $P = P(t,x)$ is the pressure per unit area acting the cross section at position x. Thus, it follows

$$\frac{d}{dt}\left(S\int_a^b \rho u\, dx\right) = S[-\rho u^2 - P]_{x=a}^{x=b} \tag{2.12}$$

From this, and by the similar method for mass conservation, we have the equation of conservation of momentum

$$(\rho u)_t + (\rho u^2 + P)_x = 0. \tag{2.13}$$

Lastly, we consider the conservation of energy. Microscopically, pressure and temperature have close relation to molecular mechanics, and they of the ideal gas with the volume V, with the mole number n satisfy the relation

$$PV = nRT \tag{2.14}$$

by the Boyle-Charle's law, where R is a constant which is independent on the kind of the gas. This relation is called the equation of the state. For a minute interval $[x, x + \Delta x]$, we have

$$V = S\Delta x, \quad n = n(t; [x, x + \Delta x]) = \frac{S\Delta x \rho(t, x)}{m}, \tag{2.15}$$

thus the equation of the state becomes

$$P(t, x)S\Delta x = \frac{S\Delta x \rho(t, x)}{m} RT(t, x), \tag{2.16}$$

and hence, it follows

$$P = \frac{R}{m}\rho T. \tag{2.17}$$

There are two kinds of energy of a gas. One is the energy by the molecular mechanics, that is, inertial energy, and the other is the mechanical energy by the fluid motion dynamics. We denote the inertial energy of the gas in a interval I by $E(I)$ or $E(t; I)$. By the theory of molecular mechanics, the inertial energy for the gas with the mole number n and with the temperature T becomes

$$E = \frac{f}{2} nRT, \tag{2.18}$$

where f is the degree of freedom of the gas molecule, for example, $f = 3$ if the molecule consists of an atom and $f = 5$ if the molecule consists of two atoms.

We define the inertial energy density per unit mass $e(t, x)$ by

$$e(t, x) = \lim_{a\uparrow x, b\downarrow x} \frac{E(t; [a, b])}{mn(t; [a, b])} = \frac{1}{\rho(t, x)} \lim_{a\uparrow x, b\downarrow x} \frac{E(t; [a, b])}{(b - a)S}, \tag{2.19}$$

then,

$$E(t; [x, x + \Delta x]) = \rho(t, x)S\Delta x e(t, x), \tag{2.20}$$

and hence,

$$E(t; [a, b]) = S \int_a^b \rho e \, dx. \tag{2.21}$$

Suppose that C_V and C_P are the molar specific heat at constant volume and constant pressure, respectively. Then,

$$\begin{aligned} C_V &= \frac{\Delta E}{n\Delta T} = \frac{f}{2}R, \\ C_P &= \frac{\Delta E + P\Delta V}{n\Delta T} = \frac{\frac{f}{2}nR\Delta T + nR\Delta T}{n\Delta T} = \frac{f + 2}{2}R. \end{aligned}$$

From this, the energy density becomes

$$e(t,x) = \lim_{a\uparrow x, b\downarrow x} \frac{E(t;[a,b])}{mn(t;[a,b])} = \frac{f}{2}\frac{R}{m}T(t,x) = \frac{C_V}{m}T(t,x). \tag{2.22}$$

The mechanical energy for the minute interval $[x, x+\Delta x]$ is

$$\frac{1}{2}mn(t;[x,x+\Delta x])u(t,x)^2 = \frac{1}{2}S\Delta x\rho(t,x)u(t,x)^2. \tag{2.23}$$

Thus, the total energy for an interval (a,b) becomes

$$E(t;[a,b]) + S\int_a^b \frac{1}{2}\rho u^2\,dx = S\int_a^b \left(\rho e + \frac{1}{2}\rho u^2\right)dx. \tag{2.24}$$

The increment of the energy flowed into the interval (a,b) is

$$\left[S\left(\rho e + \frac{1}{2}\rho u^2\right)u(t,x)\Delta t\right]_{x=b}^{x=a}. \tag{2.25}$$

Suppose that no heat flows through the cylinder, then the difference of the change of the total energy and the increment of the energy flowed into the interval is equal to the works done by the pressure at the boundary from the outside the interval. Since the works is equal to

$$-SP(t,b)u(t,b)\Delta t + SP(t,a)u(t,a)\Delta t, \tag{2.26}$$

we have

$$\frac{d}{dt}S\int_a^b \left(\rho e + \frac{1}{2}\rho u^2\right)dx = \left[S\left(\rho e + \frac{1}{2}\rho u^2 + P\right)u\right]_{x=b}^{x=a}, \tag{2.27}$$

and hence, we obtain the equation of energy conservation

$$\left(\rho e + \frac{1}{2}\rho u^2\right)_t + \left\{\left(\rho e + \frac{1}{2}\rho u^2 + P\right)u\right\}_x = 0. \tag{2.28}$$

For isentropic case, the pressure is depend only on the density. We consider this case.

In this case, microscopically, the increment of inertial energy becomes the works to outside, for the energy conservation, and for no heat flow. That is,

$$\Delta E + P\Delta V = 0. \tag{2.29}$$

Since

$$E = nC_VT, \quad V = \frac{mn}{\rho} \tag{2.30}$$

for the gas with the n mole number, it follows

$$nC_V\Delta T = -Pmn\Delta\frac{1}{\rho} = -\frac{R}{m}\rho Tmn\Delta\frac{1}{\rho}. \tag{2.31}$$

From this,

$$C_V\frac{\Delta T}{T} = -R\rho\Delta\frac{1}{\rho}, \tag{2.32}$$

and hence,

$$C_V \log T + R \log \frac{1}{\rho} \tag{2.33}$$

is a constant. From the equation of state, the pressure becomes

$$P = \frac{R}{m}\rho T = C_1 \rho^{1+R/C_V} = C_1 \rho^{\gamma}, \tag{2.34}$$

where C_1 is a constant, γ is the ratio of adiabatic heat defined by

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} = 1 + \frac{R}{C_V}. \tag{2.35}$$

By using γ, the inertial energy density becomes

$$e = \frac{C_V}{m}T = P\frac{C_V}{\rho R} = \frac{1}{\gamma - 1}\frac{P}{\rho}. \tag{2.36}$$

That is, the conservation of energy is described with ρ and u, and thus, this equation must be derived by the conservation of mass and of momentum, and we may consider the system of conservation of mass and of momentum with unknown functions ρ and u. As the pair

$$\left(\frac{1}{2}\rho u^2 + \rho e, \frac{1}{2}\rho u^3 + \rho e u + P u\right), \tag{2.37}$$

the function pair $(\eta(\rho, u), q(\rho, u))$ is called the *generalized entropy pair* if the pair satisfy the equation of conservation form

$$\eta(\rho, u)_t + q(\rho, u)_x = 0 \tag{2.38}$$

derived from the system of conservation laws. We explain the entropy pair in Chapter 11.

Chapter 3

Problems

Consider the system of equation of isentropic ideal gas motion

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P(\rho))_x = 0, \end{cases} \tag{3.1}$$

where $\rho\,(\geq 0)$ is the density, u is the velocity, m is the momentum defined by $m = \rho u$, P is the pressure of the gas. In the isentropic gas case, the pressure P has the following expression

$$P = a\rho^{\gamma}, \tag{3.2}$$

where a and γ are constants which satisfy

$$a > 0, \quad 1 < \gamma \leq 5/3. \tag{3.3}$$

In this paper, we treat the following four initial boundary value problems (I)-(IV) for the system (3.1).

Problem (I) : 1-piston problem.

We consider the motion of the gas filled to the right of a moving piston $x = x_1(t)$.

$$\begin{aligned} &\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P(\rho))_x = 0 \end{cases} \quad \text{in } D_1 = \{(t,x); t > 0,\, x > x_1(t)\}, \\ &(\rho(0,x), m(0,x)) = (\rho_0(x), \rho_0(x)u_0(x)) \qquad \text{for } x > x_1(0), \\ &m(t,x) - x_1'(t)\rho(t,x)]_{x=x_1(t)} = 0 \qquad \text{for } t > 0. \end{aligned} \tag{3.4}$$

Problem (II) : 2-piston problem.

We consider the motion of the gas filled between the moving pistons $x = x_1(t)$ on the left and the one $x = x_2(t)$ on the right.

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P(\rho))_x = 0 \end{cases}$$
$$\text{in } D_2 = \{(t,x); t > 0,\ x_1(t) < x < x_2(t)\},$$
$$(\rho(0,x), m(0,x)) = (\rho_0(x), \rho_0(x)u_0(x)) \qquad \text{for } x_1(0) < x < x_2(0),$$
$$m(t,x) - x_1'(t)\rho(t,x)]_{x=x_1(t)} = 0 \qquad \text{for } t > 0,$$
$$m(t,x) - x_2'(t)\rho(t,x)]_{x=x_2(t)} = 0 \qquad \text{for } t > 0 \tag{3.5}$$

Problem (III) : free piston problem.

We consider the motion of the gas 1 filled to the left of a piston, the motion of the gas 2 filled to the right of the piston, and the motion oft the piston which can move right and left without friction and without outre force.

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P_1)_x = 0 \end{cases} \quad \text{in } D_3 = \{(t,x); t > 0,\ x < x(t)\},$$
$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P_2)_x = 0 \end{cases} \quad \text{in } D_4 = \{(t,x); t > 0,\ x > x(t)\},$$
$$x''(t) = k\{P_1(t, x(t) - 0) - P_2(t, x(t) + 0)\} \qquad \text{for } t > 0,$$
$$(\rho(0,x), m(0,x)) = (\rho_0(x), \rho_0(x)u_0(x)) \qquad \text{for } x \in R,$$
$$x(0) = x_0, \quad x'(0) = x_0'$$
$$m(t,x) - x'(t)\rho(t,x)]_{x=x(t)\pm 0} = 0 \qquad \text{for } t > 0, \tag{3.6}$$

where P_j is the pressure of the gas j, $j = 1,2$ and let

$$P_j = a_j\rho^{\gamma_j} \qquad (a_j > 0,\ \gamma_j \in (1, 5/3],\ j = 1,2), \tag{3.7}$$

$k = S/M$, S is the sectional area of the cylinder, and M is the mass of the free piston. We note in this problem that $\rho(t,x)$ and $m(t,x)$ are not only unknown functions but also the piston path $x = x(t)$ is. That is, this problem is a free boundary value problem. And we admit that a kind of the gas 1 differ from the one of the gas2, that is, it may occur that $a_1 \neq a_2$ or $\gamma_1 \neq \gamma_2$.

Problem (IV) : spherically symmetric gas motion of 3-dimension.

We consider the gas outside of a star with the radius 1. Suppose that a motion of the gas is symmetric spherically and the gravitation with the star has an effect to the gas as a outer force. We lead the system.

$$\rho_t + \operatorname{div}(\rho\boldsymbol{v}) = 0 \ \text{ (mass conservation)},$$
$$(\rho v_j)_t + \operatorname{div}(\rho v_j\boldsymbol{v}) + P_{x_j} = \rho K_j \ \ (j = 1,2,3, \text{ momentum conservation}) \tag{3.8}$$

where $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity vector of the gas, $\boldsymbol{x} = (x_1, x_2, x_3)$ is the position vector, $\boldsymbol{K} = (K_1, K_2, K_3)$ is the outer force vector per unit mass. In this Problem (IV), the outer force

$$\rho \boldsymbol{K} = \frac{\boldsymbol{x}}{r}\frac{M}{r^2}\rho \tag{3.9}$$

by the Newton's law of universal gravitation. Since the motion of the gas is symmetric spherically, we suppose that ρ and $\boldsymbol{v}$ satisfy

$$\rho = \rho(t,r), \qquad \boldsymbol{v} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}u(t,r), \tag{3.10}$$

where $r = |x|$. Then,

$$\begin{aligned}
&\operatorname{div}(\rho\boldsymbol{v}) = \operatorname{div}\left(\frac{\boldsymbol{x}}{r}\rho u\right) = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\rho u\right) = (\rho u)_r + \frac{2}{r}\rho u,\\
&(\rho v_j)_t = \frac{x_j}{r}(\rho u)_t,\\
&\operatorname{div}(\rho v_j \boldsymbol{v}) = div\left(\frac{x_j}{r^2}\boldsymbol{x}\rho u^2\right) = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{x_j}{r^2}x_i\rho u^2\right) = \frac{x_j}{r}(\rho u^2)_r + \frac{2}{r}\frac{x_j}{r}\rho u^2,\\
&P_{x_j} = P_r\frac{x_j}{r}.
\end{aligned}$$

Therefore, the system becomes

$$\begin{cases}
\rho_t + (\rho u)_r + \dfrac{2}{r}\rho u = 0,\\
(\rho u)_t + (\rho u^2 + P)_r + \dfrac{2}{r}\rho u^2 = \rho\dfrac{M}{r^2}.
\end{cases} \tag{3.11}$$

This system is the form of the 1-dimensional conservation laws with the term of outer forth. Thus, we consider the following problem.

$$\begin{aligned}
&\begin{cases}
\rho_t + m_r = -\dfrac{2}{r}m,\\
m_t + \left(\dfrac{m^2}{\rho} + P(\rho)\right)_r = -\dfrac{2}{r}\dfrac{m^2}{\rho} - \dfrac{M}{r^2}\rho
\end{cases} \qquad \text{for } t > 0, r > 1,\\
&(\rho(0,r), m(0,r)) = (\rho_0(r), \rho_0(r)u_0(r)) \qquad \text{for } r > 1,\\
&m(t,1) = 0 \qquad \text{for } t > 0.
\end{aligned} \tag{3.12}$$

We suppose the following to the initial and boundary data for above problems.

- $\rho_0(x) \geq 0$.
- Initial velocity and density are bounded. That is, there is a positive constant C such that

$$0 \leq \rho_0(x) \leq C, \quad |u_0(x)| \leq C. \tag{3.13}$$

- The piston's velocities for Problem (I), Problem (II) are bounded. That is, functions $x_1(t)$ and $x_2(t)$ are Lipschitz continuous and there is a constant C such that

$$|x_1'(t)|, |x_2'(t)| \leq C. \tag{3.14}$$

- For Problem (II), the distance between the two pistons is positive for any time t. That is,

$$x_2(t) - x_1(t) > 0 \qquad (t > 0). \tag{3.15}$$

Chapter 4

Existence theorems

We observe the existence theorems for Problems (I)-(IV). We suppose that the initial and boundary data satisfy the assumption in Chapter 3.

Denote $U(t,x) = {}^t(\rho(t,x), m(t,x))$.

THEOREM 1 *The 1-moving piston problem* (I) *has a global weak solution $U(t,x)$ and the solution satisfies*

$$0 \le \rho(t,x) \le C, \quad |m(t,x)| \le C\rho(t,x) \qquad \textit{for } x > x_1(t),\ t > 0. \tag{4.1}$$

THEOREM 2 *The 2-piston problem* (II) *has a global weak solution $U(t,x)$ and the solution satisfies*

$$0 \le \rho(t,x) \le C_1(T), \quad |m(t,x)| \le C_2(T)\rho(t,x) \qquad \textit{for } x_1(t) < x < x_2(t),\ 0 < t < T, \tag{4.2}$$

where T is any positive number, $C_1(T)$ and $C_2(T)$ are the constant given in Propositions 13, 14.

THEOREM 3 *The free piston problem* (III) *has a global weak solution $U(t,x)$ and a free boundary $x(t)$ and the solution and the boundary satisfy*

$$\begin{aligned} &0 \le \rho(t,x) \le C, \quad |m(t,x)| \le C\rho(t,x) \qquad \textit{for } x \in R,\ t > 0, \\ &x(t) \in C^1(0,\infty), \quad x'(t) \textit{ is Lipschitz continuous}, \quad |x''(t)| \le C. \end{aligned} \tag{4.3}$$

THEOREM 4 (T.Makino, S.Takeno) *There exists a positive number T_0^∞ which depends on the initial data, the mass of the star M, γ and a such that for any number T satisfying $0 < T < T_0^\infty$, the spherically symmetric gas problem* (IV) *has a solution $U(t,r)$ for $r > 1$, $0 < t < T$ and the solution satisfies*

$$0 \le \rho(t,r) \le C(T), \quad |m(t,r)| \le C(T)\rho(t,r) \qquad \textit{for } r > 1,\ 0 < t < T. \tag{4.4}$$

Here, for the definition of weak solution, see Chapter 6.

Chapter 5

Notations

In this paper, we express the system of isentropic gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \end{cases} \tag{5.1}$$

to another form

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (m^2/\rho + P(\rho))_x = 0 \end{cases} \tag{5.2}$$

by using the momentum $m = \rho u$, or to the vector form

$$U_t + F(U)_x = 0, \tag{5.3}$$

and the system for Problem (IV) is rewritten to

$$U_t + F(U)_r = H(r, U), \tag{5.4}$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + P(\rho) \end{pmatrix} = \begin{pmatrix} m \\ m^2/\rho + P(\rho) \end{pmatrix},$$

$$H(r, U) = \begin{pmatrix} -\dfrac{2}{r}\rho u \\ -\dfrac{2}{r}\rho u^2 - \dfrac{M}{r^2}\rho \end{pmatrix} = \begin{pmatrix} -\dfrac{2}{r}m \\ -\dfrac{2}{r}\dfrac{m^2}{\rho} - \dfrac{M}{r^2}\rho \end{pmatrix}.$$

The system (5.3) is a quasi-linear system of partial differential equations.

$$U_t + \nabla F(U) U_x = 0, \tag{5.5}$$

since

$$F(U)_x = \begin{pmatrix} F_1(\rho, m) \\ F_2(\rho, m) \end{pmatrix}_x = \begin{pmatrix} F_1(\rho, m)_\rho & F_1(\rho, m)_m \\ F_2(\rho, m)_\rho & F_2(\rho, m)_m \end{pmatrix} \begin{pmatrix} \rho_x \\ m_x \end{pmatrix} = \nabla F(U) U_x,$$

where,

$$\nabla = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial m} \right). \tag{5.6}$$

$\nabla F(U)$ is the 2×2 matrix

$$\nabla F(U) = \begin{pmatrix} 0 & 1 \\ -(m/\rho)^2 + P'(\rho) & 2m/\rho \end{pmatrix} \tag{5.7}$$

and the matrix has two real eigenvalues

$$\begin{aligned} \lambda_1 = \lambda_1(U) = u - \sqrt{P'} = u - \sqrt{a\gamma}\rho^\theta, \\ \lambda_2 = \lambda_2(U) = u + \sqrt{P'} = u + \sqrt{a\gamma}\rho^\theta, \end{aligned} \tag{5.8}$$

where $\theta = (\gamma - 1)/2$. Thus, the system (5.5) is a quasi-linear hyperbolic system. The eigenvectors r_j corresponding with λ_j are

$$r_j = r_j(U) = \begin{pmatrix} 1 \\ \lambda_j \end{pmatrix} \qquad (j = 1, 2). \tag{5.9}$$

A function f(U) is called the *j-Riemann invariant* if the function satisfies the linear partial differential equation of first order

$$\nabla f(U) \cdot r_j(U) = 0. \tag{5.10}$$

For more general quasi-linear $n \times n$ hyperbolic system, there are $(n-1)$ independent solutions for the equation (5.10) from the general theory of partial differential equation. Thus, in this case, there exists only one independent j-Riemann invariant. We obtain 1-Riemann invariant w

$$w = u + \int^\rho \frac{\sqrt{P'(y)}}{y} dy = u + \frac{\sqrt{a\gamma}}{\theta}\rho^\theta, \tag{5.11}$$

and 2-Riemann invariant z

$$z = u - \int^\rho \frac{\sqrt{P'(y)}}{y} dy = u - \frac{\sqrt{a\gamma}}{\theta}\rho^\theta \tag{5.12}$$

easily. We may consider that the Riemann invariants w, z cause the change of variable on the phase space from (ρ, m) or from (ρ, u) to (w, z). From this fact, we can rewrite the eigenvalues

$$\begin{cases} \lambda_1 = \dfrac{1-\theta}{2} w + \dfrac{1+\theta}{2} z, \\ \lambda_2 = \dfrac{1+\theta}{2} w + \dfrac{1-\theta}{2} z, \end{cases} \tag{5.13}$$

and the system (5.5) becomes the diagonal system

$$\begin{cases} w_t + \lambda_2 w_x = 0, \\ z_t + \lambda_1 z_x = 0. \end{cases} \tag{5.14}$$

But, the last change of unknown functions only admit for the smooth solution. From

$$\rho = \left(\frac{\theta}{\sqrt{a\gamma}} \frac{w - z}{2} \right)^{1/\theta}, \quad u = \frac{w + z}{2}, \tag{5.15}$$

the sets $\{\rho = \text{constant}\}$ and $\{u = \text{constant}\}$ are the lines in (w, z) space. Here, the domain drawing by the slant lines means $\rho < 0$. The sets $\{\lambda_1 = \text{constant}\}$ and $\{\lambda_2 = \text{constant}\}$ also are the lines in (w, z) space.

Chapter 6

Weak solutions

In the system of gas dynamics, even if the initial data is smooth, the discontinuity of solutions arises in finite time by the nonlinearity. The singularity corresponds to the shock wave in physical phenomena. But, we must consider that discontinuous function as a solution because the physical phenomena continues after the time. By this reason, we introduce the definition of weak solutions which is defined by the weak derivative in the distribution theory. Now, we explain the weak derivative in the distribution theory simply.

Suppose $C_0^1(R)$ be the function space made of continuously differentiable functions such that each of the functions vanishes outside some bounded domain. Let $\phi(x) \in C_0^1(R)$ and L be a value such that

$$\phi(x) = 0 \qquad \text{for } x \ni (-L, L). \tag{6.1}$$

Then, for any C^1 function $v(x)$, we have

$$\begin{aligned}\int_R v'(x)\phi(x)dx &= \int_{-L}^{L} v'(x)\phi(x)dx = [v(x)\phi(x)]_{-L}^{L} - \int_{-L}^{L} v(x)\phi'(x)dx \\ &= -\int_{-L}^{L} v(x)\phi'(x)dx = -\int_R v(x)\phi'(x)dx\end{aligned}$$

by the integration by parts. The last term does not contains the derivative of $u(x)$. The weak derivative is defined by such a relation in the distribution theory. For the details, see the standard text for the distribution theory, for example, [28].

Suppose Ω be a subset in R^N. For $\phi(x) \in C(\Omega)$, we define a *support* of the function ϕ by the closure of the set

$$\{x \in \Omega; \phi(x) \neq 0\} \tag{6.2}$$

and define the function space $C_0^k(\Omega)$ by

$$C_0^k(\Omega) = \{\phi \in C^k(\Omega); \operatorname{supp}\phi \text{ is compact in } \Omega\}, \tag{6.3}$$

for example,

$$C_0^1([0,T)\times R) = \{\phi(t,x) \in C^1;\ \text{There are } M, L \text{ such that } M>0,\ T>L>0$$
$$\phi(t,x) = 0 \quad \text{for } (t,x) \notin [0,L]\times[-M,M])\}.$$

A measurable function $U(t,x)$ is a *weak solution* of the Cauchy problem

$$\begin{cases} U_t + F(U)_x = 0 & \text{for } 0<t<T,\ x\in R, \\ U(0,x) = U_0(x) & \text{for } x \in R \end{cases} \tag{6.4}$$

if $U(t,x)$ satisfies an integral equality

$$\int_0^T dt \int_R \{U(t,x)\phi(t,x)_t + F(U(t,x))\phi(t,x)_x\}dx + \int_R U_0(x)\phi(0,x)dx = 0 \tag{6.5}$$

for any test function $\phi \in C_0^1([0,T)\times R)$. This definition is by the integration by parts, and the differentiability of the function $U(t,x)$ is not necessary.

If a weak solution $U(t,x)$ is smooth, it is easy to show that the weak solution $U(t,x)$ satisfies the equation in the classical sense by the following well-known fact.

THEOREM 5 *let Ω be an open set in R^N, and $u(x)$ be a measurable function on Ω. If*

$$\int_\Omega u(x)\phi(x)dx = 0 \tag{6.6}$$

for any function $\phi(x) \in C_0^1(\Omega)$, then $u = 0$ almost everywhere on Ω.

If the smooth weak solution $U(t,x)$ is continuous at $t=0$, then $U(0,x) = U_0(x)$. Thus, being the weak solution is equivalent with being the classical solution for a smooth function.

In general, the weak solution satisfies the initial condition in a weak sense.

PROPOSITION 6 *Let $U(t,x)$ be a bounded measurable function. Then, $U(t,x)$ is a weak solution if and only if $U(t,x)$ satisfies the following two properties:*

1. *$U(t,x)$ satisfies*

$$U_t + F(U)_x = 0 \qquad \textit{for } t>0,\ x\in R, \tag{6.7}$$

in the distribution sense.

2. $$\frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt \longrightarrow U_0(x) \quad (\Delta t \downarrow 0) \textit{ in } L^\infty \textit{ weak}* \tag{6.8}$$

Here,

$$f_n \longrightarrow f \text{ in } L^\infty(\Omega) \text{ weak}* \qquad (f_n(x),\ f(x) \in L^\infty(\Omega), \Omega \subset R^N) \tag{6.9}$$

means that

$$\int_\Omega f_n g\, dx \longrightarrow \int_\Omega f g\, dx \tag{6.10}$$

for any $g \in L^1(\Omega)$. This is the convergence of a weak$*$ topology on a function space $L^\infty(\Omega)$.

We prove Proposition 6 in Chapter 13.

The definitions of weak solutions for Problems (I)-(IV) are the following.

Problem (I)

$$\begin{cases} \int_0^T dt \int_{x_1(t)}^\infty (\rho\phi_t + m\phi_x)dx + \int_{x_1(0)}^\infty \rho_0(x)\phi(0,x)dx = 0, \\ \int_0^T dt \int_{x_1(t)}^\infty \{m\psi_t + (m^2/\rho + P)\psi_x\}dx + \int_{x_1(0)}^\infty m_0(x)\psi(0,x)dx = 0 \end{cases} \tag{6.11}$$

for any $\phi, \psi \in C_0^1([0,T) \times R)$ such that $\psi = 0$ on $x = x_1(t)$.

Problem (II)

$$\begin{cases} \int_0^T dt \int_{x_1(t)}^{x_2(t)} (\rho\phi_t + m\phi_x)dx + \int_{x_1(0)}^{x_2(0)} \rho_0(x)\phi(0,x)dx = 0, \\ \int_0^T dt \int_{x_1(t)}^{x_2(t)} \{m\psi_t + (m^2/\rho + P)\psi_x\}dx \\ \qquad + \int_{x_1(0)}^{x_2(0)} m_0(x)\psi(0,x)dx = 0 \end{cases} \tag{6.12}$$

for any $\phi, \psi \in C_0^1([0,T) \times R)$ such that $\psi = 0$ on $x = x_1(t)$ and $x = x_2(t)$.

Problem (III)

$$\begin{cases} \int_0^T dt \int_{x(t)}^\infty (\rho\phi_t + m\phi_x)dx + \int_{x_0}^\infty \rho_0(x)\phi(0,x)dx = 0, \\ \int_0^T dt \int_{-\infty}^{x(t)} (\rho\phi_t + m\phi_x)dx + \int_{-\infty}^{x_0} \rho_0(x)\phi(0,x)dx = 0, \\ \int_0^T dt \int_R \{m\psi_t + (m^2/\rho + P)\psi_x\}dx + \int_R m_0(x)\psi(0,x)dx \\ \qquad + \frac{1}{k}\int_0^T \{\psi(t,x)_t + x'(t)\psi(t,x)_x\}_{x=x(t)} x'(t)dt + \frac{1}{k}x_0'\psi(0,x_0) = 0, \\ \qquad \left(P = P(t,x) = \begin{cases} P_1(\rho(t,x)) & (x < x(t), t > 0) \\ P_2(\rho(t,x)) & (x > x(t), t > 0) \end{cases} \right) \end{cases} \tag{6.13}$$

for any $\phi, \psi \in C_0^1([0,T) \times R)$.

Problem (IV)

$$\begin{cases} \int_0^T dt \int_1^\infty \left(\rho\phi_t + m\phi_r - \frac{2}{r}m\phi\right) dr + \int_1^\infty \rho_0(r)\phi(0,r)dr = 0, \\ \int_0^T dt \int_1^\infty \left\{m\psi_t + \left(\frac{m^2}{\rho} + P\right)\psi_r - \left(\frac{2}{r}\frac{m^2}{\rho} + \frac{M}{r^2}\rho\right)\psi\right\} dr \\ \qquad + \int_1^\infty m_0(r)\psi(0,r)dx = 0 \end{cases} \tag{6.14}$$

for any $\phi, \psi \in C_0^1([0,T) \times R)$ such that $\psi = 0$ on $r = 1$.

The reason why ψ vanishes on the boundary and why there are integrals on the boundary in above definitions are from the boundary conditions. It follows by the similar argument that for initial value problems, smooth weak solution is classical solution for initial boundary value problems (I)-(IV). The similar result to Proposition 6 is valid for these definition of Problems (I)-(IV).

PROPOSITION 7 *Let $U(t,x)$ be a bounded measurable function which satisfies*

$$0 \le \rho \le C, \qquad |m| \le C\rho \tag{6.15}$$

and let $x_1(t)$, $x_2(t)$ and $x(t)$ be Lipschitz continuous functions. Then, $U(t,x)$ is a weak solution of each one of Problems (I)-(IV) *if and only if $U(t,x)$ satisfies the following properties, respectively.*

Problem (I)

- *$U(t,x)$ satisfies*

$$U_t + F(U)_x = 0 \qquad \text{for } x > x_1(t), 0 < t < T \tag{6.16}$$

in the distribution sense.

- $$\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{U}(t,x)dt \longrightarrow U_0(x) \ \ (\Delta t \downarrow 0) \quad \text{in } L^\infty(x_1(0),\infty) \text{ weak}*, \tag{6.17}$$

where

$$\tilde{U}(t,x) = \begin{cases} U(t,x) & \text{if } (t,x) \in D_1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.18}$$

- $$\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x} m(t,x)dx - x_1'(t)\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x} \rho(t,x)dx \longrightarrow 0 \\ (\Delta x \downarrow 0) \quad \text{in } L^\infty(0,T) \text{ weak}*. \tag{6.19}$$

Problem (II)

- *$U(t,x)$ satisfies*

$$U_t + F(U)_x = 0 \qquad \text{for } x_1(t) < x < x_2(t),\ 0 < t < T \tag{6.20}$$

in the distribution sense.

- $$\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{U}(t,x)dt \longrightarrow U_0(x) \ \ (\Delta t \downarrow 0) \\ \text{in } L^\infty(x_1(0),x_2(0)) \text{ weak}* \tag{6.21}$$

where

$$\tilde{U}(t,x) = \begin{cases} U(t,x) & \text{if } (t,x) \in D_2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.22}$$

- $$\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x} m(t,x)dx - x_1'(t)\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x}\rho(t,x)dx \longrightarrow 0$$
$$(\Delta x \downarrow 0) \quad \textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.23}$$

- $$\frac{1}{\Delta x}\int_{x_2(t)-\Delta x}^{x_2(t)} m(t,x)dx - x_2'(t)\frac{1}{\Delta x}\int_{x_2(t)-\Delta x}^{x_2(t)}\rho(t,x)dx \longrightarrow 0$$
$$(\Delta x \downarrow 0) \quad \textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.24}$$

Problem (III)

- *$U(t,x)$ satisfies*
$$U_t + F(U)_x = 0 \qquad \textit{for } x \neq x(t),\ 0 < t < T \tag{6.25}$$
in the distribution sense.

- $$\frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt \longrightarrow U_0(x) \quad (\Delta t \downarrow 0) \quad \textit{in } L^\infty(R) \textit{ weak}*. \tag{6.26}$$

- $$\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x} m(t,x)dx - x'(t)\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x}\rho(t,x)dx \longrightarrow 0$$
$$(\Delta x \downarrow 0) \quad \textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.27}$$

- $$\frac{1}{\Delta x}\int_{x(t)-\Delta x}^{x(t)} m(t,x)dx - x'(t)\frac{1}{\Delta x}\int_{x(t)-\Delta x}^{x(t)}\rho(t,x)dx \longrightarrow 0$$
$$(\Delta x \downarrow 0) \quad \textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.28}$$

- *There is a bounded measurable function $\bar{P}_2(t)$ such that*
$$\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x}\{m^2/\rho - x'(t)m + P_2(\rho)\}dx \longrightarrow \bar{P}_2(t) \quad (\Delta x \downarrow 0)$$
$$\textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.29}$$

- *There is a bounded measurable function $\bar{P}_1(t)$ such that*
$$\frac{1}{\Delta x}\int_{x(t)-\Delta x}^{x(t)}\{m^2/\rho - x'(t)m + P_1(\rho)\}dx \longrightarrow \bar{P}_1(t) \quad (\Delta x \downarrow 0)$$
$$\textit{in } L^\infty(0,T) \textit{ weak}*. \tag{6.30}$$

- $$\frac{x(\Delta t) - x_0}{\Delta t} \longrightarrow x_0' \quad (\Delta t \downarrow 0) \tag{6.31}$$

- *$\bar{P}_1(t)$, $\bar{P}_2(t)$ and $x(t)$ satisfy*
$$x''(t) = -k\{\bar{P}_2(t) - \bar{P}_1(t)\} \qquad \textit{for } 0 < t < T \tag{6.32}$$
in the distribution sense.

Problem (IV)

- *$U(t,r)$ satisfies*

$$U_t + F(U)_r = H(r, U) \qquad \text{for } r > 1,\ 0 < t < T \tag{6.33}$$

in the distribution sense.

- $$\frac{1}{\Delta t}\int_0^{\Delta t} U(t,r)dt \longrightarrow U_0(r) \ \ (\Delta t \downarrow 0) \quad \text{in } L^\infty(1,\infty) \text{ weak}*. \tag{6.34}$$
- $$\frac{1}{\Delta r}\int_1^{1+\Delta r} m(t,r)dr \longrightarrow 0 \ \ (\Delta r \downarrow 0) \quad \text{in } L^\infty(0,T) \text{ weak}*. \tag{6.35}$$

The proof of Proposition 7 is given in Chapter 13. To show the global existence of the solutions is not easy even if we weaken the concept of the solution to above definition. But, the uniqueness of the weak solution is not clear since the range of solutions becomes widely.

It is known that there is at most one solution of the single conservation law under the Oleĭnik's admissible condition and of the 2×2 systems of the conservation laws in the class of the weak solution under the entropy condition and under some regularity assumptions ([22],[23],[7]). The entropy condition defined in Chapter 11 is considered to correspond the second law of thermodynamics. Without the regularity assumption, it seems the open problem whether the weak solution satisfying the entropy condition is unique or not. But, without the entropy condition, we can construct example breaking the uniqueness easily (cf. [24]). Hence, it is necessary to show that the weak solution satisfies the entropy condition after showing the existence of the weak solution. We prove in Chapter 13 that the weak solution given in this paper satisfies the entropy condition.

Chapter 7

Lax-Friedrichs difference scheme

We mainly use the Lax-Friedrichs difference scheme as approximate solution. This is the method to construct the approximation by the replacement of the differential equation

$$U_t + F(U)_x = 0 \tag{7.1}$$

with the difference equation

$$\begin{aligned} &\frac{U(t+\Delta t, x) - \dfrac{U(t, x-\Delta x) + U(t, x+\Delta x)}{2}}{\Delta t} + \frac{F(U(t, x+\Delta x)) - F(U(t, x-\Delta x))}{2\Delta x} \\ &= 0. \end{aligned}$$

Since,

$$\begin{aligned} U(t+\Delta t, x) &= \frac{U(t, x-\Delta x) + U(t, x+\Delta x)}{2} \\ &\quad - \frac{\Delta t}{2\Delta x}\{F(U(t, x+\Delta x)) - F(U(t, x-\Delta x))\}, \end{aligned}$$

the values at $(t+\Delta t, x)$ is calculated from the values at $(t, x-\Delta x)$ and at $(t, x+\Delta x)$. Thus, from the values at

$$(t, x) = (0, \pm\Delta x), (0, \pm 3\Delta x), (0, \pm 5\Delta x), \ldots \tag{7.2}$$

the value at

$$\begin{gathered} (\Delta t, 0), (\Delta t, \pm 2\Delta x), (\Delta t, \pm 4\Delta x), \ldots \\ (2\Delta t, \pm\Delta x), (2\Delta t, \pm 3\Delta x), (2\Delta t, \pm 5\Delta x), \ldots \end{gathered} \tag{7.3}$$

can be obtained.

But, the set of the points on which the scheme define the value

$$\{(n\Delta t, j\Delta x); n + j \text{ is odd}\} \tag{7.4}$$

is a discrete subset of $[0,\infty) \times R$, and it is not useful for the approximation of the solution.

O.A.Oleĭnik ([22]) extended the difference scheme to a step function, and she showed the convergence of the approximation to a weak solution for scalar conservation laws. That is, she set

$$U^{\Delta}(t,x) = U(n\Delta t, j\Delta x) \qquad \text{for } n\Delta t \le t < (n+1)\Delta t,\ j\Delta x \le x < (j+1)\Delta x). \tag{7.5}$$

On the other hand, DiPerna ([8]), Ding, Chen, Luo ([5]) used the method to extended the difference scheme to an approximate solution by using the solutions of the Riemann problems.

Now, let $U(t,x)$ be the solution of the Riemann problem

$$\begin{cases} U_t + F(U)_x = 0, \\ U(0,x) = \begin{cases} U_l & (x<0), \\ U_r & (x>0), \end{cases} \end{cases} \tag{7.6}$$

then the solution $U(t,x)$ is the form

$$U = V\left(\frac{x}{t}\right), \tag{7.7}$$

and there is a constant C such that

$$U(t,x) = \begin{cases} U_r & if x \ge Ct, \\ U_l & if x \le -Ct. \end{cases} \tag{7.8}$$

The last character of the solution corresponds the finite propagation for hyperbolic equations. We explain the details in Chapter 8.

We integrate the equation (7.1) on the domain $\{(t,x); 0 < t < \Delta t,\ -\Delta x < x < \Delta x\}$ under $\Delta x/\Delta t > C$. Then,we obtain

$$\begin{aligned} &\int_{-\Delta x}^{\Delta x} U(\Delta t - 0, x)dx - \int_{-\Delta x}^{\Delta x} U(+0,x)dx + \int_0^{\Delta t} F(U(t,\Delta x - 0))dt \\ &- \int_0^{\Delta t} F(U(t,-\Delta x+0))dt = 0 \end{aligned} \tag{7.9}$$

by the divergence theorem. Though the weak solution of the Riemann problem may be discontinuous, this equality is valid for such a solution. Since $\Delta x/\Delta t > C$,

$$\begin{cases} U(t,\Delta x - 0) = U_r & \text{if } 0 < t < \Delta t, \\ U(t,\Delta x + 0) = U_l & \text{if } 0 < t < \Delta t, \\ U(+0,x) = \begin{cases} U_r & (x>0), \\ U_l & (x<0). \end{cases} \end{cases} \tag{7.10}$$

Thus, the equality (7.9) becomes

$$\int_{-\Delta x}^{\Delta x} U(\Delta t - 0, x)dx - \Delta x(U_l + U_r) + \Delta t\{F(U_r) - F(U_l)\} = 0, \tag{7.11}$$

and hence, we obtain

$$\frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x} U(\Delta t-0,x)dx = \frac{U_l+U_r}{2} - \frac{\Delta t}{2\Delta x}\{F(U_r)-F(U_l)\}. \tag{7.12}$$

The right hand side of the equality (7.12) is equal to the Lax-Friedrichs difference scheme. From the fact, We can construct the approximate solution which is similar to the Lax-Friedrichs difference scheme.

1st Step We construct the approximation of the initial data $U_0(x)$ with a step function $U^\Delta(0,x)$. The values of $U^\Delta(0,x)$ is defined to be the average by the integral.

$$U^\Delta(0,x) = \frac{1}{2\Delta x}\int_{(2j-1)\Delta x}^{(2j+1)\Delta x} U_0(y)dy \tag{7.13}$$

for $x \in [(2j-1)\Delta x, (2j+1)\Delta x)$, j is integer.

2nd Step For $0 < t < \Delta t$, we define the approximate solution $U^\Delta(t,x)$ as the solution of the Riemann problem which is centered on the jump point $x = (2j+1)\Delta x$ of $U^\Delta(0,x)$ and which is considered in $(0,\Delta t) \times (2j\Delta x, (2j+2)\Delta x)$, where j is integer and Δt satisfies

$$\Delta x/\Delta t > C. \tag{7.14}$$

3rd Step On $t = \Delta t$, we set $U^\Delta(\Delta t, x)$ as the average by the integral on the interval $(2j\Delta x, (2j+2)\Delta x)$, where j is integer.

$$U^\Delta(\Delta t,x) = \frac{1}{2\Delta x}\int_{2j\Delta x}^{(2j+2)\Delta x} U^\Delta(\Delta t-0,y)dy \tag{7.15}$$

for $x \in (2j\Delta x, (2j+2)\Delta x)$.

After the steps, the positions of the jump of the step function $U^\Delta(\Delta t,x)$ have Δx shifts to the positions for $U^\Delta(0,x)$, however, the function $U^\Delta(\Delta t,x)$ has similar form to the step function $U^\Delta(0,x)$. Thus, we can repeat the steps and we can construct an approximate solution by such a way. Moreover, the values on $t = n\Delta t$ are equal to the values calculated from the Lax-Friedrichs difference scheme. That is, this approximation is an extension of the Lax-Friedrichs scheme to a domain $[0,\infty) \times R$.

Note that we can do the way if the C in (7.14) is bounded globally. Since the value C depends on the solution of the Riemann problem at that time from the nonlinearity of the system, the C may increase with increasing the approximate solution. Thus, we must be taking small mesh height Δt with increasing t, and the time with the scheme becomes slow, and hence, the scheme may not go ahead a time unfortunately. This is the case of the solution blowing up. Therefore, we must give *a priori estimate* of the boundedness of the approximate solution, which concerns closely with the existence of the global solution. We use the invariant region to get the estimate for this system, and we explain this in Chapter 9.

Chapter 8

The Riemann problem

We construct the approximate solutions of the Lax-Friedrichs difference scheme type for Problems (I)-(IV), and show the convergence of the approximation. Mentioned in Chapter 7, We use the solutions of the Riemann problems in order to construct the approximation. In this Chapter, we consider the Riemann problem and initial boundary value problems of simple forms.

8.1 The problem

The Riemann problem is a special type of initial value problem of the system of conservation laws

$$U_t + F(U)_x = 0 \qquad \text{for } x \in R,\ t > 0, \tag{8.1}$$

whose initial data is a step function having a jump at origin

$$U(0,x) = \begin{cases} U_l & (x < 0), \\ U_r & (x > 0). \end{cases} \tag{8.2}$$

P.D.Lax gave the deep result for this problem ([13]). In laboratories, the "shock tube" experiments have been done, which corresponds to the Riemann problems. There is a thin film at a center ($x = 0$) of a long and narrow cylinder and the gas fill to the left and right of the film, where the state of the left gas is different from one of the right gas. The situation corresponding the Riemann problem occurs by breaking the film suddenly ($t = 0$). The Riemann problem have a solution consist of the waves propagated with finite speed. The waves are divided to two groups, the forward wave and the backward wave, and there are two kinds of the simple wave for each of the wave groups, the rarefaction wave, and shock wave.

8.2 Rarefaction waves

A *rarefaction wave* is a wave propagated smoothly.

The rarefaction wave is a continuous solution and the density of the gas decrease through the wave.

This wave solution is the form

$$U(t,x) = V\left(\frac{x}{t}\right) \tag{8.3}$$

For the function of the above form,

$$\begin{aligned} U_t + F(U)_x &= U_t + \nabla F(U)U_x \\ &= -\frac{x}{t^2}V'(\xi) + \nabla F(V(\xi))V'(\xi)\frac{1}{t}, \end{aligned}$$

where $\xi = x/t$. Hence,

$$\nabla F(V(\xi))V'(\xi) = \xi V'(\xi). \tag{8.4}$$

Thus, ξ and $V'(\xi)$ are the eigenvalue and eigenvector of the matrix $\nabla F(V(\xi))$, respectively.

$$\begin{cases} \xi = \lambda_j(V(\xi)), \\ V'(\xi) = c(\xi)r_j(V(\xi)). \end{cases} \tag{8.5}$$

Since $r_j(V) = {}^t(1, \lambda_j(V))$,

$$\begin{cases} \dfrac{d\rho}{d\xi} = c(\xi), \\ \dfrac{dm}{d\xi} = c(\xi)\lambda_j(V(\xi)), \end{cases} \tag{8.6}$$

and hence, the $V(\xi) = {}^t(\rho(\xi), m(\xi))$ is on the curve defined by the ordinary differential equation

$$\frac{dm}{d\rho} = \lambda_j(U) \tag{8.7}$$

We call the curve the *rarefaction wave curve*. We consider the curve passing through the point $U_0 = {}^t(\rho_0, m_0)$ for $j = 1$. In this case, the equation

$$\begin{cases} \dfrac{dm}{d\rho} = \dfrac{m}{\rho} - \sqrt{P'(\rho)}, \\ m(\rho_0) = m_0, \end{cases} \tag{8.8}$$

has a solution

$$u = u_0 - \int_{\rho_0}^{\rho} \frac{P'(y)}{y}dy = u_0 - \frac{\sqrt{a\gamma}}{\theta}(\rho^\theta - \rho_0^\theta) \qquad \left(\theta = \frac{\gamma - 1}{2}, u_0 = \frac{m_0}{\rho_0}\right). \tag{8.9}$$

This solution can be written to

$$w = w_0. \tag{8.10}$$

We can obtain u and ρ as functions of $\xi = x/t$ by

$$\begin{cases} u - \sqrt{a\gamma}\rho^\theta = \xi \\ u + \dfrac{\sqrt{a\gamma}}{\theta}\rho^\theta = u_0 + \dfrac{\sqrt{a\gamma}}{\theta}\rho_0^\theta \end{cases} \tag{8.11}$$

since $\xi = \lambda_1(V(\xi))$. Now we consider the solution for $\xi \geq \xi_0$. Differentiating $\xi = \lambda_1(V(\xi))$, we obtain

$$\begin{aligned} 1 &= \frac{d}{d\xi}\lambda_1(V(\xi)) \\ &= \nabla\lambda_1(V(\xi))V'(\xi) \\ &= c(\xi)\nabla\lambda_1(V(\xi))r_1(V(\xi)). \end{aligned}$$

On the other hand, for $\lambda_1 = m/\rho - \sqrt{a\gamma}\rho^\theta$,

$$\nabla\lambda_1 \cdot r_1 = \left(-\frac{m}{\rho^2} - \theta\sqrt{a\gamma}\rho^{\theta-1}, \frac{1}{\rho}\right)\begin{pmatrix} 1 \\ \dfrac{m}{\rho} - \sqrt{a\gamma}\rho^\theta \end{pmatrix} = -(1+\theta)\sqrt{a\gamma}\rho^{\theta-1} \leq 0, \tag{8.12}$$

thus,

$$c(\xi) = \frac{\rho(\xi)}{\xi} < 0. \tag{8.13}$$

Therefore, $V(\xi)$ moves on the 1-rarefaction wave curve

$$R_1(U_0): \quad u = u_0 - \frac{\sqrt{a\gamma}}{\theta}(\rho^\theta - \rho_0^\theta) \qquad (0 \leq \rho \leq \rho_0). \tag{8.14}$$

for $\xi \geq \xi_0$.

Similarly, we obtain for j=2 that $V(\xi)$ moves on the 2-rarefaction wave curve

$$R_2(U_0): \quad u = u_0 + \frac{\sqrt{a\gamma}}{\theta}(\rho^\theta - \rho_0^\theta) \qquad (\rho \geq \rho_0) \tag{8.15}$$

for $\xi \geq \xi_0$.

The wave solution for $j = 1$ is called the *backward rarefaction wave*, or the *1-rarefaction wave*, and the solution for $j = 2$ is called the *forward rarefaction wave*, or the *2-rarefaction wave*.

Since the j-rarefaction wave satisfies

$$\lambda_j(U(t,x)) = x/t, \tag{8.16}$$

the wave solution $U(t,x)$ is constant along the line passing through the origin

$$\frac{x}{t} = \text{constant}, \tag{8.17}$$

and the line is j-characteristic curve. Suppose that values U_l and U_r in the initial data of the Riemann problem satisfies

$$U_r \in R_1(U_l). \tag{8.18}$$

Let

$$\xi_0 = \lambda_1(U_l), \quad \xi_1 = \lambda_1(U_r). \tag{8.19}$$

Then, $\xi_0 \leq \xi_1$, from the argument mentioned above. We define $U(t,x)$ for $\xi_0 \leq x/t \leq \xi_1$ by

$$\begin{cases} \lambda_1(U) = x/t, \\ U \in R_1(U_l), \end{cases} \tag{8.20}$$

that is,

$$\begin{cases} u - \sqrt{a\gamma}\rho^\theta = \dfrac{x}{t}, \\ u = u_l - \dfrac{\sqrt{a\gamma}}{\theta}(\rho^\theta - \rho_l^\theta), \end{cases} \tag{8.21}$$

and we set

$$U(t,x) = \begin{cases} U_l & (x/t \leq \xi_0), \\ U_r & (x/t \geq \xi_1). \end{cases} \tag{8.22}$$

Then, $U(t,x)$ is a weak solution of the Riemann problem. The flow of fluid particle which is the solution

$$\frac{dx}{dt} = u(t,x) \tag{8.23}$$

for the rarefaction waves are from the higher density to the lower density.

8.3 Shock waves

In nonlinear partial differential equations, or in nature, we see sometimes the formation of the discontinuous wave from the smooth initial data. Such a discontinuous wave is called a *shock wave*. This corresponds to the shock wave in physical phenomena. The radical change of the pressure occurs on the fluid through the wave by the discontinuity. Such discontinuous functions seem to be weak solutions as distribution solutions.

Suppose that the weak solution $U(t,x)$ are smooth in a neighborhood $\Omega\,(\subset \{(t,x); t > 0\})$ of (t_0, x_0) except the point on the curve $x = s(t)$, and $U(t,x)$ has discontinuities of first kind at the point on the curve $x = s(t)$.

For $\phi(t,x) \in C_0^1(\Omega)$,

$$\begin{aligned} 0 &= \iint_\Omega \{U\phi_t + F(U)\phi_x\} dxdt \\ &= \iint_{x<s(t)} \{U\phi_t + F(U)\phi_x\} dxdt + \iint_{x>s(t)} \{U\phi_t + F(U)\phi_x\} dxdt \\ &= \iint_{x<s(t)} \left((U\phi)_t + \{F(U)\phi\}_x - \phi\{U_t + F(U)_x\}\right) dxdt \\ &\quad + \iint_{x>s(t)} \left((U\phi)_t + \{F(U)\phi\}_x - \phi\{U_t + F(U)_x\}\right) dxdt. \end{aligned}$$

Since $U(t,x)$ satisfies the system in $\Omega \cap \{x \neq s(t)\}$ and $\phi = 0$ on the boundary $\partial\Omega$ of Ω,

$$\begin{aligned} 0 &= -\int_{x=s(t)} \phi(t,s(t))\{U(t,s(t)-0) - U(t,s(t)+0)\} dx \\ &\quad + \int_{x=s(t)} \phi(t,s(t))\{F(U(t,s(t)-0)) - F(U(t,s(t)+0))\} dt \\ &= \int_{x=s(t)} \phi\{[U]s'(t) - [F(U)]\} dt \end{aligned}$$

by the divergence theorem, where the symbol [ ] means the gap at the discontinuity, that is,

$$[f(t,x)] = [f(t,x)]_{x=s(t)-0}^{x=s(t)+0} = f(t,s(t)+0) - f(t,s(t)-0). \tag{8.24}$$

Since ϕ is arbitrary, we obtain the relation for the gap and the speed of the shock wave

$$[F(U)] = s'(t)[U] \tag{8.25}$$

This relation is called the *Rankine-Hugoniot condition.* The weak solution must satisfies the condition at the discontinuous point, the other way, it is easy to show that the function $U(t,x)$ is a weak solution if $U(t,x)$ is the smooth solution each sides of $x = s(t)$ and $U(t,x)$ satisfies the Rankine-Hugoniot condition on $x = s(t)$.

If the constant states U_l and U_r satisfy the equality

$$F(U_r) - F(U_l) = s(U_r - U_l) \tag{8.26}$$

for a constant s, then

$$U(t,x) = \begin{cases} U_l & (x < st), \\ U_r & (x > st) \end{cases} \tag{8.27}$$

is the weak solution of the Riemann problem

$$\begin{cases} U_t + F(U)_x = 0 \qquad \text{for } x \in R,\ t > 0, \\ U(0,x) = \begin{cases} U_l & (x < 0), \\ U_r & (x > 0). \end{cases} \end{cases} \tag{8.28}$$

The Rankine-Hugoniot relation

$$\begin{cases} m_r - m_l = s(\rho_r - \rho_l) \\ \left\{ \dfrac{m_r^2}{\rho_r} + P(\rho_r) \right\} - \left\{ \dfrac{m_l^2}{\rho_l} + P(\rho_l) \right\} = s(m_r - m_l) \end{cases} \tag{8.29}$$

give ρ_r and m_r with parameter s for fixed U_l. That is, the state U_r is on a curve passing through the state U_l for $s = 0$. The curve is called the *shock wave curve*.

We rewrite $U_0 = U_l$ and $U = U_r$, and obtain the curve. This relation

$$\begin{cases} m - m_0 = s(\rho - \rho_0), \\ \left(\dfrac{m^2}{\rho} + P \right) - \left(\dfrac{m_0^2}{\rho_0} + P_0 \right) = s(m - m_0) \end{cases} \qquad (P = P(\rho), \quad P_0 = P(\rho_0)) \tag{8.30}$$

yields

$$\rho\rho_0(u - u_0)^2 = (\rho - \rho_0)(P - P_0) \tag{8.31}$$

by the elimination of s. Thus, we obtain

$$u = u_0 \pm \sqrt{\frac{(\rho - \rho_0)(P - P_0)}{\rho\rho_0}}. \tag{8.32}$$

Though the shock waves are known as the compressible wave in physical phenomena, the relations include the rarefaction shock waves. We consider the *Lax's condition* in order to choose the admissible waves ([13]). The condition is written by

$$\begin{cases} \lambda_1(U_l) > s > \lambda_1(U_r), \\ \lambda_2(U_r) > s, \end{cases} \quad \text{or} \quad \begin{cases} s > \lambda_1(U_l), \\ \lambda_2(U_l) > s > \lambda_2(U_r) \end{cases} \tag{8.33}$$

for the system, and the condition show that characteristic curves come to the shock wave both side of the shock wave.

P.D.Lax gave the condition for the uniqueness for the solution of the Riemann problem ([13]), and the condition correspond the entropy condition (cf. [24]). The shock waves satisfying the former in (8.33) is called the *backward shock wave* or the *1-shock wave*, and the other is called the *forward shock wave* or the *2-shock wave*. Now we consider the 1-shock wave. From (8.30), it follows

$$s = \frac{\rho u - \rho_0 u_0}{\rho - \rho_0} = u_0 + \rho\frac{u - u_0}{\rho - \rho_0} = u + \rho_0\frac{u - u_0}{\rho - \rho_0}, \tag{8.34}$$

and hence,

$$-\frac{\sqrt{a\gamma}}{\rho}\rho_0^\theta = \frac{\lambda_1(U_0) - u_0}{\rho} > \frac{s - u_0}{\rho} = \frac{u - u_0}{\rho - \rho_0} = \frac{s - u}{\rho_0} > \frac{\lambda_1(U) - u}{\rho_0} = -\frac{\sqrt{a\gamma}}{\rho_0}\rho^\theta. \tag{8.35}$$

Thus, we obtain

$$\rho \geq \rho_0, \qquad u - u_0 \leq 0. \tag{8.36}$$

Therefore, we have the 1-shock curve

$$S_1(U_0): \quad u = u_0 - \sqrt{\frac{(\rho-\rho_0)(P-P_0)}{\rho\rho_0}} \qquad (\rho \geq \rho_0). \tag{8.37}$$

By the similar argument, we obtain the 2-shock wave curve

$$S_2(U_0): \quad u = u_0 - \sqrt{\frac{(\rho-\rho_0)(P-P_0)}{\rho\rho_0}} \qquad (0 < \rho \leq \rho_0). \tag{8.38}$$

The flow of the fluid particle on the shock waves are from the lower density to the higher density. Thus, the waves are the compressible shock waves.

If $\rho_0 = 0$, then the relations (8.30) become

$$m = s\rho, \qquad \frac{m^2}{\rho} + P = sm. \tag{8.39}$$

Since $m = \rho u$, we obtain

$$P = 0 \tag{8.40}$$

Thus, the shocks do not occur in this case.

8.4 Solutions of the Riemann problems

In general, a simple wave cannot connect two any constant states U_l and U_r, however, it is possible that the combination of these simple waves can connect U_l with U_r. The consideration for the characteristic velocities both side of the simple waves show that such a combination must be a form

$$U_l \to \text{ 1-wave } \to \bar{U} \to \text{ 2-wave } \to U_r, \tag{8.41}$$

where $\bar{U}$ is some constant state. If the state $\bar{U}$ is given, then the simple waves can be obtained. Therefore, we may get the state $\bar{U}$ in order to solve the Riemann problem. For example, $\bar{U}$ satisfies

$$\bar{U} \in S_1(U_l) \qquad \text{and} \qquad U_r \in R_2(\bar{U}). \tag{8.42}$$

If we define the reverse curve $\tilde{R}_2(U_0)$ of R_2 curve by

$$U \in \tilde{R}_2(U_0) \Longleftrightarrow U_0 \in R_2(U), \tag{8.43}$$

then $\bar{U}$ is obtain from

$$\bar{U} \in S_1(U_l) \cap \tilde{R}_2(U_r). \tag{8.44}$$

Thus, we can get $\bar{U}$ as the intersection point of 1-wave curve passing through U_l and $\tilde{2}$-wave curve passing through U_r, where we define the $\tilde{2}$-wave curve as a reverse curve of 2-wave curve. Because the 2-wave curve started from U_0 is a set consisting of the state which connect a 2-wave with a left state U_0, the $\tilde{2}$-wave curve started from U_0 is a set which connect a 2-wave with a right state U_0.

We give the $\tilde{2}$-wave curve.

The definition

$$U \in \tilde{S}_2(U_0) \quad \Leftrightarrow \quad U_0 \in S_2(U) \quad \Leftrightarrow \quad 0 < \rho_0 \leq \rho, \;\; u_0 = u - \sqrt{\frac{(\rho_0 - \rho)(P_0 - P)}{\rho \rho_0}} \tag{8.45}$$

gives

$$\tilde{S}_2(U_0): \quad u = u_0 + \sqrt{\frac{(\rho - \rho_0)(P - P_0)}{\rho \rho_0}} \qquad (\rho \geq \rho_0) \tag{8.46}$$

and

$$\tilde{R}_2(U_0): \quad u = u_0 + \frac{\sqrt{a\gamma}}{\theta}(\rho^\theta - \rho_0^\theta) \qquad (0 \leq \rho \leq \rho_0) \tag{8.47}$$

similarly.

Now, We define the 1-wave curve $C_1(U_0)$ and $\tilde{2}$-wave curve $\tilde{C}_2(U_0)$ by

$$\begin{aligned} C_1(U_0): &\quad \begin{cases} S_1(U_0) & (u < u_0) \\ R_1(U_0) & (u_0 \leq u < w(U_0)) \\ \rho = 0 & (w(U_0) \leq u) \end{cases} \\ \tilde{C}_2(U_0): &\quad \begin{cases} \tilde{S}_2(U_0) & (u > u_0) \\ \tilde{R}_2(U_0) & (z(U_0) < u \leq u_0) \\ \rho = 0 & (u \leq z(U_0)) \end{cases} \end{aligned} \tag{8.48}$$

for $\rho_0 > 0$, and define

$$\begin{aligned} C_1(U_0): &\;\; \rho = 0 \\ \tilde{C}_2(U_0): &\;\; \rho = 0 \end{aligned} \tag{8.49}$$

for $\rho_0 = 0$.

It is easy to show that the curves $C_1(U_0)$ and $\tilde{C}_2(U_0)$ are written by

$$\rho = f_1(u; U_0), \qquad \rho = \tilde{f}_2(u; U_0), \tag{8.50}$$

respectively. Since the curves $C_1(U_0)$ and $\tilde{C}_2(U_0)$ symmetric with respect to the line $u = u_0$, we have

$$\tilde{f}_2(u; U_0) = f_1(2u_0 - u; U_0). \tag{8.51}$$

It follows that $f_1(u;U_0)$ does not increase with u and $\tilde{f}_2(u;U_0)$ does not decrease with u.

We consider the intersection point of $C_1(U_l)$ and $\tilde{C}_2(U_r)$.

It is easy to show that the intersection point is uniquely determined in the case that the two curve interact in the subset $\{\rho > 0\}$ of phase space (cf. [20],[24]). Suppose $\rho_l > 0$. We divide the phase space to the five parts D_1-D_5.

$$\begin{cases} D_1: & \{(w,z); z(U_0) < z < w(U_0),\ w > w(U_0)\}, \\ D_2: & \{(w,z); z < z(U_0),\ u > u_0 - s(\rho,\rho_0),\ \rho > \rho_0\}, \\ D_3: & \{(w,z); u < u_0 - s(\rho,\rho_0), \}, \\ D_4: & \{(w,z); u > u_0 - s(\rho,\rho_0),\ w < w(U_0)\}, \\ D_5: & \{(w,z); z > w(U_0)\}, \end{cases} \tag{8.52}$$

where $s(\rho,\rho_0)$ is defined by

$$s(\rho,\rho_0) = \sqrt{\frac{(\rho-\rho_0)(P-P_0)}{\rho\rho 0}} \tag{8.53}$$

which appears in the definition of shock curves. For the decomposition, it is easy to show that

$$\begin{aligned} U_r \in D_1 &\Rightarrow \bar{U} \in R_1(U_l) \cap \tilde{R}_2(U_r), \\ U_r \in D_2 &\Rightarrow \bar{U} \in S_1(U_l) \cap \tilde{R}_2(U_r), \\ U_r \in D_3 &\Rightarrow \bar{U} \in S_1(U_l) \cap \tilde{S}_2(U_r), \\ U_r \in D_4 &\Rightarrow \bar{U} \in R_1(U_l) \cap \tilde{S}_2(U_r). \end{aligned} \tag{8.54}$$

For example,

$$U(t,x) = \begin{cases} U_l & (x/t < s_1), \\ \bar{U} & (s_1 < x/t < \lambda_2(\bar{U})), \\ \text{2-rare. wave} & (\lambda_2(\bar{U}) \le x/t < \lambda_2(U_r)), \\ U_r & (x/t \le \lambda_2(U_r)), \end{cases} \tag{8.55}$$

in the case $U_r \in D_2$.

The solution $U(t,x)$ takes only two values U_l and $\bar{U}$ on the curve $S_1(U_l)$, and takes all the values of the part of the curve $R_2(\bar{U})$ (or $\tilde{R}_2(U_r)$) from $\bar{U}$ to U_r. We note that the solution of the Riemann problem takes only two values on the shock wave curve, which are the constant states lying in the both sides of the shock wave, and the solution takes all the values of the part of the rarefaction wave curve, which is between two states lying in the both sides of the rarefaction wave, as stated above.

Next, We consider the case $U_r \in D_5$. In this case, we define the states $\bar{U}_l$ and $\bar{U}_r$ by

$$\bar{U}_l \in R_1(U_l) \cap \{\rho = 0\}, \qquad \bar{U}_r \in \tilde{R}_2(U_r) \cap \{\rho = 0\}, \tag{8.56}$$

that is, we set

$$\bar{\rho}_l = 0, \qquad \bar{u}_l = w(U_l), \qquad \bar{\rho}_r = 0, \qquad \bar{u}_r = z(U_r). \tag{8.57}$$

Then, the solution is obtained by the following.

$$U(t,x) = \begin{cases} U_l & (x/t < \lambda_1(U_l)), \\ \text{1-rare. wave} & (\lambda_1(U_l) \le x/t < \lambda_1(\bar{U}_l)), \\ \rho = 0, \quad m = 0 & (\lambda_1(\bar{U}_l) \le x/t < \lambda_2(\bar{U}_r)), \\ \text{2-rare. wave} & (\lambda_2(\bar{U}_r) \le x/t < \lambda_2(U_r)), \\ U_r & (\lambda_2(U_r) \le x/t). \end{cases} \tag{8.58}$$

Though, we can not define the value of u at the point of $\rho = 0$, we set the value in the interval $[\bar{u}_l, \bar{u}_r]$. This is certain weak solution of the Riemann problem. In physical phenomena, the difference of the velocity between the states U_l and U_r is much larger than the difference of the density, and, since $u_l < u_r$, the both sides of gases move to separate with high velocity, and hence, such the hight velocity flow cause the vacuum state.

Consider the case $\rho_l = 0$ and $\rho_r > 0$.

We take $\bar{U}_r$ such that

$$\bar{U}_r \in \tilde{R}_2(U_r) \cap \{\rho = 0\}, \tag{8.59}$$

that is,

$$\bar{\rho}_r = 0, \quad \bar{u}_r = z(U_r). \tag{8.60}$$

Then,

$$U(t,x) = \begin{cases} \rho = 0, \quad m = 0 & (x/t < \lambda_2(\bar{U}_r)), \\ \text{2-rare. wave} & (\lambda_2(\bar{U}_r) \le x/t < \lambda_2(U_r)), \\ U_r & (\lambda_2(U_r) \le x/t) \end{cases} \tag{8.61}$$

is the weak solution of the Riemann problem. Though u is undefined in $\{(t,x); x/t < \lambda_2(\bar{U}_r)\}$, we set u as a value between u_l and $\bar{u}_r$.

And in the case $\rho_r = 0$ and $\rho_l > 0$, by setting

$$\bar{U}_l \in R_1(U_l) \cap \{\rho = 0\}, \tag{8.62}$$

that is,

$$\bar{\rho}_l = 0, \qquad \bar{u}_l = w(U_l), \tag{8.63}$$

we have that

$$U(t,x) = \begin{cases} U_l & (x/t < \lambda_1(U_l)), \\ \text{1-rare. wave} & (\lambda_1(U_l) \le x/t < \lambda_1(\bar{U}_l)), \\ \rho = m = 0 & (\lambda_1(\bar{U}_l) \le x/t) \end{cases} \tag{8.64}$$

is the weak solution, and we set u as a value between $\bar{u}_l$ and u_r in $\{(t,x); \lambda_1(\bar{U}_l) \leq x/t\}$.

Last, in the case $\rho_l = \rho_r = 0$,

$$U(t,x) = \begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8.65}$$

is the weak solution. We suppose that the velocity u takes a value between u_l and u_r.

We remark with the velocity of simple waves. The rarefaction wave velocity is equal to the characteristic velocity, and the shock wave velocity is the value between the characteristic velocities of both sides of the wave. Thus, the absolute values of the velocities of simple waves is not greater than the value

$$\sup_{(t,x)} \max\{|\lambda_1(U(t,x))|, |\lambda_2(U(t,x))|\}. \tag{8.66}$$

8.5 Initial boundary value problems

Making an extension of the Lax-Friedrichs scheme near the boundary needs the solutions of the initial boundary value problems which corresponds the Riemann problem. These problem are the following.

$$\begin{cases} U_t + F(U)_x = 0 & (x > \alpha t, t > 0), \\ U(0,x) = U_0 & (x > 0), \\ m - \rho\alpha]_{x=\alpha t} = 0 & (t > 0). \end{cases} \tag{8.67}$$

$$\begin{cases} U_t + F(U)_x = 0 & (x < \alpha t, t > 0), \\ U(0,x) = U_0 & (x > 0), \\ m - \rho\alpha]_{x=\alpha t} = 0 & (t > 0). \end{cases} \tag{8.68}$$

The boundary conditions of both problem correspond to the piston problems. That is, these are the problem given the moving piston with constant velocity and given the stationary initial data. The 1-wave and the 2-wave lies in the solution of the Riemann problem from left to right, and the 1-wave passes through the left particle of state U_l and the 2-wave passes through the right particle of state U_r. The boundary conditions of the piston problems mean that the flows do not occur at the boundaries. Thus, for the problem (8.67), we must give the solution which consists of only a 2-wave, and for the problem (8.68), we must give the solution which consists of only a 1-wave. That is, the solution is of the form

$$\bar{U} \rightarrow \text{ 2-wave } \rightarrow U_0 \tag{8.69}$$

for the problem (8.67), and is of the form

$$U_0 \rightarrow \text{ 1-wave } \rightarrow \bar{U} \tag{8.70}$$

for the problem (8.68). The constant state $\bar{U}$ adjoining the boundary must satisfy the boundary condition

$$\bar{m} - \alpha\bar{\rho} = 0 \tag{8.71}$$

for each solution of the problems (8.67) and (8.68). Hence, if $\bar{\rho} > 0$ then $\bar{u} = \alpha$, that is, the state $\bar{U}$ is on $u = \alpha$. Therefore, we may obtain $\bar{U}$ such that

$$\begin{aligned} &\bar{U} \in \tilde{C}_2(U_0) \cap \{u = \alpha\} \quad \text{or} \quad \bar{\rho} = 0 \quad \text{for the problem (8.67)}, \\ &\bar{U} \in C_1(U_0) \cap \{u = \alpha\} \quad \text{or} \quad \bar{\rho} = 0 \quad \text{for the problem (8.68)}, \end{aligned} \tag{8.72}$$

It is clear that there exist at least one state $\bar{U}$ for each problems by the structure of the $C_1(U_0)$ and $\tilde{C}_2(U_0)$. But, the $\bar{u}$ is not defined at $\bar{\rho} = 0$ for the boundary conditions. In this case, we set $\bar{u}$ in similar way for the Riemann problem. That is, we set $\bar{u}$ as a value between α and $z(U_0)$ for the problem (8.67), and $\bar{u}$ as a value between α and $w(U_0)$ for the problem (8.68). The similar situation to the Riemann problem holds for the velocity of the wave arising in the solution, that is, the absolute value is not greater than

$$\sup_{x>\alpha t} |\lambda_2(U(t,x))| \tag{8.73}$$

for the problem (8.67), and is not greater than

$$\sup_{x<\alpha t} |\lambda_1(U(t,x))| \tag{8.74}$$

for the problem (8.68).

8.6 A free boundary value problem

Since the velocity of the free boundary is unknown for Problem (III), we have to the velocity of the boundary from the equation of motion for the free piston in order to extend the difference scheme to the approximate solution near the boundary by the methods in the section 8.5. And it is necessary that the boundary velocity is constant. On the other hand, the equation of motion for the free piston $x = x(t)$

$$Mx''(t) = SP(t, x(t) - 0) - SP(t, x(t) + 0) \qquad (t > 0) \tag{8.75}$$

is for the acceleration which is the differentiation of the velocity, and hence, the solution of the differential equation has the variable derivative in general. Thus, we replace the equation with the difference equation in order to construct the approximate solution. Although there are several kinds of the difference approximation, we use the backward difference approximation.

Thus, we use the solution of the following problem near the free boundary to construct the approximation.

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + (m^2/\rho + P_1(\rho))_x = 0 \quad (0 < t < \Delta t,\ x < u_1 t) \end{cases}$$
$$\begin{cases} \rho_t + m_x = 0 \\ m_t + (m^2/\rho + P_2(\rho))_x = 0 \quad (0 < t < \Delta t,\ x > u_1 t) \end{cases}$$
$$\frac{u_1 - u_0}{\Delta t} = k\{P_1(\rho(t, u_1 t - 0)) - P_2(\rho(t, u_1 t + 0))\} \qquad (0 < t < \Delta t) \tag{8.76}$$
$$U(0,x) = \begin{cases} U_l & (x < 0) \\ U_r & (x > 0) \end{cases}$$
$$m - u_1\rho]_{x = u_1 t \pm 0} = 0 \qquad (0 < t < \Delta t)$$

where U_l, U_r, u_0, Δt and k are given, and $U(t,x) = {}^t(\rho(t,x), m(t,x))$ and u_1 are unknown. Using the solution of the problem (8.76), we can construct the approximate solution $U(t,x)$ and the approximate boundary $x^{\Delta}(t)$ of a polygonal line type.

Since the gas of left hand side of the piston may differ from one of right hand side for this problem, there are differences between the wave curves, the Riemann invariant, the eigenvalues, and so on, on both sides. Thus, we rewrite $C_1(U_0)$ and $f_1(u; U_0)$ to $C_1(U_0; P_1)$ and $f_1(u; U_0, P_1)$, respectively, in order to clear whether the curve and function are for the left side gas or for the right side gas, and so on.

Let $\bar{U}_l$ be the state adjoining the free piston to the left and $\bar{U}_r$ be the state adjoining the piston to the right. Then, since $U(t,x)$ is the solution of (8.67) on the right and is the solution of (8.68) on the left of the piston, $\bar{U}_l$ and $\bar{U}_r$ satisfy

$$\begin{aligned} &\bar{U}_l \in C_1(U_l; P_1) \cap \{u = u_1\} \quad \text{or} \quad \bar{\rho}_l = 0, \\ &\bar{U}_r \in \tilde{C}_2(U_r; P_2) \cap \{u = u_1\} \quad \text{or} \quad \bar{\rho}_r = 0. \end{aligned} \tag{8.77}$$

Using the functions f_1 and $\tilde{f}_2$, we can write

$$\begin{cases} \bar{\rho}_l = f_1(u_1; U_l, P_1) \\ \bar{\rho}_r = \tilde{f}_2(u_1; U_r, P_2). \end{cases} \tag{8.78}$$

Thus, the equation of motion for the free boundary becomes

$$\frac{u_1 - u_0}{\Delta t} = k\{P_1(\bar{\rho}_l) - P_2(\bar{\rho}_r)\} = k\left\{P_1\big(f_1(u_1; U_l, P_1)\big) - P_2\big(\tilde{f}_2(u_1; U_r, P_2)\big)\right\}. \tag{8.79}$$

This is the equation of only u_1. Let

$$g(u) = \frac{u - u_0}{\Delta t} - k\left\{P_1\big(f_1(u; U_l, P_1)\big) - P_2\big(\tilde{f}_2(u; U_r, P_2)\big)\right\}. \tag{8.80}$$

Since $f_1(u; U_l, P_1)$ does not increase with u and $\tilde{f}_2(u; U_r, P_2)$ does not decrease, $g(u)$ increases. By

$$\lim_{u \to +\infty} g(u) = +\infty, \quad \lim_{u \to -\infty} g(u) = -\infty, \tag{8.81}$$

$g(u)$ has always a unique null point u_1. Thus, we obtain $\bar{\rho}_l$, $\bar{\rho}_r$ from (8.78), and we get $\bar{U}_l$, $\bar{U}_r$ as

$$\bar{U}_l = \begin{pmatrix} \bar{\rho}_l \\ \bar{\rho}_l u_1 \end{pmatrix}, \quad \bar{U}_r = \begin{pmatrix} \bar{\rho}_r \\ \bar{\rho}_r u_1 \end{pmatrix}, \tag{8.82}$$

moreover the solution $U(t,x)$ is constructed from the similar way in the section 8.5.

The absolute value of the wave velocities are not greater than

$$\sup_{x<u_1 t} |\lambda_1(U(t,x))|, \tag{8.83}$$

for $x < u_1 t$, and are not greater than

$$\sup_{x>u_1 t} |\lambda_2(U(t,x))|, \tag{8.84}$$

for $x > u_1 t$.

Chapter 9

The invariant region

It is necessary to show the uniform boundedness for the approximate solution and the estimate for the entropy in order to apply the compensated compactness theory. We show the estimate for the entropy in Chapter 13. In this chapter, we show the existence of the invariant regions for the solution in order to show the boundedness of the approximation. We solved the Riemann problem in the section 8.4, and we can obtain the following estimates for the solution by the method to construct the solution.

LEMMA 8 *The solution of the Riemann problem*

$$
\begin{aligned}
&U_t + F(U)_x = 0 \qquad (x \in R,\, t > 0),\\
&U(0,x) = \begin{cases} U_l & (x < 0),\\ U_r & (x > 0),\end{cases}
\end{aligned}
\tag{9.1}
$$

which constructed in the section 8.4, satisfies

$$
\begin{cases} w(U(t,x)) \le \max\{w(U_l),\, w(U_r)\},\\ z(U(t,x)) \ge \min\{z(U_l),\, z(U_r)\}.\end{cases}
\tag{9.2}
$$

Proof

First we consider $U \in \{w = z\}$. In the case $\rho_l = 0$ and $\rho_r > 0$, since we defined u by a value between u_l and $z(U_r)$,

$$
\begin{aligned}
&w(U) = u \le \max\{u_l, z(U_r)\} = \max\{w(U_l), z(U_r)\} \le \max\{w(U_l), w(U_r)\},\\
&z(U) = u \ge \min\{u_l, z(U_r)\} = \min\{z(U_l), z(U_r)\}.
\end{aligned}
\tag{9.3}
$$

In the case $\rho_l > 0$ and $\rho_r = 0$, and in the case $\rho_l = \rho_r = 0$, the estimates (9.2) are obtain by similar arguments mentioned above.

Next, let $U \in \{w > z\}$. U is on the part of the curve $C_1(U_l) \cup \tilde{C}_2(U_r)$ from U_l to U_r as we saw in the section 8.4. If $U \in C_1(U_l)$, it is clear that

$$
w(U) \le w(U_l) \tag{9.4}
$$

by the form of the curve $C_1(U_l)$ on (w,z)-space. Let $U \in \tilde{C}_2(U_r)$ and $w(U) > w(U_r)$. Then, $U \in \tilde{S}_2(U_r)$, that is, $U = \bar{U} \in C_1(U_l)$. Hence, we have

$$w(U) \leq w(U_l). \tag{9.5}$$

Therefore, the inequality with respect to w is valid. The other inequality of (9.2) with respect to z is proved similarly. ∎

This Lemma 8 show that if the states U_l and U_r are in a region

$$\Sigma = \Sigma(w_1, z_1) = \{w \geq z,\ w \leq w_1,\ z \geq z_1\} \tag{9.6}$$

which is triangular in (w,z)-space, the solution of the Riemann problem for U_l and U_r is in Σ. Thus, we call the region Σ the *invariant region* of the Riemann problem.

For the initial boundary value problems (8.67) and (8.68), the following estimates are valid.

LEMMA 9 *The solutions for the initial boundary value problems (8.67) and (8.68), which we construct in the section 8.5, satisfy the following, respectively.*

The problem (8.67)

$$\begin{cases} w(U) \leq \max\{w(U_0),\ 2\alpha - z(U_0)\}, \\ z(U) \geq \min\{z(U_0),\ \alpha\}, \end{cases} \tag{9.7}$$

The problem (8.68)

$$\begin{cases} w(U) \leq \max\{w(U_0),\ \alpha\}, \\ z(U) \geq \min\{z(U_0),\ 2\alpha - w(U_0)\}. \end{cases} \tag{9.8}$$

Proof

Consider the solution of the problem (8.67). Then, the state $\bar{U}$ adjoining the boundary satisfies

$$\bar{U} \in \tilde{C}_2(U_0) \cap \{u = \alpha\} \text{ or } r\bar{h}o = 0. \tag{9.9}$$

The solution U takes the values on the part of $\tilde{C}_2(U_0)$ from U_0 to $\bar{U}$. First, let $U \in \{w > z\}$. It is clear that the inequality $z(U) \geq z(U_0)$ is valid in this case. Suppose $w(U) > w(U_0)$. Then, $U \in \tilde{C}_2(U_0)$, that is,

$$U = \bar{U} \in \tilde{S}_2(U_0) \cap \{u = \alpha\}. \tag{9.10}$$

In this case,

$$w(U) = w(\bar{U}) = 2\alpha - z(\bar{U}) \leq 2\alpha - z(U_0). \tag{9.11}$$

Thus, the inequalities (9.7) are valid in this case. Next, let $U \in \{w = z\}$. Since we set in the section 8.5 that u took the value between α and $z(U_0)$, we have

$$\begin{aligned} w(U) &= u \leq \max\{z(U_0), \alpha\} \leq \max\{z(U_0), 2\alpha - z(U_0)\} \leq \max\{w(U_0), 2\alpha - z(U_0)\}, \\ z(U) &= u \geq \min\{z(U_0), \alpha\}. \end{aligned}$$

Therefore, the inequalities (9.7) are valid.

For the problem (8.68), we can show inequalities (9.8) similarly. ∎

This Lemma 9 shows that the triangular region $\Sigma = \Sigma(w_1, z_1)$ is the invariant region for the problems (8.67) and (8.68) if w_1 and z_1 satisfy

$$\frac{w_1 + z_1}{2} \geq \alpha \geq z_1 \quad \text{for the problem (8.67)}, \tag{9.12}$$

$$\frac{w_1 + z_1}{2} \leq \alpha \leq w_1 \quad \text{for the problem (8.68)}, \tag{9.13}$$

respectively. Because, if (9.12) and $U_0 \in \Sigma(w_1, z_1)$, then

$$\begin{aligned} &w(U) \leq \max\{w(U_0), 2\alpha - z(U_0)\} \leq \max\{w_1, 2\alpha - z_1\} = w_1, \\ &z(U) \geq \min\{z(U_0), \alpha\} \geq \min\{z_1, \alpha\} = z_1, \end{aligned} \tag{9.14}$$

and hence, $U \in \Sigma(w_1, z_1)$.

For the free piston problem, we obtain the following estimates.

LEMMA 10 *Consider the problem (8.76) and the solution constructed in the section 8.6. Let $\bar{u}_1$ and $\bar{u}_2$ be values such that*

$$\{u \in R;\ P_1(f_1(u; U_l, P_1)) = P_2(\tilde{f}_2(u; U_r, P_2))\} = [\bar{u}_1, \bar{u}_2]. \tag{9.15}$$

Then, it follows

$$\begin{aligned} &\begin{cases} w(U; P_2) \leq \max\{w(U_r; P_2),\ 2u_1 - z(U_r; P_2)\}, \\ z(U; P_2) \geq \min\{z(U_r; P_2),\ u_1\} \end{cases} && (x > u_1 t,\ 0 < t < \Delta t), \\ &\begin{cases} w(U; P_1) \leq \max\{w(U_l; P_1),\ u_1)\}, \\ z(U; P_1) \geq \min\{z(U_l; P_1),\ 2u_1 - w(U_l; P_1)\} \end{cases} && (x < u_1 t,\ 0 < t < \Delta t), \\ &\min\{\bar{u}_2,\ u_0\} \leq u_1 \leq \max\{\bar{u}_1,\ u_0\}. \end{aligned} \tag{9.16}$$

Proof

Since $U(t, x)$ is the solution of the problem (8.67) for $x > u_1 t$ and is the solution of the problem (8.68) for $x < u_1 t$, the inequalities for w and z are valid by Lemma 9. We consider the last inequalities. Let

$$l(u) = -k\Delta t\{P_1(f_1(u; U_l, P_1)) - P_2(\tilde{f}_2(u; U_r, P_2))\}. \tag{9.17}$$

Then, $l(u)$ does not decrease, and satisfies

$$g(u) = \{u - u_0 + l(u)\}/\Delta t, \quad \{u; l(u) = 0\} = [\bar{u}_1, \bar{u}_2]. \tag{9.18}$$

Suppose $u_1 < u_0$. Then, $u_0 > \bar{u}_2$ because

$$g(u_0) = l(u_0)/\Delta t > g(u_1) = 0, \tag{9.19}$$

and $\bar{u}_2 < u_1$ because

$$g(\bar{u}_2) = (\bar{u}_2 - u_0)/\Delta t < 0 = g(u_1). \tag{9.20}$$

Thus, we obtain

$$u_1 \geq \min\{u_0, \bar{u}_2\}. \tag{9.21}$$

Similarly, we have

$$u_1 \leq \max\{u_0, \bar{u}_1\}. \tag{9.22}$$

∎

This Lemma 10 shows that $\Sigma_1 = \Sigma(w_1, z_1; P_1)$ and $\Sigma_2 = \Sigma(w_2, z_2; P_2)$ are invariant regions for $x < u_1 t$ and for $x > u_1 t$, respectively, if

$$\frac{w_1 + z_1}{2} \leq u_1 \leq w_1 \quad \text{and} \quad z_2 \leq u_1 \leq \frac{w_2 + z_2}{2}. \tag{9.23}$$

But, since u_1 is unknown, it seems not suitable to call Σ_1 and Σ_2 the invariant regions for this problem. However, if $U_l \in \Sigma_1$, $U_r \in \Sigma_2$ and

$$\frac{w_1 + z_1}{2} \leq z_2 \leq u_0 \leq w_1 \leq \frac{w_2 + z_2}{2}, \tag{9.24}$$

then the solution $U(t, x)$ and u_1 satisfy

$$\begin{aligned} &U \in \begin{cases} \Sigma_1 & (x < u_1 t), \\ \Sigma_2 & (x > u_1 t), \end{cases} \\ &z_2 \leq u_1 \leq w_1. \end{aligned} \tag{9.25}$$

We show the last inequality.

Now,

$$P_1(f_1(w_1; U_l, P_1)) = P_1(0) = 0 \tag{9.26}$$

since $U_l \in \Sigma_1$. Hence, we have $w_1 \geq \bar{u}_1$ because $l(w_1) \geq 0$. Therefore, we obtain

$$u_1 \leq \max\{\bar{u}_1, u_0\} \leq w_1. \tag{9.27}$$

Similarly, it follows

$$u_1 \geq \min\{\bar{u}_2, u_0\} \geq z_2. \tag{9.28}$$

Thus, the inequalities of the last inequalities of (9.25) hold.

We observe that Σ_1 and Σ_2 are the invariant regions of the problem (8.76) as above sense under the condition (9.24).

Since the triangular region Σ becomes a closed convex set in (ρ, m)-space, we obtain easily the following lemma by the Jensen's inequality.

LEMMA 11 *If*

$$^t(\rho(s), m(s)) \in \Sigma \quad \text{for } s \in [a, b], \tag{9.29}$$

then

$$^t\left(\frac{1}{b-a}\int_a^b \rho(s)ds, \frac{1}{b-a}\int_a^b m(s)ds\right) \in \Sigma. \tag{9.30}$$

This Lemma 11 shows that Σ is the invariant region for taking the average by the integral.

Chapter 10

Constructions of approximate solutions

In Chapter 7, we saw how to construct the approximate solution for the initial value problem. In this chapter, we construct the approximate solutions for the initial boundary value problems (I)-(IV), and give the estimates for the approximations.

10.1 Problem (I)

Since the boundary $x = x(t)$ is Lipschitz continuous, the derivative

$$u_1(t) = x_1'(t) \tag{10.1}$$

is in the function space $L^\infty(0, \infty)$. Thus, we suppose

$$0 \leq \rho_0(x) \leq \rho_M, \qquad |u_0(x)| \leq u_M, \qquad u_1^m \leq u_1(t) \leq u_1^M, \tag{10.2}$$

where ρ_M, u_M, u_1^m and u_1^M are constants. We fix w_1 and z_1 such that

$$z_1 \leq u_1^m \leq u_1^M \leq \frac{w_1 + z_1}{2}, \qquad u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta \leq \min\{w_1, -z_1\}. \tag{10.3}$$

The last inequality says

$$U_0(x) \in \Sigma(w_1, z_1). \tag{10.4}$$

We can take such values w_1 and z_1 by the following way in fact. First, we take z_1 such that

$$z_1 \leq \min\left\{u_1^m, -u_M - \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta\right\}. \tag{10.5}$$

Next, we may take w_1 such that

$$w_1 \geq \max\left\{2u_1^M - z_1, u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta\right\}. \tag{10.6}$$

Let

$$\Sigma_1 = \Sigma(w_1, z_1), \qquad \Lambda_0 = \Lambda(\Sigma_1), \tag{10.7}$$

where $\Lambda(\Omega)$ defined by

$$\Lambda(\Omega) = \sup_{U\in\Omega} \max\{|\lambda_1(U)|, |\lambda_2(U)|\}. \tag{10.8}$$

We fix a constant Λ_1 such that

$$\Lambda_1 > \Lambda_0 \tag{10.9}$$

and we set sufficiently small parameters Δx and Δt such that

$$\frac{\Delta x}{\Delta t} = \Lambda_1. \tag{10.10}$$

For simplicity, we suppose $x_1(0) = 0$, and we set

$$J_1^n = \{j \in Z; n + j \text{ is odd}\}, \quad E_j^n = ((j-1)\Delta x, (j+1)\Delta x) \qquad (j \in J_1^n). \tag{10.11}$$

1st Step We make an approximation of initial data and a moving boundary. Let

$$\bar{U}_j^0 = \frac{1}{m(E_j^0)} \int_{E_j^0} U_0(y)dy \qquad (j \in J_1^0), \tag{10.12}$$

where $m(\cdot)$ is a 1-dimensional Lebesgue measure. Let $U^\Delta(0, x)$ be a step function

$$U^\Delta(0, x) = \bar{U}_j^0 \qquad (x \in E_j^0, j \in J_1^0). \tag{10.13}$$

And we construct the approximate boundary with the polygonal line $x = x_1^\Delta(t)$.

$$\begin{aligned} x_1^\Delta(t) &= x_1(n\Delta t) + \frac{t - n\Delta t}{\Delta t}\{x_1((n+1)\Delta t) - x_1(n\Delta t)\}, \\ &\quad (t \in (n\Delta t, (n+1)\Delta t], \ n = 0, 1, 2, \ldots), \\ x_1^\Delta(0) &= 0. \end{aligned} \tag{10.14}$$

Since $U_0(x) \in \Sigma_1$, it follows

$$U^\Delta(0, x) \in \Sigma_1 \tag{10.15}$$

by Lemma 11.

2nd Step We define the approximate solution $U^\Delta(t, x)$ for $0 < t < \Delta t$. In a domain

$$\{(t, x); x_1^\Delta(t) < x < \Delta x, \ 0 < t < \Delta t\}, \tag{10.16}$$

since $U^\Delta(0,x)$ is a constant state $\bar{U}_1^0$ and $x = x_1^\Delta(t)$

$$x_1^\Delta(t) = \frac{t}{\Delta t}x_1(\Delta t) = t\frac{1}{\Delta t}\int_0^{\Delta t} u_1(s)ds \tag{10.17}$$

is a boundary with constant velocity, we define $U(t,x)$ as the solution of Problem (8.67). And in a each domain

$$(0,\Delta t)\times E_j^1 \qquad (j = 2,4,6,\ldots), \tag{10.18}$$

since

$$U^\Delta(0,x) = \begin{cases} \bar{U}_{j-1}^0 & ((j-1)\Delta x < x < j\Delta x), \\ \bar{U}_{j+1}^0 & (j\Delta x < x < (j+1)\Delta x), \end{cases} \tag{10.19}$$

we define $U^\Delta(t,x)$ as the solution of the Riemann problem centered $x = j\Delta x$. Here, since the boundary velocity

$$1/\Delta t\int_0^{\Delta t} u_1(s)ds \tag{10.20}$$

satisfies

$$u_1^m \le \frac{1}{\Delta t}\int_0^{\Delta t} u_1(s)ds \le u_1^M \tag{10.21}$$

and $U^\Delta(0,x) \in \Sigma_1$, we obtain

$$U^\Delta(t,x) \in \Sigma_1 \qquad (x > x_1^\Delta(t),\ 0 < t < \Delta t) \tag{10.22}$$

by Lemmas 8, 9. The waves which appear in the construction of $U^\Delta(t,x)$ can not meet for $0 < t < \Delta t$ because the absolute values of the velocities are smaller than $\Lambda_1 = \Delta x/\Delta t$ and the start points of the waves separate with, at least, $2\Delta x$ distance.

3rd Step We define $U^\Delta(t,x)$ for $[n\Delta t,(n+1)\Delta t)$ inductively, where $n = 1,2,\ldots$. Suppose that $U^\Delta(t,x)$ is defined in a region

$$\{(t,x); x \ge x_1^\Delta(t),\ 0 \le t < n\Delta t\} \tag{10.23}$$

and satisfies

$$U^\Delta(t,x) \in \Sigma_1 \tag{10.24}$$

in the region. We take an integer $j_0 \in J_1^n$ such that

$$x_1^{\Delta}(n\Delta t) = x_1(n\Delta t) \in ((j_0 - 3)\Delta x, (j_0 - 1)\Delta x], \tag{10.25}$$

and we set

$$\begin{aligned} \tilde{E}_j^n &= \begin{cases} (x_1^{\Delta}(n\Delta t), (j_0 + 1)\Delta t) & (j = j_0), \\ E_j^n & (j \in J_1^n,\ j \geq j_0 + 2), \end{cases} \\ \tilde{J}_1^n &= J_1^n \cap \{j \in Z; j \geq j_0\}. \end{aligned} \tag{10.26}$$

We define $U^{\Delta}(n\Delta t, x)$ as the step function

$$U^{\Delta}(n\Delta t, x) = \bar{U}_j^n = \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U^{\Delta}(n\Delta t - 0, y) dy \qquad (x \in \tilde{E}_j^n,\ j \in \tilde{J}_1^n). \tag{10.27}$$

In the region

$$\{(t, x); x_1^{\Delta}(t) < x < j_0\Delta x,\ n\Delta t < t < (n+1)\Delta t\}, \tag{10.28}$$

since $U^{\Delta}(n\Delta t, x)$ equals a constant state $\bar{U}_{j_0}^n$ and $x = x_1^{\Delta}(t)$ is a line with a velocity

$$\frac{x_1((n+1)\Delta t) - x_1(n\Delta t)}{\Delta t} = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} u_1(s) ds, \tag{10.29}$$

we define $U^{\Delta}(t, x)$ as the solution of Problem (8.67) similarly in Step 10.1. And in the region

$$(n\Delta t, (n+1)\Delta t) \times E_j^{n+1} \qquad (j \in J_1^{n+1},\ j \geq j_0 + 1), \tag{10.30}$$

since

$$U^{\Delta}(n\Delta t, x) = \begin{cases} \bar{U}_{j-1}^n & ((j-1)\Delta x < x < j\Delta x), \\ \bar{U}_{j+1}^n & (j\Delta x < x < (j+1)\Delta x), \end{cases} \tag{10.31}$$

we define $U^{\Delta}(t, x)$ as the solution of the Riemann problem centered at $x = j\Delta x$. By Lemma 11,

$$\bar{U}_j^n \in \Sigma_1 \qquad (j \in J_1^n) \tag{10.32}$$

and the velocity of the boundary

$$\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} u_1(s) ds \tag{10.33}$$

satisfies

$$u_1^m \leq \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} u_1(s)ds \leq u_1^M, \tag{10.34}$$

it follows that

$$U^\Delta(t,x) \in \Sigma_1 \tag{10.35}$$

by Lemmas 8, 9 and by the assumption (10.3). The waves which appear here can not meet since $U^\Delta(t,x) \in \Sigma_1$ and the start positions of the waves have distances which are not less than $2\Delta x$.

Therefore, by repeating Step 10.1, we can construct the approximate solution $U^\Delta(t,x)$ in the region

$$\{(t,x); x \geq x_1^\Delta(t),\ t \geq 0\} \tag{10.36}$$

and $U^\Delta(t,x) \in \Sigma_1$ in this region.

For $t < 0$, or $x < x_1^\Delta(t)$, we define $U^\Delta(t,x)$ by

$$U^\Delta(t,x) = {}^t(0,0). \tag{10.37}$$

PROPOSITION 12 *We can construct the approximate solution of Problem* (I) *by such a way and $U^\Delta(t,x)$ has the uniformly bounded estimates*

$$0 \leq \rho^\Delta(t,x) \leq C, \quad |u^\Delta(t,x)| \leq C \qquad \text{for } x > x_1^\Delta(t),\ t > 0. \tag{10.38}$$

10.2 Problem (II)

Since the boundaries $x_1(t)$ and $x_2(t)$ are Lipschitz continuous,

$$u_1(t) = x_1'(t), \qquad u_2(t) = x_2'(t) \tag{10.39}$$

are in $L^\infty(0,\infty)$. Thus, we suppose

$$0 \leq \rho_0(x) \leq \rho_M, \qquad |u_0(x)| \leq u_M, \qquad u_1^m \leq u_1(t) \leq A, \qquad B \leq u_2(t) \leq u_2^M, \tag{10.40}$$

where ρ_M, u_M, u_1^m and u_2^M are constants,

$$A = \operatorname*{ess.\,sup}_{t>0} u_1(t), \qquad B = \operatorname*{ess.\,inf}_{t>0} u_2(t) \tag{10.41}$$

We consider the two cases of $A \leq B$ and $A > B$ separately.

10.2.1 In the case $A \leq B$

We fix the values w_1 and z_1 such that

$$z_1 \leq u_1^m \leq A \leq \frac{w_1 + z_1}{2} \leq B \leq u_2^M \leq w_1, \qquad u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta \leq \min\{w_1, -z_1\}. \tag{10.42}$$

We can take such values w_1 and z_1 by the following. Let

$$P = \min\left\{u_1^m, -u_M - \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta\right\}, \qquad Q = \max\left\{u_2^M, u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta\right\}. \tag{10.43}$$

Then, since we may take w_1 and z_1 such that

$$z_1 \leq P, \quad w_1 \geq Q, \quad 2A \leq w_1 + z_1 \leq 2B, \tag{10.44}$$

if $P + Q \geq 2B$, we fix w_1 such that $w_1 \geq Q$ and we may take z_1 such that

$$2A - w_1 \leq z_1 \leq 2B - w_1. \tag{10.45}$$

And if $P + Q < 2B$, we fix z_1 such that $z_1 \leq P$ and we may take w_1 such that

$$\max\{Q, 2A - z_1\} \leq w_1 \leq 2B - z_1. \tag{10.46}$$

In the last case, we can take w_1 because

$$Q < 2B - P \leq 2B - z_1. \tag{10.47}$$

Since $A \leq B$, the distance between the two pistons

$$x_2(t) - x_1(t) = \int_0^t (u_2(s) - u_1(s))ds + x_2(0) - x_1(0) \geq x_2(0) - x_1(0) + (B - A)t \tag{10.48}$$

increases.

Let

$$\Sigma_2 = \Sigma(w_1, z_1) \tag{10.49}$$

and fix the constant Λ_2 such that

$$\Lambda_2 > \Lambda(\Sigma_2). \tag{10.50}$$

We set sufficiently small Δx and Δt such that

$$\frac{\Delta x}{\Delta t} = \Lambda_2. \tag{10.51}$$

For simplicity, we suppose $x_1(0) = 0$ and we set

$$J_2^n = \{j \in Z; n + j \text{ is odd}\}, \qquad E_j^n = ((j-1)\Delta x, (j+1)\Delta x) \qquad (j \in J_2^n). \tag{10.52}$$

And we construct the approximate boundaries $x_1(t)$, $x_2(t)$ with polygonal lines $x_1^\Delta(t)$, $x_2^\Delta(t)$

$$\begin{cases} x_1^\Delta(t) = x_1(n\Delta t) + \dfrac{t - n\Delta t}{\Delta t}\{x_1((n+1)\Delta t) - x_1(n\Delta t)\}, \\ x_2^\Delta(t) = x_2(n\Delta t) + \dfrac{t - n\Delta t}{\Delta t}\{x_2((n+1)\Delta t) - x_2(n\Delta t)\} \end{cases}$$
$$(n\Delta t < t < (n+1)\Delta t), \tag{10.53}$$
$$x_1^\Delta(0) = x_1(0) = 0,\ x_2^\Delta(0) = x_2(0),$$

respectively.

We define $U^\Delta(t,x)$ for $n\Delta t \le t < (n+1)\Delta t$ inductively, where $n = 0, 1, \ldots$, similarly in the section 10.1.

Suppose that $U^\Delta(t,x)$ is defined in a region

$$\{(t,x); x_1^\Delta(t) < x < x_2^\Delta(t),\ 0 \le t < n\Delta t\} \tag{10.54}$$

and satisfies

$$U^\Delta(t,x) \in \Sigma_2 \tag{10.55}$$

in this region.

Now, we take integers $j_0,\ j_1 \in J_2^n$ such that

$$\begin{cases} x_1^\Delta(n\Delta t) = x_1(n\Delta t) \in ((j_0 - 3)\Delta x, (j_0 - 1)\Delta x], \\ x_2^\Delta(n\Delta t) = x_2(n\Delta t) \in [(j_1 + 1)\Delta x, (j_1 + 3)\Delta x). \end{cases} \tag{10.56}$$

Since

$$x_2(n\Delta t) - x_1(n\Delta t) \ge x_2(0) - x_1(0) + (B - A)n\Delta t, \tag{10.57}$$

it follows that $j_0 < j_1$ for sufficiently small Δx. Let

$$\tilde{E}_j^n = \begin{cases} (x_1^\Delta(n\Delta t), (j_0+1)\Delta x) & (j = j_0), \\ E_j^n & (j_0 < j < j_1,\ j \in J_2^n), \\ ((j_1 - 1)\Delta x, x_2^\Delta(n\Delta t)) & (j = j_1), \end{cases}$$
$$\tilde{J}_2^n = J_2^n \cap \{j \in Z; j_0 \le j \le j_1\}. \tag{10.58}$$

First, we define $U^\Delta(n\Delta t, x)$ by the step function

$$U^\Delta(n\Delta t, x) = \bar{U}_j^n = \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U^\Delta(n\Delta t - 0, y)dy \qquad (x \in \tilde{E}_j^n,\ j \in \tilde{J}_2^n). \tag{10.59}$$

Then, it follows

$$U^\Delta(n\Delta t, x) \in \Sigma_2 \tag{10.60}$$

by Lemma 11.

Next, we define $U^\Delta(t,x)$ as the solutions of the problems (8.67), (8.68) and the Riemann problems similarly to Problem (I). That is, $U^\Delta(t,x)$ is defined by the solution of the problem (8.67) in the region

$$\{(t,x); x_1^\Delta(t) < x < j_0\Delta x,\ n\Delta t < t < (n+1)\Delta t\}, \tag{10.61}$$

defined by the solution of the Riemann problems in regions

$$(n\Delta t, (n+1)\Delta t) \times (j\Delta x, (j+2)\Delta x) \qquad (j = j_0, j_0+2, \ldots, j_1-2), \tag{10.62}$$

and defined by the solution of the problem (8.68) in the region

$$\{(t,x); j_1\Delta x < x < x_2^\Delta(t),\ n\Delta t < t < (n+1)\Delta t\}. \tag{10.63}$$

Since the left boundary velocity

$$\frac{x_1((n+1)\Delta t) - x_1(n\Delta t)}{\Delta t} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} u_1(s)ds, \tag{10.64}$$

and the right boundary velocity

$$\frac{x_2((n+1)\Delta t) - x_2(n\Delta t)}{\Delta t} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} u_2(s)ds, \tag{10.65}$$

satisfy

$$u_1^m \le \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} u_1(s)ds \le A, \qquad B \le \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} u_2(s)ds \le u_2^M, \tag{10.66}$$

it follows

$$U^\Delta(t,x) \in \Sigma_2 \tag{10.67}$$

for $x_1^\Delta(t) < x < x_2^\Delta(t)$, $n\Delta t < t < (n+1)\Delta t$ by Lemmas 8, 9.

We can construct the approximate solution $U^\Delta(t,x)$ by repeating this step, globally.

And we define $U^\Delta(t,x)$ by

$$U^\Delta(t,x) = {}^t(0,0) \tag{10.68}$$

for $t < 0$, $x < x_1^\Delta(t)$, or $x > x_2^\Delta(t)$.

PROPOSITION 13 *We can construct the approximate solution $U^\Delta(t,x)$ of Problem (II) in the case $A \le B$ by such a way and $U^\Delta(t,x)$ has the uniformly bounded estimates*

$$0 \le \rho^\Delta(t,x) \le C, \quad |u^\Delta(t,x)| \le C \qquad (x_1^\Delta(t) < x < x_2^\Delta(t),\ t > 0). \tag{10.69}$$

10.2.2 In the case $A > B$

In this case, since we can not take the values w_1, z_1 as before, it is difficult that we obtain the approximate solution in a invariant region. Thus, we consider the sequence of the invariant regions and give the estimate for the growth of the regions.

Let

$$\delta(t) = \min_{s\in[0,t]}\{x_2(t) - x_1(t)\}. \tag{10.70}$$

Since $x_1(t)$ and $x_2(t)$ are Lipschitz continuous and

$$x_2(t) - x_1(t) > 0, \tag{10.71}$$

the function $\delta(t)$ is continuous, does not increase, and satisfies

$$x_2(t) - x_1(t) \geq \delta(t) > 0. \tag{10.72}$$

Though the construction of approximate solution is done by almost similar way in the case $A \leq B$, we may take the smaller value Δt step by step because the wave velocities may become faster with the growth the invariant region.

We take the values w_1, z_1 such that

$$\frac{w_1 + z_1}{2} = A, \quad u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^{\theta} \leq \min\{w_1, -z_1\}, \quad z_1 \leq u_1^m, \quad u_2^M \leq w_1. \tag{10.73}$$

In order to take the values, we may do similarly in the case $A \leq B$.

Let Δx be sufficiently small, and we fix the positive values k_0, k_1. We construct the approximate solution $U^{\Delta}(t, x)$ for $0 < t < T$, where T satisfies

$$(8 + k_0)\Delta x \leq \delta(T), \quad \Delta x(A - B)\int_0^T \frac{dt}{\delta(t)^2} \leq k_1. \tag{10.74}$$

Let $T(\Delta x)$ be the max value of the value which satisfies (10.74), then

$$\lim_{\Delta x\downarrow 0} T(\Delta x) = \infty \tag{10.75}$$

since $\delta(t) > 0$ for all $t > 0$. Thus, if we can construct the approximate solution to such a time T, we can construct for any time by taking a sufficiently small Δx.

The sequence of the increasing invariant regions are defined the following. We set w_m and z_m as

$$w_m = \begin{cases} w_{m-1} & (m \text{ is even}), \\ 2A - z_{m-1} & (m \text{ is odd}), \end{cases} \tag{10.76}$$

$$z_m = \begin{cases} 2B - w_{m-1} & (m \text{ is even}), \\ z_{m-1} & (m \text{ is odd}). \end{cases} \tag{10.77}$$

inductively. Then,

$$w_1 = w_2 \leq w_3 = w_4 \leq w_5 = \cdots, \qquad z_1 \geq z_2 = z_3 \geq z_4 = \cdots$$
$$\frac{w_m + z_m}{2} = \begin{cases} A & (m \text{ is odd}), \\ B & (m \text{ is even}), \end{cases} \tag{10.78}$$

and hence, for any odd m, we have

$$w_{m+1} = w_m = w_1 + (m-1)(A-B), \qquad z_m = z_{m-1} = z_1 - (m-1)(A-B). \tag{10.79}$$

We define the sequence of regions

$$\Sigma(w_1, z_1) \subset \Sigma(w_2, z_2) \subset \Sigma(w_3, z_3) \subset \cdots \tag{10.80}$$

Since (5.13) and

$$\max\{|\lambda_1(U)|, |\lambda_2(U)|\} = \max\{-\lambda_1(U), \lambda_2(U)\}, \tag{10.81}$$

we have

$$\Lambda(\Sigma(w_0, z_0)) = \max\{w_0, -z_0\}. \tag{10.82}$$

Let

$$\sigma_m = \Lambda(w_m, z_m). \tag{10.83}$$

Then, for odd m,

$$\sigma_m = \max\{w_m, -z_m\} = (m-1)(A-B) + w_1 - A + |A| \tag{10.84}$$

and for even m,

$$\sigma_m = (m-1)(A-B) + w_1 - A + |B|. \tag{10.85}$$

We fix $c_0 > 0$, and let $\bar{\sigma}_m$ be a value such that

$$\sigma_m < \bar{\sigma}_m < \sigma_m + c_0. \tag{10.86}$$

Under these assumptions, we defined the approximate solution $U^\Delta(t, x)$ inductively.
Let

$$t_n = \sum_{i=1}^{n} \Delta t_i \, (< T), \tag{10.87}$$

where each Δt_i is positive number, which is the t-direction mesh length for each step, and they satisfy

$$\Delta t_1 \geq \Delta t_2 \geq \Delta t_3 \geq \cdots. \tag{10.88}$$

The approximate boundaries $x_1^\Delta(t)$ and $x_2^\Delta(t)$ are defined by

$$\begin{cases} x_1^\Delta(t) = x_1(t_{k-1}) + \dfrac{t-t_{k-1}}{\Delta t_k}\{x_1(t_k) - x_1(t_{k-1})\}, \\ x_2^\Delta(t) = x_2(t_{k-1}) + \dfrac{t-t_{k-1}}{\Delta t_k}\{x_2(t_k) - x_2(t_{k-1})\} \end{cases}$$
$$(t_{k-1} < t \leq t_k,\ k = 1, 2, \ldots, n),$$
$$x_1^\Delta(0) = x_1(0) = 0, \quad x_2^\Delta(0) = x_2(0) \tag{10.89}$$

and we suppose the approximate solution $U^\Delta(t,x)$ is defined in the region

$$\{(t,x); x_1^\Delta(t) < x < x_2^\Delta(t),\ 0 \leq t < t_n\}. \tag{10.90}$$

We define $U^\Delta(t_n, x)$ as the step function

$$U^\Delta(t_n, x) = \bar{U}_j^n = \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U^\Delta(t_n - 0, y)dy \qquad (x \in \tilde{E}_j^n,\ j \in \tilde{J}_2^n), \tag{10.91}$$

where $j_0 = j_0^n$, $j_1 = j_1^n$ are in

$$J_2^n = \{j \in Z; n + j \text{ is odd}\} \tag{10.92}$$

such that

$$x_1^\Delta(t_n) \in ((j_0^n - 3)\Delta x, (j_0^n - 1)\Delta x], \qquad x_2^\Delta(t_n) \in [(j_1^n + 1)\Delta x, (j_1^n + 3)\Delta x) \tag{10.93}$$

and we set $\tilde{E}_j^n$, $\tilde{J}_2^n$, as

$$\tilde{E}_j^n = \begin{cases} (x_1^\Delta(t_n), (j_0^n + 1)\Delta x) & (j = j_0^n), \\ ((j-1)\Delta x, (j+1)\Delta x) & (j \in J_2^n, j_0^n < j < j_1^n), \\ ((j_1^n - 1)\Delta x, x_2^\Delta(t_n)) & (j = j_1^n), \end{cases}$$
$$\tilde{J}_2^n = J_2^n \cap \{j \in Z; j_0^n \leq j \leq j_1^n\}. \tag{10.94}$$

We take Δt_{n+1} sufficiently small. Then, we can construct $x_1^\Delta(t)$ and $x_2^\Delta(t)$ by

$$\begin{cases} x_1^\Delta(t) = x_1(t_n) + \dfrac{t-t_n}{\Delta t_{n+1}}\{x_1(t_{n+1}) - x_1(t_n)\}, \\ x_2^\Delta(t) = x_2(t_n) + \dfrac{t-t_n}{\Delta t_{n+1}}\{x_2(t_{n+1}) - x_2(t_n)\} \end{cases}$$
$$(t_n < t < t_{n+1} = t_n + \Delta t_{n+1}) \tag{10.95}$$

and we can define $U^\Delta(t,x)$ by the solutions of the problems (8.67), (8.68) and of the Riemann problems for $t_n < t < t_n + \Delta t_n = t_{n+1}$.

We must take Δt_n such that the simple waves do not spread over the range Δx. In order to show that we can construct $U^\Delta(t,x)$ for $0 < t < T$, we consider the estimate of $U^\Delta(t,x)$ and decide Δt_i.

From the definition of w_1 and z_1 by Lemma 11,

$$U_0(x) \in \Sigma(w_1, z_1), \quad U^\Delta(0,x) \in \Sigma(w_1, z_1) \qquad (0 < x < x_2^\Delta(0)). \tag{10.96}$$

For $0 < t < t_1$, since

$$z_1 \leq u_1^m \leq A = \frac{w_1 + z_1}{2}, \tag{10.97}$$

$\Sigma(w_1, z_1)$ is the invariant region of the solution of the problem (8.67) near the left boundary $x_1^\Delta(t)$ and the Riemann problem by Lemmas 8, 9. Thus, the value U^Δ which may be outside $\Sigma(w_1, z_1)$ are the solution of the problem (8.68) near the right boundary. However, from

$$\frac{w_2 + z_2}{2} = B \leq u_2^M \leq w_1 = w_2, \tag{10.98}$$

the solution is in $\Sigma(w_2, z_2)$ by Lemma 9. More precisely, the wave which lies in the solution start from $x = x_2(0)$ and the velocity of the wave is faster than $-\sigma_2$.

Thus, it follows that

$$\begin{cases} U^\Delta(t,x) \in \Sigma(w_1, z_1) & (x_1^\Delta(t) < x < x_2(0) - \bar{\sigma}_2 t,\ 0 < t < t_1), \\ U^\Delta(t,x) \in \Sigma(w_2, z_2) & (x_2(0) - \bar{\sigma}_2 t < x < x_2^\Delta(t),\ 0 < t < t_1). \end{cases} \tag{10.99}$$

Since $\Sigma(w_2, z_2) \supset \Sigma(w_1, z_1)$, we may take Δt_1 such that

$$\frac{\Delta x}{\Delta t_1} = \bar{\sigma}_2. \tag{10.100}$$

On $t = t_1$, the values of the step function $U^\Delta(t_1, x)$ are in $\Sigma(w_1, z_1)$ except the values on the right end interval in $\{\tilde{E}_j^n\}_{j \in \tilde{J}_2^n}$, and the values are in $\Sigma(w_2, z_2)$. That is,

$$\bar{U}_j^1 \in \Sigma(w_1, z_1) \ \ (j \in \tilde{J}_2^1,\ j < j_1^1), \qquad \bar{U}_{j_1^1}^1 \in \Sigma(w_2, z_2). \tag{10.101}$$

The integer j_1^1 which decide the right end interval

$$\tilde{E}_{j_1^1}^1 = ((j_1^1 - 1)\Delta x, x_2^\Delta(t_1)) \tag{10.102}$$

is defined by

$$x_2^\Delta(t_1) = x_2(t_1) \in [(j_1^1 + 1)\Delta x, (j_1^1 + 3)\Delta x). \tag{10.103}$$

Thus,

$$(j_1^1 - 1)\Delta x > x_2(t_1) - 4\Delta x = x_2(0) + \int_0^{t_1} u_2(s) ds - 4\Delta x \geq x_2(0) + B t_1 - 4\Delta x. \tag{10.104}$$

Since

$$B = \frac{w_2 + z_2}{2} \geq z_2 \geq -\bar{\sigma}_2, \tag{10.105}$$

it follows

$$(j_1^1 - 1)\Delta x \geq x_2(0) - 4\Delta x - \bar{\sigma}_2 t_1. \tag{10.106}$$

Therefore, we have

$$\begin{cases} U^\Delta(t_1,x) \in \Sigma(w_1,z_1) & (x \le x_2(0) - 4\Delta x - \bar{\sigma}_2 t_1), \\ U^\Delta(t_1,x) \in \Sigma(w_2,z_2) & (x \ge x_2(0) - 4\Delta x - \bar{\sigma}_2 t_1). \end{cases} \tag{10.107}$$

For $t_1 < t < t_2$, the value $U^\Delta(t,x)$ which is determined independently with the values on the interval $\tilde{E}^1_{j^1_1}$ is in $\Sigma(w_1,z_1)$. In the region

$$(t_1,t_2) \times ((j^1_1 - 2)\Delta x, j^1_1 \Delta x), \tag{10.108}$$

$U^\Delta(t,x)$ has a relation with the values on the interval $\tilde{E}^1_{j^1_1}$, and hence, the value $U^\Delta(t,x)$ is in $\Sigma(w_2,z_2)$. More precisely, the value $U^\Delta(t,x)$ is in $\Sigma(w_1,z_1)$ for $x \le (j^1_1 - 1)\Delta x - \bar{\sigma}_2(t - t_1)$ because the value is equal to $\bar{U}^1_{j^1_1-2}$.

Since

$$(j^1_1 - 1)\Delta x - \bar{\sigma}_2(t-t_1) \ge x_2(0) - 4\Delta x - \bar{\sigma}_2 t_1 - \bar{\sigma}_2(t - t_1) = x_2(0) - 4\Delta x - \bar{\sigma}_2 t, \tag{10.109}$$

we have

$$\begin{cases} U^\Delta(t,x) \in \Sigma(w_1,z_1) & (x^\Delta_1(t) < x < x_2(0) - 4\Delta x - \bar{\sigma}_2 t), \\ U^\Delta(t,x) \in \Sigma(w_2,z_2) & (x_2(0) - 4\Delta x - \bar{\sigma}_2 t < x < x^\Delta_2(t)) \end{cases} \tag{10.110}$$

for $0 < t < t_2$. We may take Δt_2 such that

$$\frac{\Delta x}{\Delta t_2} = \bar{\sigma}_2. \tag{10.111}$$

The line

$$x = l_1(t) = x_2(0) - 4\Delta x - \bar{\sigma}_2 t \tag{10.112}$$

shows the propagation of the influence of the right boundary. Thus, we call this the *line of influence*. According to the above argument, we can set Δt_i such that

$$\frac{\Delta x}{\Delta t_i} = \bar{\sigma}_2 \tag{10.113}$$

and we obtain

$$\begin{cases} U^\Delta(t,x) \in \Sigma(w_1,z_1) & (x^\Delta_1(t) < x < l_1(t)), \\ U^\Delta(t,x) \in \Sigma(w_2,z_2) & (l_1(t) < x < x^\Delta_2(t)) \end{cases} \tag{10.114}$$

until the line of influence comes near the left boundary $x = x^\Delta_1(t)$.

The time when the influence comes to left is t_n such that

$$\tilde{E}^n_{j^n_0} \ni l_1(t_n) \tag{10.115}$$

firstly. Let $T_1 = t_n$. Then,

$$l_1(T_1) < (j^n_0 + 1)\Delta x < x_1(T_1) + 4\Delta x \tag{10.116}$$

in this time. Since

$$\begin{aligned}
&x_1(T_1) = \int_0^{T_1} u_1(s)ds \le AT_1, \\
&l_1(T_1) = x_2(0) - 4\Delta x - \bar{\sigma}_2 T_1, \\
&A + \bar{\sigma}_2 > A + \sigma_2 > B + \sigma_2 = w_1 + |B| \ge u_M + \frac{\sqrt{a\gamma}}{\theta}\rho_M^\theta + |B| \ge 0,
\end{aligned} \tag{10.117}$$

we have

$$T_1 > \frac{x_2(0) - 8\Delta x}{A + \bar{\sigma}_2}. \tag{10.118}$$

The line of influence are defined by

$$l_2(t) = x_1(T_1) + 4\Delta x + \bar{\sigma}_3(t - T_1), \tag{10.119}$$

and then we have

$$\begin{aligned}
&\begin{cases} U^\Delta(t,x) \in \Sigma(w_2, z_2) & (x \ge l_2(t), T_1 < t < T_1 + \Delta t_{n+1}), \\ U^\Delta(t,x) \in \Sigma(w_3, z_3) & (x \le l_2(t), T_1 < t < T_1 + \Delta t_{n+1}) \end{cases} \\
&\Delta t_{n+1} = \Delta x / \bar{\sigma}_3
\end{aligned} \tag{10.120}$$

We repeat this way. Let T_m be the time when the effect with the line of influence $x = l_m(t)$ propagate to the opposite boundary. Then,

$$\begin{aligned}
&0 = T_0 < T_1 < T_2 < T_3 < \cdots, \\
&l_m(t) = \begin{cases} x_2(T_{m-1}) - 4\Delta x - \bar{\sigma}_{m+1}(t - T_{m-1}) & (m \text{ is odd}), \\ x_1(T_{m-1}) + 4\Delta x + \bar{\sigma}_{m+1}(t - T_{m-1}) & (m \text{ is even}). \end{cases}
\end{aligned} \tag{10.121}$$

If m is odd,

$$l_m(T_m) < (j_0^n + 1)\Delta x < x_1(T_m) + 4\Delta x < x_1(T_{m-1}) + A(T_m - T_{m-1}) + 4\Delta x, \tag{10.122}$$

and we have

$$T_m > T_{m-1} + \frac{x_2(T_{m-1}) - x_1(T_{m-1}) - 8\Delta x}{A + \bar{\sigma}_{m+1}}. \tag{10.123}$$

If m is even,

$$l_m(T_m) > (j_1^n - 1)\Delta x > x_2(T_m) - 4\Delta x > x_2(T_{m-1}) + B(T_m - T_{m-1}) - 4\Delta x, \tag{10.124}$$

and we have

$$T_m > T_{m-1} + \frac{x_2(T_{m-1}) - x_1(T_{m-1}) - 8\Delta x}{\bar{\sigma}_{m+1} - B} \tag{10.125}$$

since

$$\bar{\sigma}_{m+1} - B > \sigma_{m+1} - B \ge \sigma_1 - B = w_1 - A + |A| - B \ge u_2^M - B + |A| - A \ge 0. \tag{10.126}$$

For $T_{m-1} < t < T_m$, we may take Δt_i such that

$$\frac{\Delta x}{\Delta t_k} = \bar{\sigma}_{m+1} \tag{10.127}$$

and we obtain

$$\begin{cases} U^{\Delta}(t,x) \in \Sigma(w_m, z_m) & (\text{under } x = l_m(t),\ T_{m-1} < t < T_m), \\ U^{\Delta}(t,x) \in \Sigma(w_{m+1}, z_{m+1}) & (\text{over } x = l_m(t)\, T_{m-1} < t < T_m). \end{cases} \tag{10.128}$$

We show that T_m reach the time T.

It follows

$$\bar{\sigma}_{m+1} + A < \sigma_{m+1} + c_0 + A < m(A-B) + c_1 \qquad (c_1 = w_1 - A + |A| + |B|) \tag{10.129}$$

if m is odd, and

$$\bar{\sigma}_{m+1} - B < \sigma_{m+1} + c_0 - B < m(A-B) + c_1 \tag{10.130}$$

if m is even. Hence, we obtain

$$T_m > T_{m-1} + \frac{\delta(T_{m-1}) - 8\Delta x}{m(A-B) + c_1} \tag{10.131}$$

for any $m \geq 1$. Since

$$\delta(t) \geq (8 + k_0)\Delta x > 8\Delta x \tag{10.132}$$

for $0 \leq t \leq T$, we have

$$\sum_{j=1}^{m} \frac{1}{j(A-B) + c_1} \leq \sum_{j=1}^{m} \frac{T_j - T_{j-1}}{\delta(T_{j-1}) - 8\Delta x} \leq \int_0^{T_m} \frac{dt}{\delta(t) - 8\Delta x} \tag{10.133}$$

for $T_m \leq T$. Since

$$\delta(t) - 8\Delta x = \frac{8}{8+k_0}\{\delta(t) - (8+k_0)\Delta x\} + \frac{k_0}{8+k_0}\delta(t) \geq \frac{k_0}{8+k_0}\delta(t) \tag{10.134}$$

by the assumption for T, we obtain

$$\frac{1}{\delta(t) - 8\Delta x} = \frac{1}{\delta(t)} + \frac{8\Delta x}{\delta(t)(\delta(t) - 8\Delta x)} \leq \frac{1}{\delta(t)} + 8\Delta x \frac{8+k_0}{k_0} \frac{1}{\delta(t)^2}. \tag{10.135}$$

On the other hand, it follows that

$$\sum_{j=1}^{m} \frac{1}{j(A-B) + c_1} \geq \int_1^{m+1} \frac{dx}{x(A-B) + c_1} = \frac{1}{A-B} \log \frac{(m+1)(A-B) + c_1}{A - B + c_1}. \tag{10.136}$$

Thus, (10.133) gives

$$\begin{aligned} \frac{1}{A-B} \log \frac{(m+1)(A-B) + c_1}{A-B+c_1} &\leq \int_0^{T_m} \frac{dt}{\delta(t)} + 8\Delta x \frac{8+k_0}{k_0} \int_0^{T_m} \frac{dt}{\delta(t)^2} \\ &\leq \int_0^{T_m} \frac{dt}{\delta(t)} + 8 \frac{8+k_0}{k_0} \frac{k_1}{A-B}. \end{aligned} \tag{10.137}$$

Therefore, we obtain

$$(m+1)(A-B)+c_1 \leq c_2 \exp\left\{(A-B)\int_0^{T_m} \frac{dt}{\delta(t)}\right\} \tag{10.138}$$

where

$$c_2 = (A-B+c_1)\exp\left(8k_1\frac{8+k_0}{k_0}\right). \tag{10.139}$$

The left hand side tend to infinity as $m \to \infty$. Thus, it does not occur that T_m can not take a larger value than $T-\varepsilon$ with some positive number ε. Therefore, we can construct the approximate solution for $0<t<T$ in this way.

We define $U^{\Delta}(t,x)$ by ${}^t(0,0)$ outside

$$\{(t,x); x_1^{\Delta}(t) < x < x_2^{\Delta}(t),\, 0 \leq t < T\}. \tag{10.140}$$

Consider the bounded estimate for the approximate solution. Since

$$U^{\Delta}(t,x) \in \Sigma(w_{m+1}, -z_{m+1}) \subset \Sigma(w_{m+2}, -z_{m+2}) \tag{10.141}$$

for $0<t<T$, we have

$$\begin{aligned} |u^{\Delta}(t,x)| &\leq \max_{U\in\Sigma(w_{m+2},-z_{m+2})} \frac{|w+z|}{2} \leq \sigma_{m+2}, \\ 0 \leq \rho^{\Delta}(t,x) &\leq \max_{U\in\Sigma(w_{m+2},-z_{m+2})} \left(\frac{\theta}{\sqrt{a\gamma}}\frac{w-z}{2}\right)^{1/\theta} \leq \left(\frac{\theta\sigma_{m+2}}{\sqrt{a\gamma}}\right)^{1/\theta}. \end{aligned} \tag{10.142}$$

Now, since

$$\sigma_{m+2} = \max\{w_{m+2}, -z_{m+2}\} \leq (m+1)(A-B)+c_1 \leq c_2\exp\left\{(A-B)\int_0^T \frac{dt}{\delta(t)}\right\}, \tag{10.143}$$

it follows the following proposition.

PROPOSITION 14 . *In the case $A>B$, for sufficiently small Δx, there is the positive number T such that the approximate boundary $x_1^{\Delta}(t)$, $x_2^{\Delta}(t)$ and the approximate solution $U^{\Delta}(t,x)$ for Problem (II) is constructed for $0<t<T$, and $U^{\Delta}(t,x)$ satisfies*

$$0 \leq \rho^{\Delta}(t,x) \leq C_1(T), \quad |u^{\Delta}(t,x)| \leq C_2(T) \qquad (x_1^{\Delta}(t) < x < x_2^{\Delta}(t),\, 0<t<T), \tag{10.144}$$

where $C_1(T)$ and $C_2(T)$ satisfy

$$C_1(T)^{\theta}, C_2(T) \leq C_3\exp\left\{(A-B)\int_0^T \frac{dt}{\delta(t)}\right\}. \tag{10.145}$$

We can take T greater than the any number by taking Δx sufficiently small.

10.3 Problem (III)

We suppose

$$0 \leq \rho_0(x) \leq \rho_M, \qquad |u_0(x)| \leq u_M, \tag{10.146}$$

where ρ_M and u_M are constants. We fix w_1, z_1, w_2 and z_2 such that

$$\begin{aligned} &u_M + \frac{\sqrt{a_1\gamma_1}}{\theta_1}\rho_M^{\theta_1} \leq \min\{w_1, -z_1\}, \qquad u_M + \frac{\sqrt{a_2\gamma_2}}{\theta_2}\rho_M^{\theta_2} \leq \min\{w_2, -z_2\}, \\ &\frac{w_1+z_1}{2} \leq z_2 \leq x_0' \leq w_1 \leq \frac{w_2+z_2}{2} \\ &\qquad (\theta_j = \frac{\gamma_j - 1}{2}) \end{aligned} \tag{10.147}$$

We note that we can always take these values in order to satisfy the relations (10.147). Because, first, we can take w_1 and z_2 such that

$$w_1 \geq \max\left\{x_0',\, u_M + \frac{\sqrt{a_1\gamma_1}}{\theta_1}\rho_M^{\theta_1}\right\}, \tag{10.148}$$

$$z_2 \leq \min\left\{x_0',\, -u_M - \frac{\sqrt{a_2\gamma_2}}{\theta_2}\rho_M^{\theta_2}\right\}, \tag{10.149}$$

next, we may take w_2 and z_1 such that

$$w_2 \geq \max\left\{2w_1 - z_2,\, u_M + \frac{\sqrt{a_2\gamma_2}}{\theta_2}\rho_M^{\theta_2}\right\}, \tag{10.150}$$

$$z_1 \leq \min\left\{2z_2 - w_1,\, -u_M - \frac{\sqrt{a_1\gamma_1}}{\theta_1}\rho_M^{\theta_1}\right\}. \tag{10.151}$$

Let

$$\Sigma_1 = \Sigma(w_1, z_1; P_1), \qquad \Sigma_2 = \Sigma(w_2, z_2; P_2) \tag{10.152}$$

and we fix a constant Λ_3 such that

$$\Lambda_3 > \max\{\Lambda(\Sigma_1; P_1), \Lambda(\Sigma_2; P_2)\}. \tag{10.153}$$

We set

$$\frac{\Delta x}{\Delta t} = \Lambda_3. \tag{10.154}$$

For simplicity, we suppose $x_0 = 0$ and let

$$J_3^n = \{j \in Z; n+j \text{ is odd}\}, \qquad E_j^n = ((j-1)\Delta x, (j+1)\Delta x) \quad (j \in J_3^n). \tag{10.155}$$

We define the approximate solution $U^\Delta(t,x)$ and the approximate free boundary $x^\Delta(t)$ inductively.

Suppose that $U^\Delta(t,x)$ is defined in

$$\{(t,x); 0 \le t < n\Delta t,\ x \in R\} \tag{10.156}$$

and satisfies

$$\begin{cases} U^\Delta(t,x) \in \Sigma_1 & (x < x^\Delta(t),\ 0 \le t < n\Delta t), \\ U^\Delta(t,x) \in \Sigma_2 & (x > x^\Delta(t),\ 0 \le t < n\Delta t). \end{cases} \tag{10.157}$$

For the boundary, we suppose that $x^\Delta(t)$ is defined as polygonal line

$$\begin{gathered} x^\Delta(t) = x^\Delta((k-1)\Delta t) + \frac{t-(k-1)\Delta t}{\Delta t}\{x^\Delta(k\Delta t) - x^\Delta((k-1)\Delta t)\} \\ ((k-1)\Delta t \le t < k\Delta t,\ k = 1, 2, \dots, n) \end{gathered} \tag{10.158}$$

and satisfies

$$z_2 \le \frac{d}{dt}x^\Delta(t) \le w_1 \tag{10.159}$$

for $0 < t < n\Delta t$.

First, we take an integer $j_0 = j_0^n \in J_3^n$ such that

$$x^\Delta(n\Delta t) \in ((j_0-1)\Delta x, (j_0+1)\Delta x]. \tag{10.160}$$

If $x^\Delta(n\Delta t) = (j_0+1)\Delta x$, then we set

$$\begin{array}{ll} \tilde{J}_{3,-}^n = J_3^n \cap \{j \in Z; j \le j_0\}, & j_- = j_0, \\ \tilde{J}_{3,+}^n = J_3^n \cap \{j \in Z; j \ge j_0+2\}, & j_+ = j_0+2, \\ \tilde{E}_j^n = E_j^n \quad (j \in J_3^n), & \end{array} \tag{10.161}$$

and if $x^\Delta(n\Delta t) \in ((j_0-1)\Delta x, (j_0+1)\Delta x) = E_{j_0}^n$, then we set

$$\begin{array}{ll} \tilde{J}_{3,-}^n = J_3^n \cap \{j \in Z; j \le j_0\}, & j_- = j_0 - 2, \\ \tilde{J}_{3,+}^n = J_3^n \cap \{j \in Z; j \ge j_0+2\}, & j_+ = j_0+2 \\ \tilde{E}_j^n = \begin{cases} E_j^n & (j < j_-, j > j_+), \\ ((j_- - 1)\Delta x,\ x^\Delta(n\Delta t)) & (j = j_-), \\ (x^\Delta(n\Delta t),\ (j_+ + 1)\Delta x) & (j = j_+). \end{cases} & \end{array} \tag{10.162}$$

We define $U^\Delta(n\Delta t, x)$ as the step function

$$U^\Delta(n\Delta t, x) = \bar{U}_j^n = \frac{1}{m(\tilde{E}_j^n)}\int_{\tilde{E}_j^n} U^\Delta(n\Delta t - 0, y)dy \qquad (x \in \tilde{E}_j^n,\ j \in \tilde{J}_{3,-}^n \cup \tilde{J}_{3,+}^n) \tag{10.163}$$

Then,

$$\begin{cases} U^\Delta(n\Delta t,x) \in \Sigma_1 & (x < x^\Delta(n\Delta t)), \\ U^\Delta(n\Delta t,x) \in \Sigma_2 & (x > x^\Delta(n\Delta t)), \end{cases} \tag{10.164}$$

from Lemma 11. In regions

$$\begin{aligned}&(n\Delta t,(n+1)\Delta t)\times((j-2)\Delta x,j\Delta x)\quad(j\in\tilde{J}^n_{3,-}),\\&(n\Delta t,(n+1)\Delta t)\times(j\Delta x,(j+2)\Delta x)\quad(j\in\tilde{J}^n_{3,+}),\end{aligned}\tag{10.165}$$

we define $U^\Delta(t,x)$ as the solution of the Riemann problem, and in the region

$$(n\Delta t,(n+1)\Delta t)\times(j_-\Delta x,j_+\Delta x),\tag{10.166}$$

we define $U^\Delta(t,x)$ as the solution of the problem (8.76), where

$$u_0=[\frac{d}{dt}x^\Delta(t)]_{t=n\Delta t-0}=\frac{x^\Delta(n\Delta t)-x^\Delta((n-1)\Delta t)}{\Delta t},\qquad U_l=\bar{U}^n_{j_-},\ \ U_r=\bar{U}^n_{j_+}.\tag{10.167}$$

And we define

$$x^\Delta(t)=x^\Delta(n\Delta t)+(t-n\Delta t)u_1\tag{10.168}$$

for $n\Delta t\le t<(n+1)\Delta t$.

It follows

$$\begin{cases}U^\Delta(t,x)\in\Sigma_1\quad(x<j_-\Delta x),\\U^\Delta(t,x)\in\Sigma_2\quad(x>j_+\Delta x)\end{cases}\tag{10.169}$$

from Lemma 8, and it follows

$$\begin{cases}U^\Delta(t,x)\in\Sigma_1\quad(j_-\Delta x<x<x^\Delta(t)),\\U^\Delta(t,x)\in\Sigma_2\quad(x^\Delta(t)<x<j_+\Delta x)\end{cases}\tag{10.170}$$

and

$$z_2\le u_1\le w_1,\tag{10.171}$$

from Lemma 10. That is,

$$z_2\le\frac{d}{dt}x^\Delta(t)\le w_1\qquad(n\Delta t<t<(n+1)\Delta t).\tag{10.172}$$

PROPOSITION 15 *We can construct the approximate solution $U^\Delta(t,x)$ and the approximate free boundary $x^\Delta(t)$ for Problem (III), and they have the uniformly bounded estimates*

$$\begin{aligned}&0\le\rho^\Delta(t,x)\le C,\ \ |u^\Delta(t,x)|\le C\qquad(t>0,\ x\in R),\\&\left|\frac{d}{dt}x^\Delta(t)\right|\le C\qquad(t>0).\end{aligned}\tag{10.173}$$

10.4 Problem (IV)

The reason why there is the outer force term in the equation for the Problem (IV), we must change how to construct the approximate solution for another problems. We user a method which is well-known in numerical calculation and is called the fractional step method (cf. [6],[18]).

We suppose that

$$0 \leq \rho_0(r) \leq \rho_M, \qquad |u_0(r)| \leq u_M \tag{10.174}$$

where ρ_M and u_M are constants, and Δr, Δt are sufficiently small. Let

$$J_4^n = \{j \in Z; n+j \text{ is odd}\}, \quad E_j^n = (1+(j-1)\Delta r, 1+(j+1)\Delta r) \qquad (j \in J_4^n) \tag{10.175}$$

We define the approximate solution $U^\Delta(t,r)$ inductively.

Suppose that $U^\Delta(t,r)$ is defined in

$$\{(t,r); r>1,\, 0 \leq t < n\Delta t\} \tag{10.176}$$

and Δt is so small that the waves appearing in the approximation do not meet one another. Let

$$\begin{aligned} &\tilde{J}_4^n = J_4^n \cap \{j \in Z; j \geq 1\}, \\ &\tilde{E}_j^n = \begin{cases} E_j^n \quad (j \in \tilde{J}_4^n) & (n \text{ is even}), \\ \begin{cases} (1, 1+3\Delta r) & (j=2), \\ E_j^n & (j \geq 4) \end{cases} & (n \text{ is odd}). \end{cases} \end{aligned} \tag{10.177}$$

We define $U^\Delta(n\Delta t, r)$ as the step function

$$U^\Delta(n\Delta t, r) = \bar{U}_j^n = \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U^\Delta(n\Delta t - 0, y)dy \qquad (r \in \tilde{E}_j^n,\, j \in \tilde{J}_4^n). \tag{10.178}$$

In the region

$$\{(t,r); r>1,\, n\Delta t < t < (n+1)\Delta t\}, \tag{10.179}$$

we set $U_0^\Delta(t,r)$ by the similar method for Problem (I), that is, in the region

$$(n\Delta t, (n+1)\Delta t) \times (1, 1+\Delta r), \tag{10.180}$$

$U_0^\Delta(t,r)$ is defined as the solution of the problem (8.67), and in the region

$$(n\Delta t, (n+1)\Delta t) \times (1+j\Delta r, 1+(j+2)\Delta r) \qquad (j \in \tilde{J}_4^n) \tag{10.181}$$

$U_0^\Delta(t,r)$ is defined as the solution of the Riemann problem. For this, we define $U^\Delta(t,r)$ by

$$U^\Delta(t,r) = U_0^\Delta(t,r) + H(r, U_0^\Delta(t,r))(t - n\Delta t) \qquad (n\Delta t \leq t < (n+1)\Delta t,\, r>1). \tag{10.182}$$

These step are the method called the *fractional step method.*

We consider the estimate for the approximate solution and obtain when we can not do this step. These method is by the paper [6], however, the global uniform boundedness of the approximation and the global existence of the weak solution is not given in the paper for general outer force.

Since

$$\rho^{\Delta} = \rho_0^{\Delta}\left\{1 - \frac{2}{r}u_0^{\Delta}(t - n\Delta t)\right\} \tag{10.183}$$

from (10.182), $\rho^{\Delta} = 0$ and $\rho_0^{\Delta} = 0$ are occur at the same point under the assumption

$$\frac{2}{r}u_0^{\Delta}(t - n\Delta t) < 1. \tag{10.184}$$

If $\rho_0^{\Delta} > 0$, we have

$$u^{\Delta} = u_0^{\Delta} - \frac{(t - n\Delta t)M/r^2}{1 - 2(t - n\Delta t)u_0^{\Delta}/r}. \tag{10.185}$$

If $\rho_0^{\Delta} = 0$, then $\rho^{\Delta} = 0$ under the above assumption, and $U^{\Delta} = 0$. In this case, we defined the value u_0^{Δ} as a suitable value for the boundedness in Chapter 8. Thus, we define u^{Δ} by (10.185) and from the value u_0^{Δ} in the case $\rho_0^{\Delta} = 0$.

Then, we have

$$\begin{aligned}
w(U^{\Delta}) &= u^{\Delta} + \frac{\sqrt{a\gamma}}{\theta}(\rho^{\Delta})^{\theta} \\
&= u_0^{\Delta} - \frac{(t - n\Delta t)M/r^2}{1 - 2(t - n\Delta t)u_0^{\Delta}/r} + \frac{\sqrt{a\gamma}}{\theta}(\rho_0^{\Delta})^{\theta}\{1 - 2u_0^{\Delta}(t - n\Delta t)/r\}^{\theta} \\
&= w(U_0^{\Delta}) - \frac{(t - n\Delta t)M/r^2}{1 - 2(t - n\Delta t)u_0^{\Delta}/r} \\
&\quad + \frac{\sqrt{a\gamma}}{\theta}(\rho_0^{\Delta})^{\theta}\left(\left\{1 - \frac{2}{r}u_0^{\Delta}(t - n\Delta t)\right\}^{\theta} - 1\right), \\
z(U^{\Delta}) &= z(U_0^{\Delta}) - \frac{(t - n\Delta t)M/r^2}{1 - 2(t - n\Delta t)u_0^{\Delta}/r} \\
&\quad - \frac{\sqrt{a\gamma}}{\theta}(\rho_0^{\Delta})^{\theta}\left(\left\{1 - \frac{2}{r}u_0^{\Delta}(t - n\Delta t)\right\}^{\theta} - 1\right).
\end{aligned}$$

We fix δ such that $0 < \delta < 1$ and we suppose

$$2|u_0^{\Delta}|\Delta t < \delta, \tag{10.186}$$

then, for $n\Delta t < t < (n+1)\Delta t$, we have

$$\begin{aligned}
w(U^{\Delta}) &\leq w(U_0^{\Delta}) - \frac{\sqrt{a\gamma}}{\theta}(\rho_0^{\Delta})^{\theta}\frac{2u_0^{\Delta}}{r}(t - n\Delta t)\theta \\
&= w(U_0^{\Delta}) - \frac{w(U_0^{\Delta}) - z(U_0^{\Delta})}{2}\frac{w(U_0^{\Delta}) + z(U_0^{\Delta})}{2}\frac{2\theta}{r}(t - n\Delta t), \\
z(U^{\Delta}) &\geq z(U_0^{\Delta}) - \frac{1}{1-\delta}\frac{M}{r^2}\Delta t + \frac{w(U_0^{\Delta}) - z(U_0^{\Delta})}{2}\frac{w(U_0^{\Delta}) + z(U_0^{\Delta})}{2}\frac{2\theta}{r}(t - n\Delta t)
\end{aligned}$$

since

$$(1-x)^{\theta}-1 \leq -\theta x \qquad \text{for } x \leq 1. \tag{10.187}$$

Let

$$\sigma(U)=\max\{w(U),-z(U)\}. \tag{10.188}$$

Now, from $w(U) \geq z(U)$, it follows

$$z(U)^2-w(U)^2 \leq \sigma(U)^2. \tag{10.189}$$

Since

$$\begin{aligned} w(U^{\Delta}) &\leq \sigma(U_0^{\Delta})+\sigma(U_0^{\Delta})^2\frac{\theta}{2}\Delta t, \\ -z(U^{\Delta}) &\leq \sigma(U_0^{\Delta})+\frac{M}{1-\delta}\Delta t+\sigma(U_0^{\Delta})^2\frac{\theta}{2}\Delta t, \end{aligned}$$

we obtain

$$\sigma(U^{\Delta}) \leq \sigma(U_0^{\Delta})+\left\{\frac{M}{1-\delta}+\frac{\theta}{2}\sigma(U_0^{\Delta})^2\right\}\Delta t. \tag{10.190}$$

Let

$$B_n=\sup_{r>1,(n-1)\Delta t\leq t<n\Delta t}\sigma(U^{\Delta}(t,r)), \qquad B_0=\sup_{r>1}\sigma(U_0(r)). \tag{10.191}$$

Then,

$$\sup_{r>1}\sigma(U^{\Delta}(0,r)) \leq B_0 \tag{10.192}$$

by Lemma 11, and

$$\sigma(U_0^{\Delta}(t,r)) \leq \sup_{r>1}\sigma(U^{\Delta}(0,r)) \leq B_0 \tag{10.193}$$

for $0<t<\Delta t$ and $r>1$ by Lemmas 8, 9. From (10.190), we obtain

$$B_1 \leq B_0+\left(\frac{M}{1-\delta}+\frac{\theta}{2}B_0^2\right)\Delta t. \tag{10.194}$$

On $t=\Delta t$, by Lemma 11, we get

$$\sigma(U^{\Delta}(\Delta t,r)) \leq B_1. \tag{10.195}$$

Thus, by repeating the above argument, we have

$$B_n \leq B_{n-1}+\left(\frac{M}{1-\delta}+\frac{\theta}{2}B_{n-1}^2\right)\Delta t \tag{10.196}$$

for $n \geq 1$.

We note that we suppose

$$2B_n \Delta t < \delta \tag{10.197}$$

for (10.186) and Δt need to be so small that the waves do not meet one another on $U_0^{\Delta}(t,r)$.

If we compare B_n, which satisfies (10.196), with the solution $\beta = \beta(t) = \beta(t; B_0, M, \delta, \theta)$ of an ordinary differential equation

$$\begin{cases} \dfrac{d\beta}{dt} = \dfrac{M}{1-\delta} + \dfrac{\theta}{2}\beta^2, \\ \beta(0) = B_0. \end{cases} \tag{10.198}$$

Then, it follows

$$B_n \leq \beta(n\Delta t) \qquad (n = 0, 1, 2, \ldots). \tag{10.199}$$

Proof

Suppose the inequality is valid for $(n-1)$ $(n \geq 1)$. Since $d\beta/dt \geq 0$ from (10.198),

$$\beta(t) \geq \beta((n-1)\Delta t) \geq B_{n-1} \geq 0 \tag{10.200}$$

for $(n-1)\Delta t < t < n\Delta t$. Thus,

$$\begin{aligned} \beta(n\Delta t) &= \beta((n-1)\Delta t) + \int_{(n-1)\Delta t}^{n\Delta t} \beta'(t)dt \\ &= \beta((n-1)\Delta t) + \int_{(n-1)\Delta t}^{n\Delta t} \left\{ \frac{M}{1-\delta} + \frac{\theta}{2}\beta(t)^2 \right\} dt \\ &\geq B_{n-1} + \left(\frac{M}{1-\delta} + \frac{\theta}{2}B_{n-1}^2 \right) \Delta t \\ &\geq B_n. \end{aligned}$$

This complete the proof. ∎

The solution $\beta(t)$ of (10.198) is given by

$$\beta(t) = \sqrt{\frac{M}{1-\delta}\frac{2}{\theta}} \tan\left(\sqrt{\frac{M}{1-\delta}\frac{\theta}{2}}t + \arctan\left(\sqrt{\frac{1-\delta}{M}\frac{\theta}{2}}B_0 \right) \right) \tag{10.201}$$

Thus, $\beta(t)$ has finite value for

$$0 < t < T_1^{\infty} = \sqrt{\frac{1-\delta}{M}\frac{2}{\theta}} \left\{ \frac{\pi}{2} - \arctan\left(\sqrt{\frac{1-\delta}{M}\frac{\theta}{2}}B_0 \right) \right\}, \tag{10.202}$$

and

$$\lim_{t \uparrow T_1^{\infty}} \beta(t) = \infty. \tag{10.203}$$

Let

$$T_0^\infty = \lim_{\delta \downarrow 0} T_1^\infty = \sqrt{\frac{2}{\theta M}} \left\{ \frac{\pi}{2} - \arctan \left(\sqrt{\frac{\theta}{2M}} B_0 \right) \right\}, \tag{10.204}$$

and let T be any number such that $0 < T < T_0^\infty$. Then, we show that we can construct the approximate solution for $0 < t < T$.

First, we fix sufficiently small δ such that

$$T < T_1^\infty < T_0^\infty \tag{10.205}$$

and let B be the value which satisfies

$$B > \beta(T). \tag{10.206}$$

We define

$$\Sigma_4 = \Sigma(B, -B) \tag{10.207}$$

and take Δt such that

$$2B\Delta t < \delta, \qquad \frac{\Delta r}{\Delta t} \geq 2\Lambda(\Sigma_4) = 2B. \tag{10.208}$$

Then, while $B_n \leq B$, it follows $U^\Delta \in \Sigma_4$.

Since $B_n \leq \beta(n\Delta t)$ and $B > \beta(T)$, $B_n \leq B$ hold for $n\Delta t \leq T$. Thus, we can construct the approximate solution U^Δ for $0 < t < T$ and $U^\Delta \in \Sigma_4$.

PROPOSITION 16 *For any T such that $0 < T < T_0^\infty$, we construct an approximate solution for Problem* (IV) *in*

$$\{(t, r); r > 1,\, 0 < t < T\} \tag{10.209}$$

and satisfies

$$0 \leq \rho^\Delta(t, r) \leq C_1(T), \quad |u^\Delta(t, r)| \leq C_2(T) \qquad (r > 1,\, 0 < t < T). \tag{10.210}$$

Chapter 11

Entropies

The entropy in fluid mechanics is S which appears in the second law of thermodynamics

$$\frac{\delta Q}{T} = dS, \tag{11.1}$$

where T is the absolute temperature, Q is the heat quantity, and δQ is the change of the heat quantity. This becomes

$$dS = \frac{\delta Q}{T} = nC_V \frac{dT}{T} + \frac{Pmn}{T} d\left(\frac{1}{\rho}\right) = nC_V \frac{dT}{T} + nR\rho d\left(\frac{1}{\rho}\right) \tag{11.2}$$

from the first law of thermodynamics. Thus, by integrating (11.2), we have

$$S - nC_V \log T - nR \log \frac{1}{\rho} = \text{constant}. \tag{11.3}$$

Let $S = 0$ if $n = 0$. Then, it follows

$$S = nC_V \log \frac{P}{\rho^\gamma} + nC_V \log \frac{m}{R} \tag{11.4}$$

from the equation of state. This gives the entropy per unit mass of each point

$$\eta_0(t, x) = \lim_{|I(x)| \to 0} \frac{S(t; I(x))}{mn(t; I(x))} = \frac{C_V}{m} \log \frac{P}{\rho^\gamma} + \frac{C_V}{m} \log \frac{m}{R} \tag{11.5}$$

Now, we set η_0 which we remove the last constant term in (11.5).

$$\eta_0(t, x) = \frac{C_V}{m} \log \frac{P}{\rho^\gamma} \tag{11.6}$$

For isentropic gas, $\eta_0(t, x)$ becomes constant because

$$P = a\rho^\gamma. \tag{11.7}$$

We show that η_0 satisfies the equation

$$(\rho\eta_0)_t + (\rho u \eta_0)_x = 0. \tag{11.8}$$

Since

$$(\rho\eta_0)_t + (\rho u\eta_0)_x = \{\rho_t + (\rho u)_x\}\eta_0 + \rho\{(\eta_0)_t + u(\eta_0)_x\} = \rho\{(\eta_0)_t + u(\eta_0)_x\}, \tag{11.9}$$

we may show

$$(\eta_0)_t + u(\eta_0)_x = 0. \tag{11.10}$$

Since we saw

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} \tag{11.11}$$

in Chapter 2, the conservation of energy becomes

$$\left(\frac{1}{\gamma-1}P + \frac{1}{2}\rho u^2\right)_t + \left(\frac{\gamma P}{\gamma-1}u + \frac{1}{2}\rho u^3\right)_x = 0 \tag{11.12}$$

Thus, the conservation laws of mass, momentum, and energy become

$$\begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 1 \\ \frac{u^3}{2} & \frac{\gamma}{\gamma-1}P + \frac{3}{2}\rho u^2 & \frac{\gamma}{\gamma-1}u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_x = 0. \tag{11.13}$$

This yields

$$\begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma P & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_x = 0. \tag{11.14}$$

And hence, it follows

$$\begin{aligned} & (\eta_0)_t + u(\eta_0)_x \\ = & \begin{pmatrix} (\eta_0)_\rho & 0 & (\eta_0)_P \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_t + u \begin{pmatrix} (\eta_0)_\rho & 0 & (\eta_0)_P \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_x \\ = & -\begin{pmatrix} (\eta_0)_\rho & 0 & (\eta_0)_P \end{pmatrix} \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & 1/\rho \\ 0 & \gamma P & 0 \end{pmatrix} \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_x \\ = & \ 0 \end{aligned}$$

since

$$\begin{pmatrix} (\eta_0)_\rho & 0 & (\eta_0)_P \end{pmatrix} = \begin{pmatrix} -\frac{C_V}{m}\frac{\gamma}{\rho} & 0 & \frac{C_V}{m}\frac{1}{P} \end{pmatrix}. \tag{11.15}$$

Therefore, the equation (11.10) is satisfied. The equation (11.10) is the conservation form, and then, such equations of the conservation form are called the *additional conservation laws.* P.D.Lax named such a function pair the generalized entropy pair ([14]).

For the system of conservation laws

$$U_t + F(U)_x = 0, \tag{11.16}$$

a pair of functions of U $(\eta(U), q(U))$ is an *entropy pair* if the pair satisfies the additional conservation law

$$\eta(U)_t + q(U)_x = 0. \tag{11.17}$$

The function $\eta(U)$ is called *entropy* and the function $q(U)$ is called *entropy flux.* Since

$$\eta(U)_t = \nabla\eta(U)U_t = -\nabla\eta(U)F(U)_x = -\nabla\eta(U)\nabla F(U)U_x, \quad q(U)_x = \nabla q(U)U_x, \tag{11.18}$$

(η, q) is an entropy pair if and only if (η, q) satisfies the system of linear partial differential equations

$$\nabla q(U) = \nabla\eta(U)\nabla F(U). \tag{11.19}$$

For general conservation laws

$$U = {}^t(u_1, \ldots, u_n), \quad F(U) = {}^t(f_1(U), \ldots, f_n(U)), \tag{11.20}$$

the gradient of $F(U)$

$$\nabla F(U) = \begin{pmatrix} \nabla f_1(U) \\ \vdots \\ \nabla f_n(U) \end{pmatrix} = \begin{pmatrix} f_1(U)_{u_1} & \cdots & f_1(U)_{u_n} \\ \vdots & & \vdots \\ f_n(U)_{u_1} & \cdots & f_n(U)_{u_n} \end{pmatrix} \tag{11.21}$$

is a $n \times n$ matrix, and hence the system (11.19) is for the n-th equations concerned with only the two unknown functions. Thus, it is not clear that whether the solution of the system of the equations (11.19) exist except the obvious solutions

$$(\eta, q) = \text{constant}, \tag{11.22}$$

or

$$(\eta, q) = (g(u_j), g(f_j(u))) \tag{11.23}$$

where $g(y)$ is any function, $j = 1, \ldots n,$. For example, we can show that the system of conservation of mass, momentum, and energy have entropy pairs which are equivalent to $(\rho\eta_0, \rho u\eta_0)$. The proof of the fact is given at the end of this chapter. However, for 2×2 system (i.e. $n = 2$), it is known that there exists many kinds of the entropy pair ([14]). The compensated compactness theory use many kinds of the entropy pair.

Consider the system for isentropic gas dynamics.

The form of the system of partial differential equations (11.19) turns by the change of variable. For (ρ, m), (ρ, u) and (w, z), $\nabla F(U)$ becomes

$$\begin{aligned}
&\nabla_{(\rho,m)}F(U)=\begin{pmatrix}0 & 1\\ 2u & -u^2+P'(\rho)\end{pmatrix},\nabla_{(\rho,u)}F(U)=\begin{pmatrix}u & \rho\\ P'(\rho)/\rho & \rho\end{pmatrix},\\
&\nabla_{(w,z)}F(U)=\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_1\end{pmatrix},
\end{aligned} \tag{11.24}$$

respectively. Considering by (w,z), the system of equations is

$$\begin{cases} q_w=\lambda_2\eta_w,\\ q_z=\lambda_1\eta_z. \end{cases} \tag{11.25}$$

This gives the second order linear partial equation for η, called the *Euler-Poisson-Darboux type*

$$\eta_{wz}+\frac{\tau}{w-z}(\eta_w-\eta_z)=0 \qquad \left(\tau=\frac{1-\theta}{\theta}=\frac{1}{2}\frac{3-\gamma}{\gamma-1}\right). \tag{11.26}$$

Though the equation (11.26) has a singularity on $w=z$, the equation has solutions

$$\eta=\int_z^w\{(w-s)(s-z)\}^\tau\phi(s)ds \qquad (\phi\in C^2) \tag{11.27}$$

which vanish on $w=z$. The expression (11.27) is called the *Darboux's formula.* The entropy flux q corresponding the entropy η given by the Darboux's formula is the following.

$$\begin{aligned}
q &= \theta\int_z^w\{(w-s)(s-z)\}^\tau\{s+\tau(w+z)\}\phi(s)ds\\
&= \lambda_1\eta+\theta\int_z^w(w-s)^\tau(s-z)^{\tau+1}\phi(s)ds\\
&= \lambda_2\eta-\theta\int_z^w(w-s)^{\tau+1}(s-z)^\tau\phi(s)ds
\end{aligned}$$

The entropy given by (11.27) satisfies

$$\eta(\rho,\rho u)]_{\rho=0}=0, \tag{11.28}$$

thus, the entropy is called a *weak entropy.* On the other side,

$$(1,1),\ \left(u,\frac{u^2}{2}+\sigma_0\right),\ \left(\frac{u^2}{2}+\sigma_1,\frac{u^3}{3}+(\sigma_0+\sigma_1)u\right) \tag{11.29}$$

are the *strong entropy* pair which does not vanish on $\rho=0$, where $\sigma_0=\sigma_0(\rho)$ and $\sigma_1=\sigma_1(\rho)$ is defined by

$$\rho\sigma_0'=P'(\rho),\quad \rho\sigma_1'-\sigma_1=\sigma_0, \tag{11.30}$$

that is,

$$\sigma_0=\frac{a\gamma}{\gamma-1}\rho^{\gamma-1},\quad \sigma_1=-\frac{a\gamma}{\gamma-1}\frac{1}{2-\gamma}\rho^{\gamma-1}. \tag{11.31}$$

The entropy pairs which DiPerna, Ding, Chen, Luo used are the pairs given by (11.27), where the function ϕ are the followings (cf. [1],[2],[9]).

1. $$\phi(s) = k^{\tau+1} e^{\pm ks} \qquad (k = 1, 2, \ldots). \tag{11.32}$$

2. $\phi(s) = \psi_n(s)$ and $\psi_n'(s)$, where
$$\psi_n(s) \in C_0^\infty(R), \quad \psi_n(s) \longrightarrow \delta(s-a). \tag{11.33}$$

3. $$\phi(s) = \phi_0^{(\tau+3)}(s), \quad \phi_0(s) \in C_0^\infty(-1,1) \tag{11.34}$$
Here, if k is not integer, the k-th derivative is defined by the following. We set the distribution $f_\alpha(x)$, whose support is in $[0,\infty)$, by
$$f_\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} \mathcal{X}_{(0,\infty)}(x) & (\alpha > 0), \\ f'_{\alpha+1}(x) & (\alpha \le 0) \end{cases} \tag{11.35}$$
for any $\alpha \in R$. For this, we define the β-th derivative of a function $\phi(s) \in C_0^\infty(R)$ by the convolution
$$\phi^{(\beta)}(s) = (f_{-\beta} * \phi)(s) = < f_{-\beta}(y), \phi(s-y) > \tag{11.36}$$
for any $\beta \in R$. Of course, if β is a positive integer, then the derivative by this definition is equal to the derivative in standard sense.

4. $$\phi(s) = s^2 \tag{11.37}$$

Now we calculate for $\phi(s) = s^j$ $(j = 0, 1, 2)$.

$$\begin{aligned}
&\int_z^w \{(w-s)(s-z)\}^\tau ds \\
&= (w-z)^{2\tau+1} B(\tau+1, \tau+1) \\
&= c_0 \rho, \\
&\quad \left(2\tau + 1 = \frac{1}{\theta}, \, c_0 = \left(\frac{2\sqrt{a\gamma}}{\theta} \right)^{1/\theta} B(\tau+1, \tau+1) \right), \\
&\int_z^w \{(w-s)(s-z)\}^\tau s\, ds \\
&= (w-z)^{2\tau+2} B(\tau+1, \tau+2) + (w-z)^{2\tau+1} z B(\tau+1, \tau+1) \\
&= c_0 \rho u, \\
&\int_z^w \{(w-s)(s-z)\}^\tau s^2\, ds \\
&= (w-z)^{2\tau+3} B(\tau+1, \tau+3) + 2z(w-z)^{2\tau+2} B(\tau+1, \tau+2) \\
&\quad + z^2 (w-z)^{2\tau+1} B(\tau+1, \tau+1) \\
&= c_0 \rho \left\{ wz + \frac{1}{2} \frac{\tau+2}{2\tau+3} (w-z)^2 \right\} \\
&= c_0 \left(\rho u^2 + \frac{2}{\gamma-1} a \rho^\gamma \right) \\
&= 2c_0 \left(\rho e + \frac{1}{2} \rho u^2 \right)
\end{aligned}$$

Thus, we obtain the conservation quantities

$$\begin{aligned}&(\rho,\ \rho u),\qquad (\rho u,\ \rho u^2+P),\\&(\rho e+\rho u^2/2,\ (\rho e+P+\rho u^2/2)u)\end{aligned} \tag{11.38}$$

In particular, we denote

$$\eta_* = \frac{1}{2}\rho u^2 + \rho e, \quad q_* = \left(\frac{1}{2}\rho u^2 + \rho e + P\right) u. \tag{11.39}$$

For the entropy par and the invariant region Σ, the following lemma holds.

LEMMA 17 *Let $\Sigma = \Sigma(w_0, z_0)$ be a trigonal region defined in Chapter 9, $\phi(s)$ be a C^2 function, and η be the entropy given by the Darboux's formula and by the ϕ. Then, the following estimates hold.*

$$\begin{cases} |\nabla\eta(U)| \le C_1(\Sigma,\phi), \\ \nabla^2\eta_*(U) \ge C_2(\Sigma,\phi) > 0 \quad (\textit{as quadratic form}), \\ |\nabla^2\eta(U)| \le C_3(\Sigma,\phi)\nabla^2\eta_*(U) \quad (\textit{as quadratic form}) \end{cases} \tag{11.40}$$
$$(\nabla = (\partial_\rho, \partial_m)).$$

The proof are done easily by using

$$\eta = C\rho\int_0^1 \{y(1-y)\}^\tau \phi\left(u + (2y-1)\frac{\sqrt{a\gamma}}{\theta}\rho^\theta\right) dy. \tag{11.41}$$

DiPerna, Ding, Chen, Luo show the following theorem ([1],[5],[8],[9]).

THEOREM 18 *Consider the system*

$$U_t + F(U)_x = 0 \qquad (U = {}^t(\rho,\rho u),\ F(U) = {}^t(\rho u, \rho u^2 + P),\ P = a\rho^\gamma, 1 < \gamma \le 5/3). \tag{11.42}$$

Let Q be an open domain in $R_t \times R_x$. If the approximate solution U_n of the system is uniformly bounded on Q

$$0 \le \rho_n(t,x) \le C,\ |u_n(t,x)| \le C \qquad \textit{for } (t,x) \in Q, \tag{11.43}$$

and the set

$$\{\eta(U_n(t,x))_t + q(U_n(t,x))_x\}_n \tag{11.44}$$

is relatively compact in $H^{-1}_{loc}(Q)$ for any entropy pair (η, q) given by the Darboux's formula, then there exist a subsequence $\{U_{n_j}\}_j$ of $\{U_n\}_n$, which converges almost everywhere on Q

$$U_{n_j} \longrightarrow \bar{U} \quad a.e.\ in\ Q. \tag{11.45}$$

For the proof, see [1],[2],[4],[5],[8],[9],[10],[26],[27].

By using Theorem 18, we show the existence of the weak solution for each Problem (I)-(IV). The boundedness for the approximate solution is given in Chapter 9, thus, we may obtain the compactness of $\eta_t + q_x$ in order to show the convergence of the approximation. We show the compactness in Chapter 13. And we have to prove the limit is a weak solution. This is shown in Chapter 13. Moreover, we have to show that the weak solution satisfies the entropy condition for the admissibility of the weak solution. Here, we explain the entropy conditions.

The weak solution of the initial value problem

$$\begin{cases} U_t + F(U)_x = 0 & (0 < t < T,\, x \in R) \\ U(0,x) = U_0(x) & (x \in R) \end{cases} \tag{11.46}$$

satisfies the *entropy condition* if for any smooth weak entropy pair (η, q), where η is convex, the weak solution satisfies the entropy inequality

$$\eta(U)_t + q(U)_x \le 0 \qquad (t > 0) \tag{11.47}$$

in the distribution sense, that is, for any $\phi \in C_0^\infty((0,T) \times R)$, $\phi(t,x) \ge 0$, U satisfies

$$\iint_{0<t<T} \{\eta(U)\phi_t + q(U)\phi_x\} dx dt \ge 0. \tag{11.48}$$

For example, the solutions of the Riemann problems which we constructed in the section 8.4 satisfy conditions. About the relation between the solution of the Riemann problem and the entropy condition, see [14],[24].

We define the entropy condition for each of the Problems (I)-(IV). For the initial boundary value problems the entropy condition is defined as the solutions satisfy the inequality

$$\eta_t + q_x \le 0 \tag{11.49}$$

in the distribution sense for any smooth entropy pair (η, q), where the entropy η is convex, on inner domain of the problem. That is, for Problem (I),

$$\eta(U)_t + q(U)_x \le 0 \qquad (x > x_1(t),\, t > 0), \tag{11.50}$$

for Problem (II),

$$\eta(U)_t + q(U)_x \le 0 \qquad (x_1(t) < x < x_2(t),\, t > 0), \tag{11.51}$$

and for Problem (III),

$$\begin{cases} \eta(U; P_1)_t + q(U; P_1)_x \le 0 & (x < x(t),\, t > 0), \\ \eta(U; P_2)_t + q(U; P_2)_x \le 0 & (x > x(t),\, t > 0). \end{cases} \tag{11.52}$$

However, for Problem (IV), we must establish the entropy condition considering the outer force term.

For the purpose, we see how to yield the entropy condition of the initial value problem without the outer force. First, we consider the limit of the ε-perturbation. According to [24], if the weak solution is the limit of the solution of the ε-perturbed equation, called the *viscosity method.*

Let U^ε be the solution of the equation

$$U_t + F(U)_x = \varepsilon U_{xx} \tag{11.53}$$

where $\varepsilon > 0$. We suppose the solution U^ε of the parabolic equation (11.53) vanishes at infinity of x and has regularity. Let (η, q) be a smooth weak entropy pair and let η is convex. Then,

$$\begin{aligned}
\eta((U^\varepsilon)_t + q(U^\varepsilon)_x &= \nabla\eta(U^\varepsilon)U_t^\varepsilon + \nabla q(U^\varepsilon)U_x^\varepsilon \\
&= \nabla\eta(U^\varepsilon)U_t^\varepsilon + \nabla\eta(U^\varepsilon)\nabla F(U^\varepsilon)U_x^\varepsilon \\
&= \nabla\eta(U^\varepsilon)\{U_t^\varepsilon + F(U^\varepsilon)_x\} \\
&= \varepsilon\nabla\eta(U^\varepsilon)U_{xx}^\varepsilon \\
&= \varepsilon\eta(U^\varepsilon)_{xx} - \varepsilon^t U_x^\varepsilon \nabla^2\eta(U^\varepsilon)U_x^\varepsilon.
\end{aligned}$$

Suppose $U^\varepsilon \to U$ as $\varepsilon \downarrow 0$. Then, for $\phi \in C_0^\infty((0,T)\times R)$, $\phi(t,x) \geq 0$, since

$$\begin{aligned}
&\iint_{0<t<T}\{\eta(U^\varepsilon)\phi_t + q(U^\varepsilon)\phi_x\}dxdt \\
= &-\iint_{0<t<T}\phi\{\eta(U^\varepsilon)_t + q(U^\varepsilon)_x\}dxdt \\
= &-\iint_{0<t<T}\phi\{\varepsilon\eta(U^\varepsilon)_{xx} - \varepsilon^t U_x^\varepsilon\nabla^2\eta(U^\varepsilon)U_x^\varepsilon\}dxdt \\
\geq &-\varepsilon\iint_{0<t<T}\phi_{xx}\eta(U^\varepsilon)dxdt,
\end{aligned}$$

it follows

$$\iint_{0<t<T}\{\eta(U)\phi_t + q(U)\phi_x\}dxdt \geq 0. \tag{11.54}$$

In these way, we have the entropy inequality

$$\eta_t + q_x \leq 0. \tag{11.55}$$

We do the similar way for the equation with the outer force term

$$U_t + F(U)_x = H(t,x,U). \tag{11.56}$$

Let U^ε be the solution of

$$U_t + F(U)_x = H(t,x,U) + \varepsilon U_{xx}. \tag{11.57}$$

In this case, since

$$\eta(U^\varepsilon)_t + q(U^\varepsilon)_x = \nabla\eta(U^\varepsilon)H(t,x,U^\varepsilon) + \varepsilon\eta(U^\varepsilon)_{xx} - \varepsilon^t U_x^\varepsilon \nabla^2\eta(U^\varepsilon)U_x^\varepsilon, \tag{11.58}$$

it follows

$$\begin{aligned}
&\iint_{0<t<T} \{\eta(U^\varepsilon)\phi_t + q(U^\varepsilon)\phi_x\}dxdt \\
= &-\iint_{0<t<T} \phi\{\eta(U^\varepsilon)_t + q(U^\varepsilon)_x\}dxdt \\
= &-\iint_{0<t<T} \phi\{\nabla\eta H + \varepsilon\eta_{xx} - \varepsilon^t U_x^\varepsilon\nabla^2\eta(U^\varepsilon)U_x^\varepsilon\}dxdt \\
\geq &-\iint_{0<t<T} \{\phi\nabla\eta(U^\varepsilon)H(t,x,U^\varepsilon) + \varepsilon\phi_{xx}\eta(U^\varepsilon)\}dxdt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&-\iint_{0<t<T} \{\phi\nabla\eta(U^\varepsilon)H(t,x,U^\varepsilon) + \varepsilon\phi_{xx}\eta(U^\varepsilon)\}dxdt \\
&\longrightarrow\ -\iint_{0<t<T} \phi\nabla\eta(U)H(t,x,U)dxdt
\end{aligned}$$

as $\varepsilon \downarrow 0$. Therefore, we obtain

$$\eta_t + q_x \leq \nabla\eta H \tag{11.59}$$

in this case.

Next, we consider the energy nonincreasing. Since the mechanical energy η_* is convex, an entropy condition

$$(\eta_*)_t + (q_*)_x \leq 0 \tag{11.60}$$

holds. Suppose the vanishing of the solution at infinity of x. Then, it follows

$$\begin{aligned}
0 \ \geq\ & \iint_{0<t<T} \{(\eta_*)_t + (q_*)_x\}dxdt \\
=\ & \int_R \eta_*(U(T,x))dx - \int_R \eta_*(U_0(x))dx
\end{aligned}$$

Thus, we have

$$\int_R \eta_*(U(T,x))dx \leq \int_R \eta_*(U_0(x))dx. \tag{11.61}$$

This means that the energy does not increase. We consider this nonincreasing law for energy for Problem (IV).

The conservation of energy for Problem (IV) is described by

$$\left\{r^2\left(\frac{1}{2}\rho u^2 + \rho e\right)\right\}_t + \left\{r^2\left(\frac{1}{2}\rho u^2 + \rho e + P\right)u\right\}_r = -\rho u M. \tag{11.62}$$

The term r^2 is from the spherical symmetry

$$\iiint_{r>1} \left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho e\right) dx_1 dx_2 dx_3 = \int_1^\infty r^2 \left(\frac{1}{2}\rho u^2 + \rho e\right) dr. \tag{11.63}$$

Thus, the nonincreasing law seems to be the form

$$(r^2\eta_*)_t + (r^2 q_*)_r \leq -\rho u M. \tag{11.64}$$

This implies

$$(\eta_*)_t + (q_*)_r \leq -\frac{\rho u}{r^2}M - \frac{2}{r}q_*. \tag{11.65}$$

Now, since

$$\begin{aligned} \nabla\eta_* &= (\partial_\rho, \partial_m)\left(\frac{1}{2}\frac{m^2}{\rho} + \frac{a}{\gamma-1}\rho^\gamma\right) \\ &= \left(-\frac{u^2}{2} + \frac{a\gamma}{\gamma-1}\rho^{\gamma-1}, u\right), \\ H &= {}^t\left(-\frac{2}{r}\rho u, -\frac{2}{r}\rho u^2 - \frac{M}{r^2}\rho\right), \end{aligned}$$

it follows

$$\nabla\eta_* H = -\frac{\rho u}{r^2}M - \frac{2}{r}q_*. \tag{11.66}$$

Thus, we obtain

$$(\eta_*)_t + (q_*)_r \leq \nabla\eta_* H. \tag{11.67}$$

Therefore, for Problem (IV), we define the entropy condition by the following:

> The weak solution of Problem (IV) satisfies the entropy inequalities
>
> $$\eta_t + q_r \leq \nabla\eta H \qquad (r > 1,\, t > 0) \tag{11.68}$$
>
> in the distribution sense for any smooth entropy pair (η, q), where the entropy η is convex.

Last, we solve the equations (11.19) for the system of conservation of mass, momentum, and energy. For the variable (ρ, u, P), the system (11.19) becomes

$$q_\rho = u\eta_\rho \tag{11.69}$$

$$q_u = \rho\eta_\rho + u\eta_u + \gamma P \eta_P \tag{11.70}$$

$$q_P = \frac{1}{\rho}\eta_u + u\eta_P \tag{11.71}$$

since

$$\nabla F(U) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma P & u \end{pmatrix}. \tag{11.72}$$

By the compatibility of (11.69) and (11.71)

$$q_{\rho P} = u\eta_{\rho P} = \left(\frac{1}{\rho}\eta_u\right)_\rho + u\eta_{\rho P}, \tag{11.73}$$

we have

$$\left(\frac{1}{\rho}\eta_u\right)_\rho = 0. \tag{11.74}$$

Thus, it follows

$$\frac{1}{\rho}\eta = \alpha(\rho, P) + \beta(u, P). \tag{11.75}$$

From (11.69), q becomes

$$q = u\eta + \delta(u, P) = \rho u\alpha(\rho, P) + \rho u\beta(u, P) + \delta(u, P). \tag{11.76}$$

This and (11.71) imply

$$\begin{aligned} q_P &= \rho u\alpha_P + \rho u\beta_P + \delta_P \\ &= \frac{1}{\rho}\eta_u + u\eta_P = \beta_u + \rho u\alpha_P + \rho u\beta_P. \end{aligned}$$

Hence,

$$\delta(u, P)_P = \beta(u, P)_u. \tag{11.77}$$

It follows that there is a function $\varepsilon(u, P)$ such that

$$\varepsilon(u, P)_u = \delta(u, P), \quad \varepsilon_P(u, P) = \beta(u, P). \tag{11.78}$$

Therefore, η and q are written by α and ε

$$\left\{ \begin{array}{l} \eta = \rho\alpha(\rho, P) + \rho\varepsilon_P(u, P), \\ q = \rho u\alpha(\rho, P) + \rho u\varepsilon_P(u, P) + \varepsilon_u(u, P). \end{array} \right. \tag{11.79}$$

They satisfy (11.69) and (11.71). Thus, we obtain α and ε from (11.70). We may write (11.70) to the following form

$$\begin{aligned} (q - u\eta)_u &= q_u - u\eta_u - \eta = \rho\eta_\rho + \gamma P\eta_P - \eta \\ &= \rho^2\left(\frac{1}{\rho}\eta\right)_\rho + \gamma P\eta_P. \end{aligned}$$

Then, it follows

$$\begin{aligned} &(q - u\eta)_u = \varepsilon_{uu}(u, P), \\ &= \rho^2\left(\frac{1}{\rho}\eta\right)_\rho + \gamma P\eta_P = \rho^2\alpha_\rho + \gamma\rho P(\alpha_P + \varepsilon_{PP}) \end{aligned} \tag{11.80}$$

from (11.79). Thus, we obtain

$$\varepsilon_{uu}(u,P) - \gamma\rho P\varepsilon_{PP}(u,P) = \rho^2\alpha_\rho(\rho,P) + \gamma\rho P\alpha_P(\rho,P) \tag{11.81}$$

Since the right hand side of this equality is independent on u, we have

$$\varepsilon_{uuu}(u,P) - \gamma\rho P\varepsilon_{uPP}(u,P) = 0. \tag{11.82}$$

If $\varepsilon_{uPP} \neq 0$, it follows

$$\frac{\varepsilon_{uuu}(u,P)}{\gamma P\varepsilon_{uPP}(u,P)} = \rho. \tag{11.83}$$

But this is irrational. Thus, we obtain

$$\varepsilon(u,P) = a_1u^2P + a_2u^2 + a_3uP + a_4u + h(P) \tag{11.84}$$

since

$$\varepsilon_{uPP} = \varepsilon_{uuu} = 0, \tag{11.85}$$

where a_1, a_2, a_3, and a_4 are constants, and h is any function. This yield the 1-order linear partial differential equation with α

$$\rho\alpha(\rho,P)_\rho + \gamma P\alpha(\rho,P)_P = 2a_1\frac{P}{\rho} + 2a_2\frac{1}{\rho} - \gamma Ph''(P) \tag{11.86}$$

from (11.81). It is easy to solve and the solution becomes

$$\alpha(\rho,P) = \frac{2a_1}{\gamma-1}\frac{P}{\rho} - \frac{2a_2}{\rho} - h'(P) + g\left(\frac{P}{\rho^\gamma}\right) \tag{11.87}$$

where g is any function. Therefore, we obtain

$$\begin{aligned}
\eta &= \rho(\alpha+\varepsilon_P) \\
&= 2a_1\left(\frac{1}{2}\rho u^2 + \rho e\right) + a_3\rho u + \rho g\left(\frac{P}{\rho^\gamma}\right) - a_2 \\
q &= u\eta + \varepsilon_u \\
&= 2a_1\left(\frac{1}{2}\rho u^2 + \rho e + P\right)u + a_3(\rho u^2 + P) + \rho u g\left(\frac{P}{\rho^\gamma}\right) + a_4 \\
&\qquad \left(e = \frac{1}{\gamma-1}\frac{P}{\rho}\right).
\end{aligned}$$

This means that entropy pairs are

$$(\rho u, \rho u^2 + P),\quad (\rho u^2/2 + \rho e, (\rho u^2/2 + \rho e + P)u),\quad (\rho g(P/\rho^\gamma), \rho u g(P/\rho^\gamma)). \tag{11.88}$$

Therefore the entropy which is not obvious is the last one. But, this is almost equivalent to the physical entropy pair

$$(\rho\eta_0, \rho u\eta_0) \qquad \left(\eta_0 = \frac{C_V}{m}\log\frac{P}{\rho^\gamma}\right). \tag{11.89}$$

The proof of Theorem 18 is done by the compensated compactness theory and using many kinds of the entropy. So, it seems a key that there exists many kinds of the entropy, however, the 3×3 system lacks the variety of such entropies. Therefore, it seems difficult to analyze the 3×3 system by the compensated compactness theory.

Chapter 12

Other approximations

12.1 Godunov difference scheme

Though we have considered the approximate solutions by the Lax-Friedrichs difference scheme, the approximation by the Godunov difference scheme can be constructed for each of Problem (I)-(IV) and we can show the convergence. Moreover, the proofs for the Godunov scheme are almost the same for the Lax-Friedrichs scheme (cf. [25]). In the way for the Lax-Friedrichs scheme, the intervals where we calculate the average are slipped with the length Δx at the time after Δt. In the Godunov scheme, the average are calculated of intervals of the same position at the time after Δt.

Consider the initial boundary value problem. Let

$$E_j^n = ((j-1/2)\Delta x, (j+1/2)\Delta x), \quad J_n = \{\pm 1/2, \pm 3/2, \pm 5/2, \ldots\}. \tag{12.1}$$

$U^\Delta(n\Delta t, x)$ is defined as the step function

$$U^\Delta(n\Delta t, x) = \bar{U}_j^n = \frac{1}{m(E_j^n)} \int_{E_j^n} U^\Delta(n\Delta t - 0, y)dy \qquad (x \in E_j^n, j \in J_n), \tag{12.2}$$

and $U^\Delta(t,x)$ is defined by the solution of the Riemann problems. Each interval where the average is calculated lies on the two solutions of the Riemann problems because the intervals are on the same positions with x. Thus, the influence spread with with the speed $\Delta x/\Delta t$ which is the same speed for the Lax-Friedrichs scheme, however, the mesh width is the half length for the Lax-Friedrichs scheme.

12.2 Other approximation of u for Problem (IV)

We showed a method of calculate for the outer force in Problem (IV) in the section 10.4. However, there are other methods. For example, we can set $U^\Delta(t,x)$ by

$$U^\Delta(t,r) \;=\; U_0^\Delta(t,r) + \tilde{H}(r, t-n\Delta t, U_0^\Delta(t,r))(t-n\Delta t) \qquad (n\Delta t \le t < (n+1)\Delta t)$$

$$\tilde{H}(r,s,U) = H(r,U) + {}^t(0, 2Msm/r^3)$$

for the same U_0^Δ in the section 10.4. This definition implies

$$\begin{aligned} \rho^\Delta &= \rho_0^\Delta - (t - n\Delta t)2\rho_0^\Delta u_0^\Delta / r, \\ u^\Delta &= u_0^\Delta - (t - n\Delta t)M/r^2. \end{aligned}$$

The form of u^Δ is more simple than one in the section 10.4, and hence, we can give the estimate more easily. The convergence can be shown by the similar way in the both case of the Lax-Friedrichs scheme and the Godunov scheme.

12.2.1 Improvement for the boundary

Though we have considered the existence of the weak solutions, the research of the character also is important. It is the useful method for the purpose that we carry out the numerical simulation on the computer by the difference scheme.

But methods in the sections 10.1, 10.2, 10.3 and 10.4 are include some difficulties near the boundary. The difficulties have relations to the boundary conditions. Consider Problem (I).

We suppose $x_1(t) = 0$ for simplicity. We saw in Chapter 7 that the values $\bar{U}_j^n$ is obtained directly by the Lax-Friedrichs scheme at a distance from the boundary. For example, we have

$$\bar{U}_4^1 = \frac{\bar{U}_5^0 + \bar{U}_3^0}{2} - \frac{\Delta t}{2\Delta x}\{F(\bar{U}_5^0) - F(\bar{U}_3^0)\}. \tag{12.3}$$

We consider the value $\bar{U}_2^1$. This value is defined by

$$\bar{U}_2^1 = \frac{1}{3\Delta x}\int_0^{3\Delta x} U^\Delta(\Delta t - 0, y)dy. \tag{12.4}$$

Now, by the integration of the equation $U_t + F(U)_x = 0$ in the region $(0, \Delta t) \times (0, 3\Delta x)$, it follows

$$\begin{aligned} 0 &= \int_0^{3\Delta x} U(\Delta t, x)dx - \int_0^{3\Delta x} U(+0, x)dx \\ &\quad + \int_0^{\Delta t} F(U(t, 3\Delta x - 0))dt - \int_0^{\Delta t} F(U(t, +0))dt \\ &= 3\Delta x\bar{U}_2^1 - 2\Delta x\bar{U}_1^0 - \Delta x\bar{U}_3^0 + F(\bar{U}_3^0)\Delta t - F(\bar{U}_0^{1/2})\Delta t, \end{aligned}$$

where

$$\bar{U}_0^{1/2} = U(\Delta t/2, +0) \tag{12.5}$$

is the state at the boundary. Thus, we have

$$\bar{U}_2^1 = \frac{2\bar{U}_1^0 + \bar{U}_3^0}{3} - \frac{\Delta t}{3\Delta x}\{F(\bar{U}_3^0) - F(\bar{U}_0^{1/2})\}. \tag{12.6}$$

In order to obtain $\bar{U}_2^1$ by (12.6), we must know the value $\bar{U}_0^{1/2}$ on the boundary $x = 0$. The boundary condition only says

$$\bar{m}_0^{1/2} = 0. \tag{12.7}$$

The value $\bar{\rho}_0^{1/2}$ is not obtain if we do not solve the problem (8.67). Thus, this approximation is not well for the numerical computations.

Consider the value $\bar{U}_0^{1/2}$ on the boundary precisely. We saw in the section 8.5 that the value is given by

$$\bar{U}_0^{1/2} \in \tilde{C}_2(\bar{U}_1^0) \cap \{u = 0\}. \tag{12.8}$$

On the other hand, the curve $C_1(U_0)$ and the curve $\tilde{C}_2(U_0)$ are symmetric for each other with respect to $u = u_0$, and $C_1(U_1)$ is the translation of $C_1(U_0)$ for $\rho_0 = \rho_1$. Thus, for the state $U_1 = {}^t(\rho_1, \rho_1 u_1)$ such that

$$\rho_1 = \rho_0, \quad \frac{u_1 + u_0}{2} = \alpha, \tag{12.9}$$

the cross point of $\tilde{C}_2(U_0)$ and $u = \alpha$ is equal to the cross point of $C_1(U_1)$ and $\tilde{C}_2(U_0)$.

That is, the solution $U(t,x)$ of the problem (8.67) near the left boundary

$$\begin{cases} U_t + F(U)_x = 0 & (x > \alpha t,\, t > 0), \\ U(0,x) = U_0 & (x > 0), \\ m - \rho\alpha]_{x=\alpha t} = 0 & (t > 0) \end{cases} \tag{12.10}$$

coincide with the solution $\tilde{U}(t,x)$ of the Riemann problem

$$\begin{cases} U_t + F(U)_x = 0 & (x \in R,\, t > 0), \\ U(0,x) = \begin{cases} U_0 & (x > 0), \\ U_1 & (x < 0) \end{cases} \end{cases} \tag{12.11}$$

in the region $\{(t,x); x > \alpha t,\, t > 0\}$ under $\rho_1 = \rho_0$, $u_1 = 2\alpha - u_0$.

This means that the existence of the piston is equivalent to the collision between the this side gas and the opposite side reversed gas. Therefore, giving such a reverse state outside the boundary, we consider that $U(t,x)$ is the solution of the Riemann problem, and hence, we can calculate simply by using the Lax-Friedrichs scheme.

For stopping piston problem, that is, Problem (I) with $x_1(t) = 0$, we set $\bar{U}_{-1}^0$ by $\bar{U}_1^0$, calculate $\bar{U}_0^1$ from $\bar{U}_1^0$ and $\bar{U}_{-1}^0$, and we can obtain $\bar{U}_1^2$ from $\bar{U}_0^1, \bar{U}_2^1$. Hence, we may obtain the boundary states

$$\bar{U}_0^1, \bar{U}_0^3, \bar{U}_0^5, \ldots. \tag{12.12}$$

Also in the moving piston problem (I), by analogy, we construct the approximate boundary having the piecewise constant speed on $2\Delta t$ length interval and we obtain the boundary value at $t = \Delta t$, $3\Delta t$, $5\Delta t$,.... That is, we set

$$\begin{aligned} x_1^{\Delta}(t) &= x_1(2n\Delta t) + \frac{x_1((2n+2)\Delta t) - x_1(2n\Delta t)}{2\Delta t}(t - 2n\Delta t) \\ & \quad (2n\Delta t \le t < (2n+2)\Delta t), \end{aligned}$$

the reverse state outside the boundary on $t = 2n\Delta t$, we solved the Riemann problem in $2n\Delta t < t < (2n+1)\Delta t$, and define the boundary state on $t = (2n+1)\Delta t$. For $(2n+1)\Delta t < t < (2n+2)\Delta t$, the solution is defined by the boundary state near the boundary. To construct the approximate solution such a method, the boundary state must satisfies the boundary condition.

Consider more precisely.

LEMMA 19 *Let*

$$\rho_l = \rho_r, \quad (u_l + u_r)/2 = \alpha. \tag{12.13}$$

Then, the solution of the Riemann problem

$$\begin{cases} U_t + F(U)_x = 0 \qquad (x \in R,\, t > 0), \\ U(0,x) = \begin{cases} U_r & (x > 0), \\ U_l & (x < 0), \end{cases} \end{cases} \tag{12.14}$$

which we construct in the section 8.4, is symmetric with respect $x = \alpha t$, that is,

$$\rho(t, \alpha t - y) = \rho(t, \alpha t + y), \quad \frac{u(t, \alpha t - y) + u(t, \alpha t + y)}{2} = \alpha. \tag{12.15}$$

Note that we suppose that the relation (12.15) means

$$\frac{m(t, \alpha t - y) + m(t, \alpha t + y)}{2} = \alpha\rho(t, \alpha t - y), \quad \rho(t, \alpha t - y) = \rho(t, \alpha t + y) \tag{12.16}$$

instead of the relation for u in the case $\rho = 0$.

Proof

In the case $u_l = u_r = \alpha$ or $\rho_l = \rho_r = 0$, it follows obviously. We suppose $\rho_l = \rho_r > 0$ and $u_l \ne u_r$. First, we consider the case $u_l > u_r$. In this case, the Riemann problem has a solution which consists of 1-shock wave and 2-shock wave. Let $\bar{U} \in S_1(U_l) \cap \tilde{S}_2(U_r)$. Since the curve $S_1(U_l)$ and $\tilde{S}_2(U_r)$ are symmetric with respect to $u = \alpha$, it follows $\bar{u} = \alpha$. Thus, we have

$$\begin{cases} \alpha = u_l - \sqrt{\dfrac{(\bar{\rho} - \rho_l)(\bar{P} - P_l)}{\bar{\rho}\rho_l}} \quad (\bar{\rho} > \rho_l), \\ \alpha = u_r + \sqrt{\dfrac{(\bar{\rho} - \rho_r)(\bar{P} - P_r)}{\bar{\rho}\rho_r}} \quad (\bar{\rho} > \rho_r). \end{cases} \tag{12.17}$$

Consider the shock speeds σ_1 and σ_2. From the Rankine-Hugoniot condition,

$$\begin{aligned}\sigma_1 &= \frac{\bar\rho\bar u - \rho_l u_l}{\bar\rho - \rho_l} = \alpha + \rho_l\frac{\alpha - u_l}{\bar\rho - \rho_l} \\ &= \alpha - \sqrt{\frac{\rho_l}{\bar\rho}\frac{\bar P - P_l}{\bar\rho - \rho_l}}, \\ \sigma_2 &= \frac{\bar\rho\bar u - \rho_r u_r}{\bar\rho - \rho_r} = \alpha + \rho_r\frac{\alpha - u_r}{\bar\rho - \rho_r} \\ &= \alpha + \sqrt{\frac{\rho_r}{\bar\rho}\frac{\bar P - P_r}{\bar\rho - \rho_r}}\end{aligned}$$

Since $\rho_l = \rho_r$, we have $(\sigma_1 + \sigma_2)/2 = \alpha$. Let $U(t,x)$ be the solution of the Riemann problem. Then,

$$U(t,x) = \begin{cases} U_l & (x < \sigma_1 t), \\ \bar U & (\sigma_1 t < x < \sigma_2 t), \\ U_r & (\sigma_2 t < x). \end{cases} \tag{12.18}$$

Thus, this lemma is hold in this case since

$$(\sigma_1 t + \sigma_2 t)/2 = \alpha t. \tag{12.19}$$

Next, we consider the case $u_l < u_r$. In this case, the solution of the Riemann problem consists of 1-rarefaction wave and 2-rarefaction wave. Let $\bar U \in R_1(U_l) \cap \tilde R_2(U_r)$. Then, $\bar u = \alpha$ or $\bar\rho = 0$. Suppose $\bar\rho > 0$. Then, the solution of the Riemann problem becomes

$$U(t,x) = \begin{cases} U_l & (x/t < \lambda_1(U_l)), \\ 1-\text{rarefaction wave} & (\lambda_1(U_l) \le x/t < \lambda_1(\bar U)), \\ \bar U & (\lambda_1(\bar U) \le x/t < \lambda_2(\bar U)), \\ 2-\text{rarefaction wave} & (\lambda_2(\bar U) \le x/t < \lambda_2(U_r)), \\ U_r & (\lambda_2(U_r) < x/t). \end{cases} \tag{12.20}$$

Since

$$\begin{aligned}\frac{\lambda_1(U_l) + \lambda_2(U_r)}{2} &= \frac{u_l - \sqrt{a\gamma}\rho_l^\theta + u_r + \sqrt{a\gamma}\rho_r^\theta}{2} = \alpha, \\ \frac{\lambda_1(\bar U) + \lambda_2(\bar U)}{2} &= \frac{\bar u - \sqrt{a\gamma}\bar\rho^\theta + \bar u + \sqrt{a\gamma}\bar\rho^\theta}{2} = \alpha,\end{aligned}$$

the two rarefaction waves are lie symmetrically. Thus, we may consider the solution on the wave. Let

$$U_1 = U(t, \alpha t - y), \quad U_2 = U(t, \alpha t + y), \quad \lambda_1(U_l)t \le \alpha t - y < \lambda_1(\bar U)t. \tag{12.21}$$

Since U_1 satisfies

$$\lambda_1(U_1) = \frac{\alpha t - y}{t}, \quad w(U_1) = w(U_l), \tag{12.22}$$

it follows

$$\begin{cases} u_1 - \sqrt{a\gamma}\rho_1^\theta = \alpha - \frac{y}{t} \\ u_1 + \frac{\sqrt{a\gamma}}{\theta}\rho_1^\theta = w(U_l), \end{cases} \tag{12.23}$$

and hence, we obtain

$$\begin{cases} u_1 = \frac{1}{1+\theta}\{\theta w(U_l) + \alpha - \frac{y}{t}\}, \\ \sqrt{a\gamma}\rho_1^\theta = \frac{\theta}{1+\theta}\{w(U_l) - \alpha + \frac{y}{t}\}. \end{cases} \tag{12.24}$$

Similarly, we have

$$\begin{cases} u_2 = \frac{1}{1+\theta}\{\theta z(U_r) + \alpha + \frac{y}{t}\}, \\ \sqrt{a\gamma}\rho_2^\theta = \frac{\theta}{1+\theta}\{-z(U_r) + \alpha + \frac{y}{t}\}, \end{cases} \tag{12.25}$$

for U_2. Assumptions $\rho_l = \rho_r$ and $u_l + u_r = 2\alpha$ give

$$w(U_l) + z(U_r) = u_l + \frac{\sqrt{a\gamma}}{\theta}\rho_l^\theta + u_r - \frac{\sqrt{a\gamma}}{\theta}\rho_r^\theta = 2\alpha \tag{12.26}$$

and show

$$w(U_l) - \alpha + \frac{y}{t} = -z(U_r) + \alpha + \frac{y}{t}. \tag{12.27}$$

Thus, it follows $\rho_1 = \rho_2$, and,

$$\frac{u_1 + u_2}{2} = \frac{1}{1+\theta}(\theta\alpha + \alpha) = \alpha \tag{12.28}$$

Therefore, this lemma holds.

In the case $\bar{\rho} = 0$, similarly obtained. ■

For such a solution $U(t, x)$,

$$\int_{\alpha t-\mu}^{\alpha t+\mu} U(t,x)dx = C\begin{pmatrix} 1 \\ \alpha \end{pmatrix}. \tag{12.29}$$

In fact,

$$\begin{aligned} &\int_{\alpha t-\mu}^{\alpha t+\mu} U(t,x)dx \\ = \; &\int_0^\mu U(t,\alpha t + y)dy + \int_0^\mu U(t, \alpha t - y)dy \\ = \; &\int_0^\mu \begin{pmatrix} \rho(t,\alpha t + y) \\ m(t, \alpha t + y) \end{pmatrix} dy + \int_0^\mu \begin{pmatrix} \rho(t,\alpha t + y) \\ 2\alpha\rho(t,\alpha t + y) - m(t,\alpha t + y) \end{pmatrix} dy \\ = \; &2\int_0^\mu \rho(t,\alpha t + y)dy \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ = \; &\int_{\alpha t-\mu}^{\alpha t+\mu} \rho(t,x)dx \begin{pmatrix} 1 \\ \alpha \end{pmatrix}. \end{aligned}$$

Thus, the boundary state satisfies the boundary condition if we set the boundary state as the integral average on the interval which is symmetric with respect to the boundary point.

We consider the boundary state from the view point of the numerical analysis.

Now, let $x = x_1^\Delta(t) = st$ $(0 < t < 2\Delta t)$. We make a reverse state $\bar{U}_{-1}^0$ of $\bar{U}_1^0$ with respect to $x_1^\Delta(t)$, and we obtain the boundary state $\tilde{U}^1$ by

$$\tilde{U}^1 = \frac{1}{2\mu}\int_{s\Delta t-\mu}^{s\Delta t+\mu} U(\Delta t, y)dy. \tag{12.30}$$

From the integration of $U_t + F(U)_x = 0$ in $(0, \Delta t) \times (s\Delta t - \mu, s\Delta t + \mu)$, it follows

$$\begin{aligned} 0 &= \int_0^{\Delta t} dt \int_{s\Delta t-\mu}^{s\Delta t+\mu} \{U_t + F(U)_x\}dx \\ &= 2\mu\tilde{U}^1 - (s\Delta t + \mu)\bar{U}_1^0 - (-s\Delta t + \mu)\bar{U}_{-1}^0 + \Delta t\{F(\bar{U}_1^0) - F(\bar{U}_{-1}^0)\}. \end{aligned}$$

Thus, we have

$$\tilde{U}^1 = \frac{(\mu + s\Delta t)\bar{U}_1^0 + (\mu - s\Delta t)\bar{U}_{-1}^0}{2\mu} - \frac{\Delta t}{2\mu}\{F(\bar{U}_1^0) - F(\bar{U}_{-1}^0)\} \tag{12.31}$$

We may take $\mu = \Delta x - s\Delta t$ in order to adjoin the right end $s\Delta t + \mu$ of the interval to the left end of the next interval. Here, we have to take $\Delta x/\Delta t$ more large value than in the section 10.1 for no meetings of waves. For example, we may take

$$\frac{\Delta x}{\Delta t} > 4\Lambda(\Sigma_1). \tag{12.32}$$

Then, $\tilde{U}^1$ becomes

$$\tilde{U}^1 = \frac{\Delta x\bar{U}_1^0 + (\Delta x - 2s\Delta t)\bar{U}_{-1}^0}{2(\Delta x - s\Delta t)} - \frac{\Delta t}{2(\Delta x - s\Delta t)}\{F(\bar{U}_1^0) - F(\bar{U}_{-1}^0)\}. \tag{12.33}$$

Let $\bar{U}_0^1$ be the value obtained from $\bar{U}_1^0$ and $\bar{U}_{-1}^1$ by the Lax-Friedrichs scheme

$$\bar{U}_0^1 = \frac{\bar{U}_1^0 + \bar{U}_{-1}^0}{2} - \frac{\Delta t}{2\Delta x}\{F(\bar{U}_1^0) - F(\bar{U}_{-1}^0)\}. \tag{12.34}$$

Then, we may write

$$\tilde{U}^1 = \frac{\Delta x\bar{U}_0^1 - s\Delta t\bar{U}_{-1}^0}{\Delta x - s\Delta t}. \tag{12.35}$$

This means that one the line $t = \Delta t$, $\tilde{U}^1$ is the value at $x = s\Delta t$ by linear interpolation from the value $\bar{U}_{-1}^0$ at $x = -\Delta x + s\Delta t$ and the value $\bar{U}_0^1$ at $x = 0$. The reason why the position of $\bar{U}_{-1}^0$ is $-\Delta x + s\Delta t$ seems that the value $\bar{U}_{-1}^0$ is the reverse state with respect to $x = st$.

Such approximate solutions for simple calculation have a uniform boundedness and satisfy the compactness for $\eta_t + q_x$, the existence of the convergent subsequence to a weak solution are shown for this approximation similarly for the approximation in 10.

The similar method can be applied to Problem (II).

In Problem (IV), there is the other difficulty for the numerical calculation. That is for the outer force by the fractional step method

$$\bar{U}_j^n = \bar{U}_{0,j}^n + \frac{\Delta t}{2\Delta x}\int_{1+(j-1)\Delta r}^{1+(j+1)\Delta r} H(r, U_0^\Delta(n\Delta t - 0, r))dr. \tag{12.36}$$

It is not easy to calculate the integral term. Thus, we reform the approximate solution.

Suppose that $U^\Delta(t,r)$ is defined for $r > 1, 0 \le t \le 2n\Delta t$ and $U^\Delta(2n\Delta t, r)$ is a step function

$$U^\Delta(2n\Delta t, r) = \bar{U}_j^{2n} \qquad (r \in E_j^{2n}, j = 1,3,5,\ldots). \tag{12.37}$$

Let $\bar{U}_{-1}^{2n}$ be the reverse value of $\bar{U}_1^{2n}$.

$$\bar{\rho}_{-1}^{2n} = \bar{\rho}_1^{2n}, \quad \bar{m}_{-1}^{2n} = -\bar{m}_1^{2n} \tag{12.38}$$

and in

$$(2n\Delta t, (2n+1)\Delta t) \times (1, 1+\Delta r), \tag{12.39}$$

$U^\Delta(t,r)$ is defined as the solution of the Riemann problem

$$\begin{cases} U_t + F(U)_x = 0 \quad (2n\Delta t, (2n+1)\Delta t) \times (1-\Delta r, 1+\Delta r), \\ U^\Delta(2n\Delta r, r) = \begin{cases} \bar{U}_{-1}^{2n} & (1-\Delta r < r < 1), \\ \bar{U}_1^{2n} & (1 < r < 1+\Delta r), \end{cases} \end{cases} \tag{12.40}$$

and for $r > 1+\Delta r$, we define $U^\Delta(t,r)$ by solutions of the Riemann problem. on $t = (2n+1)\Delta t$, we define

$$U^\Delta((2n+1)\Delta t, r) \begin{cases} = \bar{U}_j^{2n+1} = \bar{U}_{0,j}^{2n+1} + \Delta t H(1+j\Delta r, \bar{U}_{0,j}^{2n+1}) \\ \quad (r \in E_j^{2n+1}, j = 2,4,6,\ldots), \\ = \bar{U}_0^{2n+1} = \bar{U}_{0,0}^{2n+1} \\ \quad (r \in (1, 1+\Delta r)), \end{cases} \tag{12.41}$$

where

$$\bar{U}_{0,j}^{2n+1} = \frac{1}{m(E_j^{2n+1})}\int_{E_j^{2n+1}} U^\Delta((2n+1)\Delta t - 0, r)dr. \tag{12.42}$$

The reason why $\bar{U}_0^{2n+1}$ has no outer force is for the boundary condition, and we can consider that as the reaction of the outer force from the wall $r = 1$.

For $(2n+1)\Delta t < t \le (2n+2)\Delta t$, we define $U^\Delta(t,r)$ by solutions of the Riemann problems, and on $t = (2n+2)\Delta t$, we set

$$\begin{aligned} &U^\Delta((2n+2)\Delta t, r) = \bar{U}_j^{2n+2} = \bar{U}_{0,j}^{2n+2} + \Delta t H(1+j\Delta r, \bar{U}_{0,j}^{2n+2}) \\ &(r \in E_j^{2n+2}, j = 1,3,5,\ldots). \end{aligned} \tag{12.43}$$

This approximation differs from the approximation in the section 10.4 at three points. The first point is the consideration for the simple calculation at the boundary, the second one is that we add the outer force term inserted the average value instead of the average value of the outer force term to the solution of homogeneous system, and another one is that the times when we add the term concerned with the outer force only mesh points.

It is easy to calculate such approximations by computer. And we can show the existence of convergent subsequence of the approximate solutions to a weak solution. The proof is done in Chapter 13.

Unfortunately, for Problem (III), we have not known the other simple approximation which has convergent subsequence to a weak solution.

Chapter 13

Proofs

In this chapter, we prove the existence of weak solutions of Problem (I)-(IV), and show that solutions satisfy entropy conditions mentioned in Chapter 11. Moreover, we prove Propositions 6, 7 for the definitions of the weak solutions of the initial value problem and initial boundary value problems (I)-(IV) at first in this chapter.

13.1 The weak solution

We show the Propositions 6, 7 in Chapter 6.

Though we give the range of t as $[0, T)$ in the definition in Chapter 6, the proof for the range $[0, T)$ is not different from the proof for $[0, \infty)$ because of the test functions ϕ, ψ which arise in the definitions have compact support. Thus, we suppose $T = \infty$ in this section.

First, we show that we can take the test functions ϕ and ψ from more wide function space than $C_0^1([0,\infty) \times R)$.

LEMMA 20 *The weak solution of the initial value problem*

$$\begin{cases} U_t + F(U)_x = 0 & (t > 0,\ x \in R), \\ U(0, x) = U_0(x) & (x \in R) \end{cases} \tag{13.1}$$

satisfies

$$\int_0^\infty dt \int_R \{U(t,x)\phi(t,x)_t + F(U(t,x))\phi(t,x)_x\} dx + \int_R U_0(x)\phi(0,x)dx = 0 \tag{13.2}$$

for any $\phi(t,x) \in \mathrm{Lip}_0([0,\infty) \times R)$, *where*

$$\mathrm{Lip}_0(Q) = \{\phi : \phi \text{ is Lipschitz continuous on } Q,\ \operatorname{supp}\phi \text{ is compact in } Q\}. \tag{13.3}$$

That is, we can weaken the regularity of the test functions from C^1 to Lipschitz continuity.

Proof

Let $\omega_0(t,x) \in C_0^\infty(R_t \times R_x)$ such that

$$\operatorname{supp}\omega_0 \subset [-1,1]\times[-1,1], \quad \omega_0 \geq 0, \quad \iint \omega_0(t,x)dtdx = 1, \tag{13.4}$$

and we define $\omega_\varepsilon(t,x)$ by

$$\omega_\varepsilon(t,x) = \frac{1}{\varepsilon^2}\omega_0\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right). \tag{13.5}$$

We fix

$$\phi \in \mathrm{Lip}_0(R_t \times R_x). \tag{13.6}$$

Then, there is a positive constant $M > 0$ such that

$$|\phi(p) - \phi(q)| \leq M|p-q| \tag{13.7}$$

for any $p, q \in R_t \times R_x$. Thus, for any h, we have

$$\left|\frac{\phi(t+h,x)-\phi(t,x)}{h}\right| \leq M, \left|\frac{\phi(t,x+h)-\phi(t,x)}{h}\right| \leq M. \tag{13.8}$$

Hence it follows that there exist derivatives $\phi_t(t,x)$, $\phi_x(t,x)$ almost everywhere and they are bounded measurable functions such that

$$|\phi_t| \leq M, \ |\phi_x| \leq M. \tag{13.9}$$

Let ϕ_ε be a regularization of ϕ. That is,

$$\phi_\varepsilon(t,x) = (\omega_\varepsilon * \phi)(t,x) = \iint \omega_\varepsilon(s,y)\phi(t-s,x-y)dsdy. \tag{13.10}$$

Then,

$$\begin{aligned}
&\frac{\phi(t,x+h)-\phi(t,x)}{h} - (\omega_\varepsilon * \phi_x)(t,x) \\
&= \iint \omega_\varepsilon(y,s)\left\{\frac{\phi(t-s,x-y+h)-\phi(t-s,x-y)}{h} - \phi_x(t-s,x-y)\right\}dyds \\
&= \iint \omega_\varepsilon(y,s)I_\varepsilon(y,s)dyds
\end{aligned}$$

where, we set $I_\varepsilon(y,s)$ be

$$I_\varepsilon(y,s) = \frac{\phi(t-s,x-y+h)-\phi(t-s,x-y)}{h} - \phi_x(t-s,x-y) \tag{13.11}$$

Since $I_\varepsilon(y,s)$ satisfies that

$$|I_\varepsilon(y,s)| \leq 2M, \quad I_\varepsilon(y,s) \longrightarrow 0 \text{ a.e. as } h \to 0, \tag{13.12}$$

we have

$$(\phi_\varepsilon)_x = \omega_\varepsilon * \phi_x \tag{13.13}$$

by Lebesgue's bounded convergence theorem. Similarly, we obtain

$$(\phi_\varepsilon)_t = \omega_\varepsilon * \phi_t. \tag{13.14}$$

And ϕ_t, $\phi_x \in L^1_{\mathrm{loc}}(R_t \times R)$ by their boundedness. These yield

$$(\phi_\varepsilon)_x = \omega * \phi_x \longrightarrow \phi_x \quad \text{a.e.}, \qquad (\phi_\varepsilon)_t \longrightarrow \phi_t \quad \text{a.e.} \tag{13.15}$$

as $\varepsilon \downarrow 0$. Since

$$|(\phi_\varepsilon)_x| = |\omega_\varepsilon * \phi_x| \leq M, \quad |(\phi_\varepsilon)_t| \leq M, \quad \phi_\varepsilon \longrightarrow \phi \text{ as } \varepsilon \downarrow 0, \quad |\phi_\varepsilon| \leq \|\phi\|_{L^\infty}, \tag{13.16}$$

it follows that

$$\begin{aligned} 0 &= \iint_{t>0} \{U(\phi_\varepsilon)_t + F(U)(\phi_\varepsilon)_x\} dx dt + \int_R \phi_\varepsilon(0,x) U_0(x) dx \\ &\to \iint_{t>0} \{U\phi_t + F(U)\phi_x\} dx dt + \int_R \phi(0,x) U_0(x) dx \end{aligned}$$

as $\varepsilon \downarrow 0$. ■ We show that such a lemma holds for the Problem (I).

LEMMA 21 *Let $U(t,x)$ be a solution for the Problem (I) such that*

$$0 \leq \rho(t,x) \leq C, \quad |m(t,x)| \leq C\rho(t,x) \quad (x > x_1(t),\, t > 0) \tag{13.17}$$

Then, the integral equality defining the weak solution (6.11) is valid for the test functions ϕ, ψ such that

$$\phi, \psi \in \mathrm{Lip}_0([0,\infty) \times R), \qquad \psi = 0 \ \text{on } x = x_1(t) \tag{13.18}$$

Proof

The proof of the equality for the conservation of mass is done by the similar argument to Lemma 20. We show the equality for the conservation of momentum. Let $\psi \in \mathrm{Lip}_0([0,\infty) \times R)$ such that $\psi = 0$ on $x = x_1(t)$. Then, it follows

$$\begin{cases} \psi_\varepsilon \longrightarrow \psi, \\ \left.\begin{array}{l} (\psi_\varepsilon)_t \longrightarrow \psi_t, \\ (\psi_\varepsilon)_x \longrightarrow \psi_x \end{array}\right\} \text{a.e.} \\ |\psi_\varepsilon| \leq \|\psi\|_{L^\infty}, \quad |(\psi_\varepsilon)_t| \leq M, \quad |(\psi_\varepsilon)_x| \leq M \end{cases} \tag{13.19}$$

similarly to Lemma 20. We note that ψ_ε may not vanish on $x = x_1(t)$.

1st Step We make a C^∞ approximation $\bar{x}_\delta(t)$ of $x_1(t)$. Since $u_1(t) = x_1'(t)$ is bounded, there exist a C^∞ function $g_\delta(t)$ such that

$$\begin{cases} \displaystyle\int_0^t |u_1(s) - g_\delta(s)| ds < \frac{\delta}{4} \quad (t > 0), \\ |g_\delta(t)| \leq \|u_1\|_{L^\infty(0,\infty)}. \end{cases} \tag{13.20}$$

Let

$$\bar{x}_\delta(t) = \int_0^t g_\delta(s)ds. \tag{13.21}$$

Then,

$$|x_1(t) - \bar{x}_\delta(t)| < \frac{\delta}{4}, \quad \bar{x}_\delta \in C^\infty, \quad |\bar{x}'_\delta(t)| \le \|u_1\|_{L^\infty(0,\infty)}. \tag{13.22}$$

2nd Step Let $\xi_0(x) \in C^\infty$ such that

$$0 \le \xi_0 \le 1, \quad \xi_0(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x \ge 1), \end{cases} \tag{13.23}$$

and we set

$$\xi_\delta(t,x) = \xi_0\left(\frac{4}{\delta}(x - \bar{x}_\delta(t) - \delta/2)\right) \tag{13.24}$$

Then, $\xi_\delta \in C^\infty$ and satisfies that

$$\begin{aligned} &0 \le \xi_\delta \le 1, \quad \xi_\delta(t,x) = \begin{cases} 0 & (x \le x_1(t)), \\ 1 & (x \ge x_1(t) + \delta), \end{cases} \\ &|\delta(\xi_\delta)_t| = \left|4(-\bar{x}'_\delta(t))\xi'_0\left(\frac{4}{\delta}(x - \bar{x}_\delta(t) - \delta/2)\right)\right| \le C, \\ &|\delta(\xi_\delta)_x| = \left|4\xi'_0\left(\frac{4}{\delta}(x - \bar{x}_\delta(t) - \delta/2)\right)\right| \le C. \end{aligned} \tag{13.25}$$

We define $\tilde{\psi}(t,x)$ by

$$\tilde{\psi}(t,x) = \psi_\varepsilon(t,x)\xi_\delta(t,x). \tag{13.26}$$

3rd Step Since $\tilde{\psi} \in C_0^\infty$ and vanishes on $x = x_1(t)$, it follows that

$$\begin{aligned} 0 &= \iint_{D_1} \{m\tilde{\psi}_t + (m^2/\rho + P)\tilde{\psi}_x\}dxdt + \int_0^\infty m_0(x)\tilde{\psi}(0,x)dx \\ &= \iint_{D_1} \{m(\psi_\varepsilon)_t + (m^2/\rho + P)(\psi_\varepsilon)_x\}\xi_\delta dxdt + \int_0^\infty m_0(x)\xi_\delta(0,x)\psi_\varepsilon(0,x)dx \\ &\quad + \iint_{D_1} \{m(\xi_\delta)_t + (m^2/\rho + P)(\xi_\delta)_x\}\psi_\varepsilon dxdt. \end{aligned}$$

The last integral tend to

$$\begin{aligned} &\iint_{D_1} \{m\psi_t + (m^2/\rho + P)\psi_x\}\xi_\delta dxdt + \int_0^\infty m_0(x)\xi_\delta(0,x)\psi(0,x)dx \\ &\quad + \iint_{D_1} \{m(\xi_\delta)_t + (m^2/\rho + P)(\xi_\delta)_x\}\psi dxdt \end{aligned} \tag{13.27}$$

as $\varepsilon \downarrow 0$ for m and $m^2/\rho + P$ are bounded. The first two terms tend to the left hand side of the definition of the weak solution as δ tends to 0 since

$$\xi_\delta(t,x) \longrightarrow 1 \text{ as } \delta \downarrow 0 \quad (t>0, x>x_1(t)), \qquad |\xi_\delta| \le 1. \tag{13.28}$$

We may show the last term tend to zero as δ tend to 0. In order to show that, it is sufficient to prove

$$\int_0^\infty dt \int_{x_1(t)}^{x_1(t)+\delta} \{|(\xi_\delta)_t| + |(\xi_\delta)_x|\}|\psi|dx \longrightarrow 0 \qquad (\delta \downarrow 0) \tag{13.29}$$

by the boundedness of m and $m^2/\rho + P$. Let $\operatorname{supp}\psi \subset [0,L) \times R$. Then, since

$$|\psi(t,x)| = |\psi(t,x) - \psi(t,x_1(t))| \le M|x - x_1(t)|, \tag{13.30}$$

we have

$$\begin{aligned} & \int_0^\infty dt \int_{x_1(t)}^{x_1(t)+\delta} \{|(\xi_\delta)_t| + |(\xi_\delta)_x|\}|\psi|dx \\ \le & \int_0^L dt \int_{x_1(t)}^{x_1(t)+\delta} (|\delta(\xi_\delta)_t| + |\delta(\xi_\delta)_x|)\, dx \\ \le & \; CL\delta \longrightarrow 0. \end{aligned}$$

∎

The similar lemmas hold for Problems (II) and (IV), and the proofs are done by the similar ways.

But, this does not hold for Problem (III) because of the integral term on $x = x(t)$ including ψ_t, ψ_x in the definition of the weak solution.

We show Propositions 6, 7 except for Problem (III) by these lemmas. *Proof* (Proposition 6) Let $U(t,x)$ be a weak solution. We define a function $\xi_1(t)$ by

$$\xi_1(t) = \xi_1(t;\Delta t) = \begin{cases} 1 & (t \le 0), \\ \dfrac{\Delta t - t}{\Delta t} & (0 < t \le \Delta t), \\ 0 & (t > \Delta t) \end{cases} \tag{13.31}$$

For any $\psi(x) \in C_0^1(R)$, we set

$$\phi(t,x) = \xi_1(t)\psi(x) \in \mathrm{Lip}_0([0,\infty) \times R). \tag{13.32}$$

Then,

$$\begin{aligned} 0 &= \iint_{t>0} \{U(t,x)\psi(x)\xi_1'(t) + F(U(t,x))\psi'(x)\xi_1(t)\}dxdt + \int_R U_0(x)\psi(x)\xi_1(0)dx \\ &= -\int_R \psi(x)dx \int_0^{\Delta t} \frac{1}{\Delta t} U dt + \int_R \psi'(x)dx \int_0^{\Delta t} \frac{\Delta t - t}{\Delta t} F(U)dt + \int_R U_0(x)\psi(x)dx \\ &= -\int_R \psi(x)dx \int_0^{\Delta t} \frac{1}{\Delta t} U dt + \int_R U_0(x)\psi(x)dx + O(\Delta t) \end{aligned}$$

by Lemma 20. Thus, we have

$$\int_R \psi(x)\left\{\frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt - U_0(x)\right\}dx \longrightarrow 0 \qquad (\Delta t \downarrow 0). \tag{13.33}$$

Since the space $C_0^1(R)$ is dense in $L^1(R)$ and

$$\left|\frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt\right| \leq \|U\|_{L^\infty}, \tag{13.34}$$

we obtain

$$\frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt \longrightarrow U_0(x) \qquad \text{in } L^\infty \text{ weak}*. \tag{13.35}$$

Conversely, we suppose that U satisfies (6.7) and (6.8). Then, it follows

$$\int_0^\infty dt \int_R \{U(t,x)\phi(t,x)_t + F(U(t,x))\phi(t,x)_x\}dx = 0 \tag{13.36}$$

for any $\phi \in C^\infty((0,\infty)\times R)$. Since this equality include only the first derivative of ϕ, we can show that the equality holds for any $\phi \in \mathrm{Lip}_0((0,\infty)\times R)$. Moreover, using the fact that

$$\lim_{\varepsilon\downarrow 0}\iint f_1(t,x)f_2(t-\varepsilon,x)dxdt = \iint f_1(t,x)f_2(t,x)dxdt \tag{13.37}$$

for $f_1(t,x) \in L^1_{loc}([0,\infty)\times R)$ and $f_2(t,x) \in \mathrm{Lip}_0([0,\infty)\times R)$, we can show that the equality holds for $\phi \in \mathrm{Lip}_0([0,\infty)\times R)$ such that $\phi = 0$ on $t = 0$. Thus, we suppose

$$\begin{cases} \iint_{t>0}(U\phi_t + F(U)\phi_x)dxdt = 0, \\ \frac{1}{\Delta t}\int_0^{\Delta t} U(t,x)dt \longrightarrow U_0(x) \qquad \text{in } L^\infty \text{ weak}* \end{cases} \tag{13.38}$$

for $\phi \in \mathrm{Lip}_0([0,\infty)\times R)$, $\phi = 0$ on $t = 0$. We fix $\psi(t,x) \in C_0^1([0,\infty)\times R)$. We define a function $\xi_2(t)$ by

$$\begin{aligned} \xi_2(t) &= \begin{cases} 0 & (t \leq 0), \\ t/\Delta t & (0 < t \leq \Delta t), \\ 1 & (t > \Delta t), \end{cases} \\ &= 1 - \xi_1(t), \end{aligned} \tag{13.39}$$

and set $\phi(t,x) = \xi_2(t)\psi(t,x) \in \mathrm{Lip}_0([0,\infty)\times R)$. Since $\phi(0,x) = 0$, we have

$$\begin{aligned} 0 &= \iint_{t>0}\{U(\psi\xi_2)_t + F(U)(\psi\xi_2)_x\}dxdt \\ &= \iint_{t>0}\xi_2(U\psi_t + F\psi_x)dxdt + \iint_{t>0}\xi_2' U\psi dxdt \\ &= \iint_{t>0}\xi_2(U\psi_t + F\psi_x)dxdt + \int_R dx\int_0^{\Delta t}\frac{1}{\Delta t}U(t,x)\psi(t,x)dt \\ &= \iint_{t>0}(U\psi_t + F\psi_x)dxdt + \int_R \psi(0,x)dx\int_0^{\Delta t}\frac{1}{\Delta t}U(t,x)dt \\ &\quad - \int_R dx\int_0^{\Delta t}\frac{\Delta t - t}{\Delta t}(U\psi_t + F\psi_x)dxdt + \int_R dx\int_0^{\Delta t}\frac{\psi(t,x)-\psi(0,x)}{\Delta t}U(t,x)dt \\ &= \iint_{t>0}(U\psi_t + F\psi_x)dxdt + \int_R \psi(0,x)dx\int_0^{\Delta t}\frac{1}{\Delta t}U(t,x)dt + O(\Delta t) \\ &\to \iint_{t>0}(U\psi_t + F\psi_x)dxdt + \int_R U_0(x)\psi(0,x)dx. \end{aligned}$$

■ *Proof*

(for Problem (I) of Proposition 7) Let $U(t,x)$ be a weak solution of Problem (I). For any $\phi(x) \in C_0^1(0,\infty)$ and $\xi_1(t)$ used in the proof of the Proposition 6, we have

$$\begin{aligned} 0 &= \iint_{D_1} \{(\phi(x)\xi_1(t))_t\rho + (\phi(x)\xi_1(t))_x m\} dxdt + \int_0^\infty \rho_0(x)\phi(x)\xi_1(0)dxdt \\ &= \iint_{D_1} (\rho\phi\xi_1' + m\phi'\xi_1)dxdt + \int_0^\infty \rho_0\phi dx \\ &= -\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_{x_1(t)}^\infty \rho\phi dx + \int_0^{\Delta t} \frac{\Delta t - t}{\Delta t} dt \int_{x_1(t)}^\infty m\phi' dx + \int_0^\infty \rho_0\phi dx \\ &= -\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_{x_1(t)}^\infty \rho\phi dx + O(\Delta t) + \int_0^\infty \rho_0\phi dx \end{aligned}$$

by Lemma 21. Now, since

$$\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_{x_1(t)}^\infty \rho\phi dx = \int_R \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho} dt\right)\phi dx, \tag{13.40}$$

we have

$$\int_R \left\{\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}(t,x)dt - \rho_0(x)\right\}\phi(x)dx \longrightarrow 0 \qquad (\Delta t \downarrow 0) \tag{13.41}$$

for any $\phi \in C_0^1(0,\infty)$. Thus,

$$\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}(t,x)dt \longrightarrow \rho_0(x) \qquad \text{in } L^\infty(0,\infty) \text{ weak}* \tag{13.42}$$

by the similar argument in Proposition 6. For $\phi(x) \in C_0^1(0,\infty)$, $\phi(x)\xi_1(t)$ vanishes on $x = x_1(t)$ for sufficiently small Δt. Thus, for the momentum, we can prove by the similar argument for the density.

We consider the statement for the boundary condition. Let $\psi(t) \in C_0^1(0,\infty)$, and we set

$$\phi(t,x) = \psi(t)\xi_1(x - x_1(t)) \in \mathrm{Lip}_0(\{(t,x); x \geq x_1(t), t > 0\}). \tag{13.43}$$

Then, by Lemma 21, we have

$$\begin{aligned} 0 &= \iint_{D_1} \{\rho(\psi\xi_1)_t + m(\psi\xi_1)_x\} dxdt \\ &= \iint_{D_1} (\rho\psi'\xi_1 - \rho\psi\xi_1' u_1 + m\psi\xi_1')dxdt \\ &= \int_0^\infty \psi'(t)dt \int_{x_1(t)}^{x_1(t)+\Delta x} \rho\frac{x_1(t) + \Delta x - x}{\Delta x} dx \\ &\quad + \int_0^\infty \psi(t)dt \int_{x_1(t)}^{x_1(t)+\Delta x} \frac{1}{\Delta x}(\rho u_1 - m)dx \\ &= O(\Delta x) + \int_0^\infty \psi(t)dt \int_{x_1(t)}^{x_1(t)+\Delta x} \frac{1}{\Delta x}(\rho u_1 - m)dx. \end{aligned}$$

Thus, we obtain

$$\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x}(m-\rho u_1)dx \longrightarrow 0 \qquad \text{in } L^\infty(0,\infty) \text{ weak}*. \tag{13.44}$$

by the similar argument for the initial data.

We show the converse part. Similarly in the proof of Proposition 6, we can show that

$$\iint_{D_1}\{U\phi_t + F(U)\phi_x\}dxdt = 0 \tag{13.45}$$

for any $\phi(t,x) \in \mathrm{Lip}_0(R_t \times R_x)$ such that $\phi = 0$ on $t = 0$ and on $x = x_1(t)$. Let

$$\begin{aligned}
&\mu_1(x) = \xi_1(x;\Delta x), \quad \mu_2(t) = \xi_1(t;\Delta t),\\
&\theta(t,x) = \phi(t,x)\{1-\mu_1(x-x_1(t))\},\\
&\sigma(t,x) = \theta(t,x)\{1-\mu_2(t)\}
\end{aligned} \tag{13.46}$$

for any $\phi(t,x) \in C_0^1(R_t \times R_x)$. Then, since

$$\phi = \phi\mu_1 + \theta = \phi\mu_1 + \theta\mu_2 + \sigma \tag{13.47}$$

and $\sigma = 0$ on $t = 0$ and on $x = x_1(t)$,

$$\begin{aligned}
&\iint_{D_1}(\rho\phi_t + m\phi_x)\\
=\ &\iint_{D_1}(\rho\phi_t + m\phi_x)\mu_1 + \iint_{D_1}\phi(-\rho\mu_1' u_1 + m\mu_1')\\
&+\iint_{D_1}(\rho\theta_t + m\theta_x)\mu_2 + \iint_{D_1}\theta\rho\mu_2' + \iint_{D_1}(\rho\sigma_t + m\sigma_x).
\end{aligned}$$

Here,

$$\begin{aligned}
\iint_{D_1}(\rho\sigma_t + m\sigma_x) &= 0,\\
\iint_{D_1}(\rho\phi_t + m\phi_x)\mu_1 &= \int_0^\infty dt\int_{x_1(t)}^{x_1(t)+\Delta x}(\rho\phi_t + m\phi_x)\frac{x_1(t)+\Delta x - x}{\Delta x}dx\\
&= O(\Delta x),\\
\iint_{D_1}\phi(-\rho\mu_1' u_1 + m\mu_1') &= \int_0^\infty dt\int_{x_1(t)}^{x_1(t)+\Delta x}\phi(t,x)\frac{1}{\Delta x}(m-\rho u_1)dx\\
&= \int_0^\infty \phi(t,x_1(t))\left\{\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x}(m-\rho u_1)dx\right\}dt + O(\Delta x)\\
&\qquad (\phi(t,x) - \phi(t,x_1(t)) = O(\Delta x))\\
&\longrightarrow 0 \quad (\Delta x \downarrow 0),\\
\iint_{D_1}(\rho\theta_t + m\theta_x)\mu_2 &= \int_0^{\Delta t}\frac{\Delta t - t}{\Delta t}dt\int_{x_1(t)}^\infty(\rho\theta_t + m\theta_x)dx\\
&\qquad \left(\int_{x_1(t)}^\infty(\rho\theta_t + m\theta_x)dx \text{ is bounded for fixed } \Delta x\right)\\
&= O(\Delta t) \quad (\text{ for fixed } \Delta x),\\
\iint_{D_1}\theta\rho\mu_2' &= -\frac{1}{\Delta t}\int_0^{\Delta t}dt\int_{x_1(t)}^\infty \rho\theta dx.
\end{aligned}$$

Since

$$\begin{aligned}\theta(t,x)-\theta(0,x) &= \phi(t,x)\{1-\mu_1(x-x_1(t))\}-\phi(0,x)\{1-\mu_1(x)\}\\ &= \{\phi(t,x)-\phi(0,x)\}\{1-\mu_1(x)\}+\phi(t,x)\{\mu_1(x)-\mu_1(x-x_1(t))\},\end{aligned}$$

we have

$$\begin{aligned}|\theta(t,x)-\theta(0,x)| &\le t\|\phi_t\|_{L^\infty}+\frac{1}{\Delta x}|x_1(t)|\|\phi\|_{L^\infty} \qquad \left(|\mu_1'|\le\frac{1}{\Delta x}\right)\\ &\le t\left(\|\phi_t\|_{L^\infty}+\frac{1}{\Delta x}\|\phi\|_{L^\infty}\|u_1\|_{L^\infty}\right)\\ &\qquad (|x_1(t)|=|x_1(t)-x_1(0)|\le t\|u_1\|_{L^\infty})\\ &= tM\end{aligned}$$

where M is bounded for fixed Δx. Thus,

$$\begin{aligned}\iint_{D_1}\theta\rho\mu_2' &= O(1)\frac{1}{\Delta t}\int_0^{\Delta t}tdt-\frac{1}{\Delta t}\int_0^{\Delta t}dt\int_{x_1(t)}^{\infty}\rho(t,x)\theta(0,x)dx\\ &= O(\Delta t)-\int_R\theta(0,x)\left\{\frac{1}{\Delta t}\int_0^{\Delta t}\tilde\rho(t,x)dt\right\}dx\\ &= O(\Delta t)-\int_0^\infty\theta(0,x)\left\{\frac{1}{\Delta t}\int_0^{\Delta t}\tilde\rho(t,x)dt\right\}dx\\ &\qquad (\theta(0,x)=\phi(0,x)\{1-\mu_1(x)\}=0 \text{ for } x\le 0)\\ &\longrightarrow -\int_0^\infty\theta(0,x)\rho_0(x)dx \qquad (\text{ as } \Delta t\downarrow 0 \text{ for fixed } \Delta x)\end{aligned}$$

Hence, we have

$$\iint_{D_1}(\rho\phi_t+m\phi_x)=O(\Delta x)+O(\Delta x)+o(1)-\int_0^\infty\theta(0,x)\rho_0(x)dx \tag{13.48}$$

as $\Delta t\downarrow 0$ for fixed Δx. Since

$$\begin{aligned}-\int_0^\infty\theta(0,x)\rho_0(x)dx &= -\int_0^\infty\phi(0,x)\rho_0dx+\int_0^\infty\phi(0,x)\mu_1\rho_0dx\\ &= -\int_0^\infty\phi(0,x)\rho_0(x)dx+\int_0^{\Delta x}\frac{\Delta x-x}{\Delta x}\phi(0,x)\rho_0(x)dx\\ &= -\int_0^\infty\phi(0,x)\rho_0(x)dx+O(\Delta x),\end{aligned}$$

it follows

$$\iint_{D_1}(\rho\phi_t+m\phi_x)=-\int_0^\infty\phi(0,x)\rho_0(x)dx \tag{13.49}$$

as $\Delta x\downarrow 0$.

For the equation of conservation of momentum, the proof can be done similarly for $\psi\in C_0^1$ such that ψ vanishes on $x=x_1(t)$. The difference is only the term

$$\int_0^\infty\psi(t,x_1(t))\left(\frac{1}{\Delta x}\int_{x_1(t)}^{x_1(t)+\Delta x}\left\{\left(\frac{m^2}{\rho}+P\right)-mu_1\right\}dx\right)dt \tag{13.50}$$

near the boundary. However, the term vanishes by $\psi(t, x_1(t)) = 0$. ∎

The proofs for Problems (II) and (IV) are similarly obtained.

For Problem (III), the fact corresponding to Lemmas 20, 21 can not be followed, however, we prove Proposition 7 for Problem (III) by essentially equal method to other problems. First, we show the "only if" part.

Proof

(for Problem (III) of Proposition 7, "only if" part)

Let $x(t)$ be a free boundary and $U(t,x)$ a weak solution. In the definition of the weak solution for Problem (III), the terms vanish except the first and multiple integral term if $\psi = 0$ on $x = x(t)$ and $t = 0$. Thus, it follows

$$\begin{cases} U_t + F(U; P_1)_x = 0 & (x < x(t),\, t > 0), \\ U_t + F(U; P_2)_x = 0 & (x > x(t),\, t > 0), \end{cases} \tag{13.51}$$

in the distribution sense. In the definition of weak solution, we may take the test function ϕ from $\mathrm{Lip}_0(R_t \times R_x)$. This yields

$$\begin{cases} \dfrac{1}{\Delta t}\displaystyle\int_0^{\Delta t} \tilde{\rho}_r(t,x)dt \to \rho_0(x) & \text{in } L^\infty(0,\infty) \text{ weak}*, \\ \dfrac{1}{\Delta t}\displaystyle\int_0^{\Delta t} \tilde{\rho}_l(t,x)dt \to \rho_0(x) & \text{in } L^\infty(-\infty,0) \text{ weak}*, \end{cases} \tag{13.52}$$

by the similar argument for Problem (I), where

$$\tilde{\rho}_r(t,x) = \begin{cases} \rho(t,x) & (x > x(t), t > 0), \\ 0 & (x < x(t), t > 0), \end{cases} \quad \tilde{\rho}_l(t,x) = \begin{cases} 0 & (x > x(t), t > 0), \\ \rho(t,x) & (x < x(t), t > 0). \end{cases} \tag{13.53}$$

Since $\rho = \tilde{\rho}_l + \tilde{\rho}_r$, for any $\phi(x) \in\in C_0(R)$

$$\begin{aligned} & \int_R \left(\frac{1}{\Delta t}\int_0^{\Delta t} \rho(t,x)dt \right) \phi(x)dx \\ = & \int_R \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_l(t,x)dt \right) \phi(x)dx + \int_R \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_r(t,x)dt \right) \phi(x)dx \\ = & \int_{-\infty}^0 \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_l(t,x)dt \right) \phi(x)dx + \int_0^\infty \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_r(t,x)dt \right) \phi(x)dx \\ & + \int_0^\infty \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_l(t,x)dt \right) \phi(x)dx + \int_{-\infty}^0 \left(\frac{1}{\Delta t}\int_0^{\Delta t} \tilde{\rho}_r(t,x)dt \right) \phi(x)dx, \end{aligned}$$

then, the first two terms converge to

$$\int_{-\infty}^0 \phi\rho_0, \qquad \int_0^\infty \phi\rho_0 \tag{13.54}$$

respectively. Thus we may show the last two terms tend to zero. Since the free boundary $x(t)$ is Lipschitz continuous, it follows

$$|x(t)| = |x(t) - x(0)| \le Mt, \tag{13.55}$$

Thus, $\tilde{\rho}_l = 0$ for $x > Mt$. Hence, we have

$$\begin{aligned}\left|\frac{1}{\Delta t}\int_0^\infty dx \int_0^{\Delta t} \phi(x)\tilde{\rho}_l(t,x)dt\right| &= \left|\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_0^{Mt} \phi(x)\tilde{\rho}_l(t,x)dx\right| \\ \leq C\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_0^{Mt} dx &= \frac{C}{2}M\Delta t \to 0.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{\Delta t}\int_0^{\Delta t} \rho(t,x)dt \longrightarrow \rho_0(x) \text{ in } L^\infty(R) \text{ weak}* \tag{13.56}$$

Consider for the conservation of momentum.

Let $\psi(t,x) \in C_0^1(R_t \times R_x)$ such that $\psi = 0$ on $x = x(t)$. Then,

$$\int_0^\infty dt \int_R \{m\psi_t + (m^2/\rho + P)\psi_x\}dx + \int_R m_0(x)\psi(0,x)dx = 0. \tag{13.57}$$

It follows

$$\left\{\begin{array}{l}\displaystyle\int_0^\infty dt \int_{-\infty}^{x(t)} \{m\psi_t + (m^2/\rho + P)\psi_x\}dx + \int_{-\infty}^0 m_0(x)\psi(0,x)dx = 0 \\ \displaystyle\int_0^\infty dt \int_{x(t)}^\infty \{m\psi_t + (m^2/\rho + P)\psi_x\}dx + \int_0^\infty m_0(x)\psi(0,x)dx = 0\end{array}\right. \tag{13.58}$$

for any $\psi(t,x) \in \mathrm{Lip}_0(R_t \times R_x)$ such that $\psi = 0$ on $x = x(t)$ by the similar argument of Lemma 21. This show

$$\frac{1}{\Delta t}\int_0^{\Delta t} m(t,x)dt \longrightarrow m_0(x) \text{ in } L^\infty \text{ weak}* \tag{13.59}$$

similarly for ρ. And (13.58) yield the existence of functions $\bar{P}_1$ and $\bar{P}_2$. We fix a function $\mu_1(x)$ such that

$$\mu_1(x) \in C^\infty, \quad 0 \leq \mu_1(x) \leq 1, \qquad \mu_1(x) = \left\{\begin{array}{ll} 1 & (x \leq 2), \\ 0 & (x \geq 3). \end{array}\right. \tag{13.60}$$

Then, since the function

$$\psi(t,x) = \phi(t)\xi_2(x - x(t); \Delta x)\mu_1(x) \tag{13.61}$$

is an element of Lip_0 and

$$\psi(t,x(t)) = \psi(0,x) = 0, \tag{13.62}$$

for any $\phi(t) \in C_0^1(0,\infty)$, we have

$$\begin{aligned} 0 &= \int_0^\infty dt \int_{x(t)}^\infty \{m(\phi\xi_2\mu_1)_t + (m^2/\rho + P_2)(\phi\xi_2\mu_1)_x\}dx \\ &= \int_0^\infty \phi(t)\left(\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x} \{m^2/\rho + P_2 - mx'(t)\}dx\right)dt + \int_0^\infty dt \int_{x(t)}^\infty m\phi'\xi_2\mu_1 dx \\ &\quad + \int_0^\infty dt \int_{x(t)+2}^{x(t)+3} (m^2/\rho + P_2)\phi\mu_1' dx, \end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \phi(t)\left(\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x}\{m^2/\rho + P_2 - mx'(t)\}dx\right)dt \\
= &-\int_0^\infty dt\int_{x(t)}^\infty m\phi'\xi_2\mu_1 dx - \int_0^\infty dt\int_{x(t)+2}^{x(t)+3}(m^2/\rho + P_2)\phi\mu_1' dx \\
\to &-\int_0^\infty dt\int_{x(t)}^\infty m\phi'\mu_1 dx - \int_0^\infty dt\int_{x(t)+2}^{x(t)+3}(m^2/\rho + P_2)\phi\mu_1' dx \qquad (\Delta x \downarrow 0)
\end{aligned}$$

for any $\phi \in C_0^1(0,\infty)$. Thus, by the completeness of $L^\infty(0,\infty)$ with respect to the weak* topology, there is a function $\bar{P}_2(t) \in L^\infty(0,\infty)$ such that

$$\frac{1}{\Delta x}\int_{x(t)}^{x(t)+\Delta x}\{(m^2/\rho + P_2) - mx'(t)\}dx \longrightarrow \bar{P}_2(t) \qquad \text{in } L^\infty(0,\infty) \text{ weak*}. \tag{13.63}$$

Similarly, there is a function $\bar{P}_1(t) \in L^\infty(0,\infty)$ such that

$$\frac{1}{\Delta x}\int_{x(t)-\Delta x}^{x(t)}\{(m^2/\rho + P_1) - mx'(t)\}dx \longrightarrow \bar{P}_1(t) \qquad \text{in } L^\infty(0,\infty) \text{ weak*}. \tag{13.64}$$

We take a C^∞ function $\bar{x}_\delta(t)$ as in the proof Lemma 21 for $x(t)$, that is,

$$\int_0^\infty |x'(t) - \bar{x}_\delta'(t)|dt < \delta, \qquad \bar{x}_\delta(0) = 0. \tag{13.65}$$

And we set a C^∞, even function $\theta(x) = \theta(x;\delta,\varepsilon,\Delta x)$ such that

$$\begin{aligned}
&0 \le \theta(x) \le 1, \\
&\theta(x)\begin{cases} = 1 & (0 \le x \le 2\delta), \\ \ge 1-\varepsilon & (2\delta \le x \le 2\delta+\varepsilon), \\ = \text{ linear function} & (2\delta+\varepsilon \le x \le 2\delta+\varepsilon+\Delta x), \\ \le \varepsilon & (2\delta+\varepsilon+\Delta x \le x \le 2\delta+2\varepsilon+\Delta x), \\ = 0 & (2\delta+2\varepsilon+\Delta x \le x). \end{cases}
\end{aligned} \tag{13.66}$$

Then, for any $\phi(t) \in C_0^1(0,\infty)$, the function

$$\psi(t,x) = \phi(t)\theta(x - \bar{x}_\delta(t)) \tag{13.67}$$

is an element of $C_0^1((0,\infty)\times R)$ and satisfies

$$\psi_t(t,x(t)) + x'(t)\psi_x(t,x(t)) = \phi'\theta + \phi\theta'(-\bar{x}_\delta'(t)) + x'(t)\phi\theta' = \phi'(t) \tag{13.68}$$

since

$$\begin{cases} \theta(x(t) - \bar{x}_\delta(t)) = 1, \\ \theta'(x(t) - \bar{x}_\delta(t)) = 0, \end{cases} \qquad (|x(t) - \bar{x}_\delta(t)| < \delta). \tag{13.69}$$

Now, by the definition of $\bar{x}_\delta(t)$,

$$\bar{x}_\delta' \longrightarrow x' \ \text{ in } L^1(0,\infty) \qquad (\delta \downarrow 0). \tag{13.70}$$

Thus,

$$\bar{x}'_\delta(t) \longrightarrow x'(t) \quad \text{a.e.} \tag{13.71}$$

by taking the subsequence if we need. Hence,

$$\begin{aligned}
& \iint_{x>x(t)} \{m\psi_t + (m^2/\rho + P_2)\psi_x\} \\
= & \iint_{x>x(t)} m\{\phi'\theta + \phi\theta'(-\bar{x}'_\delta(t))\} + \iint_{x>x(t)} (m^2/\rho + P_2)\phi\theta' \\
\to & \int_0^\infty dt \int_{x(t)}^{x(t)+\Delta x} \left\{ m\phi' \frac{x(t)+\Delta x - x}{\Delta x} + \phi \frac{-1}{\Delta x}(m^2/\rho + P_2 - x'(t)m) \right\} dx \\
= & O(\Delta x) - \int_0^\infty \phi(t) \left\{ \frac{1}{\Delta x} \int_{x(t)}^{x(t)+\Delta x} (m^2/\rho + P_2 - x'(t)m) dx \right\} dt
\end{aligned}$$

as $\varepsilon \downarrow 0$, $\delta \downarrow 0$. This shows that

$$\begin{aligned}
0 = & \iint_{t>0} \{m\psi_t + (m^2/\rho + P)\psi_x\} + \frac{1}{k}\int_0^\infty (\psi_t + x'(t)\psi_x)_{x=x(t)} x'(t) dt \\
\to & -\int_0^\infty \phi(t) \left\{ \frac{1}{\Delta x} \int_{x(t)}^{x(t)+\Delta x} (m^2/\rho + P_2 - x'(t)m) dx \right\} dt \\
& + \int_0^\infty \phi(t) \left\{ \frac{1}{\Delta x} \int_{x(t)-\Delta x}^{x(t)} (m^2/\rho + P_1 - x'(t)m) dx \right\} dt \\
& + \frac{1}{k}\int_0^\infty \phi'(t)x'(t)dt + O(\Delta x)
\end{aligned}$$

as $\varepsilon \downarrow 0$, $\delta \downarrow 0$. By $\Delta x \downarrow 0$, we obtain

$$-\int_0^\infty \phi(t)\bar{P}_2(t)dt + \int_0^\infty \phi(t)\bar{P}_1(t)dt + \frac{1}{k}\int_0^\infty \phi'(t)x'(t)dt = 0 \tag{13.72}$$

for any $\phi \in C_0^1(0,\infty)$. This yields

$$x''(t) = k(\bar{P}_1(t) - \bar{P}_2(t)) \qquad (t > 0) \tag{13.73}$$

in the distribution sense.

We take a C^∞ function $\mu_2(t) = \mu_2(t;\varepsilon,\Delta t)$ such that

$$0 \le \mu_2(t) \le 1,$$
$$\mu_2(t) \begin{cases} = 1 & (t \le 0), \\ = \text{ linear function} & (0 \le t \le \Delta t), \\ \le \varepsilon & (\Delta t \le t \le \Delta t + \varepsilon), \\ = 0 & (\Delta t + \varepsilon \le t). \end{cases} \tag{13.74}$$

and let

$$\psi(t,x) = \mu_2(t)\phi(x) \tag{13.75}$$

for any $\phi \in C_0^1$. Then,

$$
\begin{aligned}
&\iint_{t>0} \{m\psi_t + (m^2/\rho + P)\psi_x\} \\
&= \iint_{t>0} \{m\mu_2'\phi + (m^2/\rho + P)\mu_2\phi'\} \\
&\to -\frac{1}{\Delta t}\int_0^{\Delta t} dt \int_R m(t,x)\phi(x)dx + O(\Delta t) \qquad (\varepsilon \downarrow 0) \\
&\frac{1}{k}\int_{x=x(t)} x'(t)(\psi_t + x'(t)\psi_x)dt \\
&= \frac{1}{k}\int_0^{\infty} x'(t)\{\mu_2'(t)\phi(x(t)) + x'(t)\mu_2(t)\phi'(x(t))\}dt \\
&\to -\frac{1}{k}\int_0^{\Delta t} x'(t)\frac{1}{\Delta t}\phi(x(t))dt + O(\Delta t) \qquad (\varepsilon \downarrow 0) \\
&= -\frac{1}{k}\int_0^{\Delta t} x'(t)\frac{1}{\Delta t}\phi(0)dt + O(\Delta t) \\
&= -\frac{1}{k}\frac{x(\Delta t) - x(0)}{\Delta t}\phi(0) + O(\Delta t).
\end{aligned}
$$

Thus, it follows

$$
\begin{aligned}
0 &= \iint_{t>0} \{m\psi_t + (m^2/\rho + P)\psi_x\} + \frac{1}{k}\int_{x=x(t)} x'(t)(\psi_t + x'(t)\psi_x)dt \\
&\quad + \frac{1}{k}x_0'\psi(0,0) + \int_R m_0(x)\psi(0,x)dx \\
&\to -\int_R \phi(x)\left(\frac{1}{\Delta t}\int_0^{\Delta t} m(t,x)dt - m_0(x)\right)dx + \frac{\phi(0)}{k}\left\{x_0' - \frac{x(\Delta t) - x(0)}{\Delta t}\right\} \\
&\quad + O(\Delta t) \qquad (\varepsilon \downarrow 0).
\end{aligned}
$$

Hence,

$$
\frac{1}{k}\phi(0)\left\{\frac{x(\Delta t) - x(0)}{\Delta t} - x_0'\right\} \longrightarrow 0 \qquad (\Delta t \downarrow 0). \tag{13.76}
$$

Therefore

$$
\frac{x(\Delta t) - x(0)}{\Delta t} \longrightarrow x_0'. \tag{13.77}
$$

■

Next, we show the "if" part.

The equalities for the conservation of mass are similarly obtained to Problem (I). Thus, we consider for the conservation of momentum.

Proof

(for Problem (III) of Proposition 7, "if" part)

We take $\theta(x)$ used in the proof of the "only if" part, and $\mu_3(t) = \mu(t; a, \Delta t)$ such that

$$
\begin{aligned}
&\mu_3(t) \in C^\infty(0,\infty),\\
&0 \le \mu_3(t) \le 1,\\
&\mu_3(t) \begin{cases} = 1 & (0 \le t \le a), \\ \ge 1-a & (a \le t \le 2a), \\ = \text{ linear function} & (2a \le t \le 2a+\Delta t), \\ \le a & (2a+\Delta t \le t \le 3a+\Delta t), \\ = 0 & (3a+\Delta t \le t). \end{cases}
\end{aligned}
\tag{13.78}
$$

For any $\psi \in C_0^1$, we define ψ_1, ψ_2, ψ_3 as

$$
\begin{aligned}
&\psi = \psi_1 + \psi_2 + \psi_3,\\
&\quad \psi_1 = \psi(t,x)\mu_3(t), \quad \psi_2 = \psi(t,x)(1-\mu_3(t))\theta(x-\bar{x}_\delta(t)),\\
&\quad \psi_3 = \psi(t,x)(1-\mu_3(t))(1-\theta(x-\bar{x}_\delta(t))).
\end{aligned}
\tag{13.79}
$$

Then, since

$$
\psi_2(0,x) = 0, \quad \psi_3(0,x) = \psi_3(t,x(t)) = 0, \tag{13.80}
$$

and

$$
\operatorname{supp}\psi_3 \subset \{(t,x); x > x(t), t > 0\} \cup \{(t,x); x > x(t), t > 0\}, \tag{13.81}
$$

it follows

$$
\iint_{t>0} \{m(\psi_3)_t + (m^2/\rho + P)(\psi_3)_x\} = 0 \tag{13.82}
$$

For ψ_1,

$$
\begin{aligned}
&\iint_{t>0} \{m(\psi_1)_t + (m^2/\rho + P)(\psi_1)_x\}\\
&= \iint_{t>0} \mu_3\{m\psi_t + (m^2/\rho+P)\psi_x\} + \iint_{t>0} \mu_3' m\psi\\
&\to \int_0^{\Delta t} \frac{\Delta t - t}{\Delta t} dt \int_R \{m\psi_t + (m^2/\rho + P)\psi_x\} dx - \int_0^{\Delta t} \frac{1}{\Delta t} dt \int_R m\psi dx \qquad (a \downarrow 0)\\
&= O(\Delta t) - \int_R \psi(0,x) \left(\frac{1}{\Delta t} \int_0^{\Delta t} m dt \right) dx + O(\Delta t)\\
&\to -\int_R m_0(x)\psi(0,x) dx \qquad (\Delta t \downarrow 0).
\end{aligned}
$$

And for ϕ_2, we divide

$$
\iint_{t>0} \{m(\psi_2)_t + (m^2/\rho + P)(\psi_2)_x\} = \iint_{x>x(t)} + \iint_{x<x(t)}. \tag{13.83}
$$

Then, the first term becomes

$$
\begin{aligned}
\iint_{x>x(t)} &= \iint_{x>x(t)} \theta\{m(\psi_4)_t + (m^2/\rho + P_2)(\psi_4)_x\}\\
&\quad + \iint_{x>x(t)} \{m^2/\rho + P_2 - \bar{x}_\delta'(t)m\}\theta'\psi_4\\
&\qquad (\psi_4 = \psi(t,x)(1-\mu_3(t)))
\end{aligned}
$$

Here the first term of the right hand side tends to zero as $\Delta x \downarrow 0$ after $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. As $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, The last term becomes

$$\begin{aligned}
&\iint_{x>x(t)} \{m^2/\rho + P_2 - \bar{x}'_\delta(t)m\}\theta'\psi_4 \\
&\to \ -\int_0^\infty \left\{ \frac{1}{\Delta x} \int_{x(t)}^{x(t)+\Delta x} \{m^2/\rho + P_2 - x'(t)m\}\psi_4 dx \right\} dt \\
&= \ O(\Delta x) - \int_0^\infty \psi_4(t, x(t)) \left(\frac{1}{\Delta x} \int_{x(t)}^{x(t)+\Delta x} \{m^2/\rho + P_2 - x'(t)m\} dx \right) dt \\
&\to \ -\int_0^\infty \psi_4(t, x(t))\bar{P}_2(t) dt \qquad (\Delta x \downarrow 0)
\end{aligned}$$

by taking a subsequence with δ if we need. Therefore, it follows

$$\begin{aligned}
&\iint_{t>0} \{m\psi_t + (m^2/\rho + P)\psi_x\} \\
&\to \ \iint_{t>0} \{m(\psi_1)_t + (m^2/\rho + P)(\psi_1)_x\} - \int_0^\infty \psi_4(t, x(t))\{\bar{P}_2(t) - \bar{P}_1(t)\} dt \\
&= \ \iint_{t>0} \{m(\psi_1)_t + (m^2/\rho + P)(\psi_1)_x\} - \frac{1}{k}\int_0^\infty x'(t)\frac{d}{dt}\psi_4(t, x(t)) dt \\
&\qquad \left(x''(t) = k\{\bar{P}_1(t) - \bar{P}_2(t)\} \right)
\end{aligned}$$

as $\varepsilon \downarrow 0$, $\delta \downarrow 0$ and $\Delta x \downarrow 0$. Here,

$$\frac{d}{dt}\psi_4 = (1 - \mu_3(t))(\psi_t + x'(t)\psi_x) - \mu_3'\psi \tag{13.84}$$

and

$$\begin{aligned}
&\int_0^\infty (1 - \mu_3(t))(\psi_t + x'(t)\psi_x)x'(t) dt \\
&\to \ \int_0^\infty (\psi_t + x'(t)\psi_x)x'(t) dt \qquad (a \downarrow 0, \Delta t \downarrow 0) \\
&-\int_0^\infty \mu_3'\psi x'(t) dt \\
&\to \ \frac{1}{\Delta t}\int_0^{\Delta t} \psi(t, x(t))x'(t) dt \qquad (a \downarrow 0) \\
&= \ \psi(0,0)\frac{1}{\Delta t}\int_0^{\Delta t} x'(t) dt + O(\Delta t) \\
&= \ \psi(0,0)\frac{x(\Delta t) - x(0)}{\Delta t} + O(\Delta t) \\
&\to \ x_0'\psi(0,0) \qquad (\Delta t \downarrow 0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\iint_{t>0} \{m\psi_t + (m^2/\rho + P)\psi_x\} \\
&\to \ -\int_R m_0(x)\psi(0, x) dx - \frac{1}{k}\int_{x=x(t)} x'(t)(\psi_t + x'(t)\psi_x) dt \\
&\qquad -\frac{1}{k}\psi(0,0)x_0'.
\end{aligned}$$

∎

13.2 The compactness of $\eta_t + q_x$

In this section, we show the compactness of $\eta_t + q_x$ for the weak entropy pair (η, q) in order to use Theorem 18.

First, we consider Problem (I).

Let Ω be a bounded subset of $R_t \times R_x$. We show

$$\{\eta(U^\Delta(t,x))_t + q(U^\Delta(t,x))_x\}_{\Delta x \downarrow 0} \tag{13.85}$$

is in a compact set of $H^{-1}_{loc}(\Omega)$.

Let $L_0 = L_0(\Omega)$ and $T_0 = T_0(\Omega)$ be values such that

$$\bar{\Omega} \cap [0,\infty) \times R \subset (-L_0, L_0) \times [0, T_0), \qquad \operatorname*{ess.\,sup}_{t>0} |x_1'(t)| \leq \frac{L_0}{T_0+1}, \tag{13.86}$$

and let l and m be integers such that

$$2(l-1)\Delta x < L_0 \leq 2l\Delta x, \qquad (m-1)\Delta t < T_0 \leq m\Delta t. \tag{13.87}$$

If $\Delta x \leq 1/2$, $\Delta t \leq 1$, then

$$\begin{array}{l} 2l\Delta x \leq L_0 + 2\Delta x \leq L_0 + 1, \quad m\Delta t \leq T_0 + \Delta t \leq T_0 + 1, \\ \text{ess.}\sup_{t>0} |x_1'(t)| \leq \frac{2l\Delta x}{m\Delta t}. \end{array} \tag{13.88}$$

Let $T = m\Delta t$ and $L = 2l\Delta x$. We note that $U^\Delta(t,x) = {}^t(0,0)$ for $x < x_1^\Delta(t)$ or $t < 0$. We consider the estimate for (η_*, q_*), that is, the energy estimate before the general entropy pair (η, q).

Since U^Δ is a solution of $U_t + F(U)_x = 0$ which is smooth almost everywhere, we have

$$\eta_*(U^\Delta) + q_*(U^\Delta)_x = 0 \qquad \text{a.e.}, \tag{13.89}$$

and we integrate this in

$$K = \bigcup_{n=1}^{m} \{(t,x); (n-1)\Delta t \leq t < n\Delta t,\ x_1^\Delta(t) < x < (2l+\varepsilon_n)\Delta x\}, \tag{13.90}$$

where

$$\varepsilon_n = \begin{cases} 1 & (n \text{ is odd}), \\ 0 & (n \text{ is even}). \end{cases} \tag{13.91}$$

Then,

$$\begin{aligned} 0 &= \iint_K \{\eta_*(U^\Delta)_t + q_*(U^\Delta)_x\} dx dt \\ &= \sum_{n=1}^{m} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_{x_1^\Delta(t)}^{(2l+\varepsilon_n)\Delta x} \{\eta_*(U^\Delta)_t + q_*(U^\Delta)_x\} dx \end{aligned}$$

We divide the integral to the parts. Each of the parts is on a domain where U^Δ is smooth. Then, by using the divergence theorem, we reduce the integrals to the line integrals. But, if the line is a common boundary of two domains and U^Δ is continuous neighbourhood of the line, then the integral for the line does not appear by canceling of two integrals. The rests are the line integrals on the shock waves, on $t = n\Delta t$ for $n = 0, 1, 2, \dots$, and on boundaries $x = x_1^\Delta(t)$, $x = L$, and $x = L + \Delta x$.

$$\begin{aligned} 0 &= \sum_{n=1}^{m} \left\{ \int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \eta_*(U^\Delta(n\Delta t - 0, x))dx \right. \\ &\quad \left. - \int_{x_1^\Delta((n-1)\Delta t)}^{L+\varepsilon_n\Delta x} \eta_*(U^\Delta((n-1)\Delta t + 0, x))dx \right\} + \int_0^T \sum_{\text{shock}} (\sigma[\eta_*] - [q_*])dt \\ &\quad + \int_0^T \{x_1^\Delta(t)'\eta_* - q_*\}(t, x_1^\Delta(t) + 0)dt + \sum_{n=1}^{m} \int_{(n-1)\Delta t}^{n\Delta t} q_*(U^\Delta(t, L + \varepsilon_n\Delta x))dt \end{aligned}$$

where, $\sum_{\text{shock}}$ is the sum over all shock wave in K, and σ is the velocity of each shock wave. This becomes

$$\begin{aligned} 0 &= \sum_{n=0}^{m} \int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \left[\eta_*(U^\Delta)\right]_{t=n\Delta t+0}^{t=n\Delta t-0} dx + \int_{x_1^\Delta(T)}^{L+\varepsilon_m\Delta x} \eta_*(U^\Delta(T+0, x))dx \\ &\quad - \int_0^L \eta_*(U_0(x))dx - \sum_{n=0}^{m-1} \int_{L+\varepsilon_n\Delta x}^{L+\varepsilon_{n+1}\Delta x} \eta_*(U^\Delta(n\Delta t + 0, x)))dx \\ &\quad + \int_0^T \sum_{\text{shock}} (\sigma[\eta_*] - [q_*])dt + \int_0^T \{x_1^\Delta(t)'\eta_* - q_*\}(t, x_1^\Delta(t) + 0)dt \\ &\quad + \sum_{n=1}^{m} \Delta t q_*(\bar{U}_{2l+\varepsilon_n}^{n-1}) \end{aligned}$$

Here

$$\int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \left[\eta_*(U^\Delta)\right]_{n\Delta t+0}^{n\Delta t-0} dx = \sum_{j\in \tilde{J}_1, j\le 2l} \int_{\tilde{E}_j^n} \left[\eta_*(U^\Delta)\right]_{n\Delta t+0}^{n\Delta t-0} dx, \tag{13.92}$$

and if $\rho^\Delta(n\Delta t + 0, x) > 0$ and $\rho^\Delta(n\Delta t - 0, x) > 0$, then

$$\begin{aligned} &[\eta_*(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} \\ &= \nabla\eta_*(U^\Delta(n\Delta t + 0, x)) \left[U^\Delta\right]_{t=n\Delta t+0}^{t=n\Delta t-0} \\ &+ \int_0^1 (1-y)^t[U^\Delta]\nabla^2\eta_*(U^\Delta(n\Delta t + 0, x) + y[U^\Delta])[U^\Delta]dy, \end{aligned}$$

and

$$\begin{aligned} &\int_{\tilde{E}_j^n} \left[U^\Delta\right]_{n\Delta t+0}^{n\Delta t-0} dx \\ &= \int_{\tilde{E}_j^n} \{U^\Delta(n\Delta t - 0, x) - U^\Delta(n\Delta t + 0, x)\}dx \\ &= \int_{\tilde{E}_j^n} U^\Delta(n\Delta t - 0, x)dx - \bar{U}_j^n m(\tilde{E}_j^n) \\ &= 0. \end{aligned}$$

On the other hand, since

$$\rho^\Delta(n\Delta t+0,x)=\frac{1}{m(\tilde{E}_j^n)}\int_{\tilde{E}_j^n}\rho^\Delta(n\Delta t-0,y)dy\geq 0 \qquad (x\in\tilde{E}_j^n), \tag{13.93}$$

if $\rho^\Delta(n\Delta t+0,x)=0$ then $\rho^\Delta(n\Delta t-0,x)=0$ on $\tilde{E}_j^n$. Therefore,

$$\int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon\Delta x}[\eta_*(U^\Delta)]dx=\sum_{j\in\tilde{J}_1^n, j\leq 2l,\bar{\rho}_j^n>0}\int_{\tilde{E}_j^n}dx\int_0^1(1-y)^t[U^\Delta]\nabla^2\eta_*(\bar{U}_j^n+y[U^\Delta])[U^\Delta]dy. \tag{13.94}$$

For a shock wave,

$$\begin{cases}\sigma[\eta_*]-[q_*]\geq 0,\\ |\sigma[\eta]-[q]|\leq C_3(\sigma[\eta_*]-[q_*])\end{cases} \tag{13.95}$$

hold, where C_3 is the constant in Lemma 17. For the proof, see [5]. Therefore, we have the energy estimate

$$\begin{aligned}
&\sum_{n=0}^m\sum_{j\in\tilde{J}_1^n, j\leq 2l,\bar{\rho}_j^n>0}\int_{\tilde{E}_j^n}dx\int_0^1(1-y)^t[U^\Delta]\nabla^2\eta_*(\bar{U}_j^n+y[U^\Delta])[U^\Delta]dy \\
&\quad+\int_0^T\sum_{\text{shock}}(\sigma[\eta_*]-[q_*])dt \\
&=\int_0^L\eta_*(U_0(x))dx-\int_{x_1^\Delta(T)}^{L+\varepsilon_m\Delta x}\eta_*(U^\Delta(T+0,x))dx+\sum_{n=0}^{m-1}(-1)^n\eta_*(\bar{U}_{2l+\varepsilon_{n+1}}^n)\Delta x \\
&\quad+\int_0^T\{q_*-x_1^\Delta(t)'\eta_*\}(t,x_1^\Delta(t)+0)dt-\sum_{n=1}^m\Delta t q_*(\bar{U}_{2l+\varepsilon_n}^{n-1}) \\
&\quad\leq C(\Omega,\Sigma_1,x_1,\Lambda_1)
\end{aligned} \tag{13.96}$$

since U^Δ is uniformly bounded, and

$$m\Delta x\leq m\Delta t\frac{\Delta x}{\Delta t}=T\Lambda_1. \tag{13.97}$$

Moreover, by Lemma 17, we have

$$\sum_{n=0}^m\sum_{j\in\tilde{J}_1^n, j\leq 2l}\int_{\tilde{E}_j^n}|[U^\Delta]|^2dx\leq C(\Omega,\Sigma_1,x_1,\Lambda_1) \tag{13.98}$$

Next, we consider for the (η,q). Let (η,q) be a smooth weak entropy pair satisfying Lemma 17 and let $\phi\in C_0^1(\Omega)$. Then,

$$\begin{aligned}
&\iint\{\eta(U^\Delta)\phi_t+q(U^\Delta)\phi_x\}dxdt \\
&=\iint_{x>x_1^\Delta(t)}(\eta^\Delta\phi_t+q^\Delta\phi_x)dxdt \\
&=\sum_{n=1}^m\int_{(n-1)\Delta t}^{n\Delta t}dt\int_{x_1^\Delta(t)}^{L+\varepsilon_n\Delta x}(\eta^\Delta\phi_t+q^\Delta\phi_x)dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{m}\int_{(n-1)\Delta t}^{n\Delta t} dt \int_{x_1^\Delta(t)}^{L+\varepsilon_n\Delta x} \{(\eta^\Delta\phi)_t + (q^\Delta\phi)_x\} dx \\
&= \sum_{n=1}^{m}\left\{\int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \{\eta^\Delta\phi\}_{t=n\Delta t-0} dx - \int_{x_1^\Delta((n-1)\Delta t)}^{L+\varepsilon_n\Delta x} \{\eta^\Delta\phi\}_{t=(n-1)\Delta t-0} dx\right\} \\
&\quad + \int_0^T \sum_{\text{shock}} (\sigma[\eta^\Delta] - [q^\Delta])\phi dt + \int_0^T \{(x_1^\Delta(t)'\eta^\Delta - q^\Delta)\phi\}(t, x_1^\Delta(t)) dt \\
&= \sum_{n=0}^{m-1}\int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \phi(n\Delta t, x)[\eta^\Delta]_{n\Delta t+0}^{n\Delta t-0} dx - \int_0^L \phi(0,x)\eta(U_0(x)) dx \\
&\quad + \int_0^T \sum_{\text{shock}} \phi(\sigma[\eta^\Delta] - [q^\Delta]) dt + \int_0^T \{(x_1^\Delta(t)'\eta^\Delta - q^\Delta)\phi\}(t, x_1^\Delta(t)) dt \\
&= L(\phi) + M(\phi) + \Sigma(\phi) + \Pi(\phi)
\end{aligned}
$$

where

$$
\begin{aligned}
L(\phi) &= \sum_{n=0}^{m-1}\int_{x_1^\Delta(n\Delta t)}^{L+\varepsilon_n\Delta x} \phi(n\Delta t, x)[\eta^\Delta]_{n\Delta t+0}^{n\Delta t-0} dx \\
M(\phi) &= -\int_0^L \phi(0,x)\eta(U_0(x)) dx \\
\Sigma(\phi) &= \int_0^T \sum_{\text{shock}} \phi(\sigma[\eta^\Delta] - [q^\Delta]) dt \\
\Pi(\phi) &= \int_0^T \{(x_1^\Delta(t)'\eta^\Delta - q^\Delta)\phi\}(t, x_1^\Delta(t)) dt
\end{aligned}
$$

and we denote $\eta(U^\Delta(t,x))$ by $\eta^\Delta(t,x)$, and so on. Now we divide

$$
\begin{aligned}
L(\phi) &= \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1}\int_{\bar{E}_j^n} \phi(n\Delta t, x)[\eta^\Delta] dx \\
&= L_1(\phi) + L_2(\phi),
\end{aligned}
$$

where

$$
\begin{aligned}
L_1(\phi) &= \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1} \phi_j^n \int_{\bar{E}_j^n} [\eta^\Delta] dx, \\
&= \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1^n, \bar{\rho}_j^n>0} \phi_j^n \int_{\bar{E}_j^n} dx \int_0^1 (1-y)^t[U^\Delta]\nabla^2\eta(\bar{U}_j^n + y[U^\Delta])[U^\Delta] dy, \\
L_2(\phi) &= \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1}\int_{\bar{E}_j^n} \{\phi(n\Delta t, x) - \phi_j^n\}[\eta^\Delta] dx, \\
\phi_j^n &= \phi(n\Delta t, j\Delta x).
\end{aligned}
$$

Then, from (13.97) and by Lemma 17, we have

$$
|L_1(\phi)| \le \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1^n, j\le 2l, \bar{\rho}_j^n>0} \|\phi\|_{C(\Omega)}
$$

$$
\begin{aligned}
& \times \int_{\bar{E}_j^n} dx \int_0^1 (1-y)|^t[U^\Delta]\nabla^2\eta(\bar{U}_j^n + y[U^\Delta])[U^\Delta]|dy \\
& \le C(\Omega, \Sigma_1, x_1, \Lambda_1, \eta)\|\phi\|_{C(\Omega)}, \\
|M(\phi)| & \le C(\Omega, \Sigma_1, \eta)\|\phi\|_{C(\Omega)}, \\
|\Sigma(\phi)| & \le C(\Omega, \Sigma_1, x_1, \eta, \Lambda_1)\|\phi\|_{C(\Omega)}, \\
|\Pi(\phi)| & \le C(\Omega, \Sigma_1, x_1, \eta)\|\phi\|_{C(\Omega)}.
\end{aligned}
$$

For $0 < \alpha < 1$, it follows

$$
\begin{aligned}
|L_2(\phi)| & \le \|\phi\|_{C^\alpha(\bar{\Omega})}(3\Delta x)^\alpha \sum_{n=0}^{m-1} \sum_{j \in \bar{J}_1^n, j \le 2l} \int_{\bar{E}_j^n} |[\eta^\Delta]_{n\Delta t+0}^{n\Delta t-0}| dx \\
& \le C(\Omega, \alpha, \eta, \Sigma_1)\|\phi\|_{C^\alpha(\bar{\Omega})}(\Delta x)^\alpha \sqrt{mL} \left(\sum_{n=0}^{m-1} \sum_{j \in \bar{J}_1^n, j \le 2l} \int_{\bar{E}_j^n} |[U^\Delta]|^2 dx \right)^{1/2} \\
& \le C(\Omega, \alpha, \eta, \Sigma_1, x_1, \Lambda_1)\|\phi\|_{C^\alpha(\bar{\Omega})}(\Delta x)^\alpha \sqrt{m} \\
& \qquad \left(\sqrt{m} = \sqrt{m\Delta t}\sqrt{\frac{\Delta x}{\Delta t}}(\Delta x)^{-1/2} = \sqrt{T\Lambda_1}(\Delta x)^{-1/2} \right).
\end{aligned}
$$

Then, setting the linear operator $T_1(\phi)$, $T_2(\phi)$ as

$$
T_1(\phi) = L_1(\phi) + M(\phi) + \Sigma(\phi) + \Pi(\phi), \quad T_2(\phi) = L_2(\phi), \tag{13.99}
$$

it follows that

$$
\begin{cases}
\iint \{\eta(U^\Delta)\phi_t + q(U^\Delta)\phi_x\} = T_1(\phi) + T_2(\phi) \\
|T_1(\phi)| \le C\|\phi\|_{C(\Omega)} \\
|T_2(\phi)| \le C(\Delta x)^{\alpha - 1/2}\|\phi\|_{C^\alpha(\bar{\Omega}))}
\end{cases} \tag{13.100}
$$

for any $\phi \in C_0^1(\Omega)$ and $1/2 < \alpha < 1$.

From the fact, we show the compactness of $\eta_t^\Delta + q_x^\Delta$ by using the following theorems.

THEOREM 22 *Let Ω be a bounded open subset of R^N and let $\mathcal{A}$ be a subset of*

$$
\mathcal{M}(\Omega) = \{\mu; \mu \text{ is a signed Radon measure on } \Omega\}. \tag{13.101}
$$

If

$$
\sup_{\mu \in \mathcal{A}, C_0(\Omega) \ni \phi \ne 0} \frac{|<\mu, \phi>|}{\|\phi\|_{C(\Omega)}} < \infty \tag{13.102}
$$

then there exists an imbedding

$$
A \hookrightarrow W^{-1,p}(\Omega) \, (= \text{ dual space of } W_0^{1,p/(p-1)}(\Omega)) \tag{13.103}
$$

and $\mathcal{A}$ is relatively compact in $W^{-1,p}(\Omega)$ for any p such that $1 < p < N(N-1)$

(cf. [10],[26])

THEOREM 23 (Murat's lemma) *Let q and r be the values $1 < q \leq 2 < r < \infty$ and let Ω be a bounded open subset of R^N. Then,*

$$\begin{aligned} & (compactsetof W^{-1,q}(\Omega)) \cap (boundedsetof W^{-1,r}(\Omega)) \\ \subset \; & (compactsetof H^{-1}_{loc}(\Omega)) \end{aligned}$$

(cf. [5],[26])

Since Ω is bounded, the Sobolev's imbedding theorem

$$W_0^{1,p}(\Omega) \hookrightarrow C^{\beta}(\bar{\Omega}) \qquad (p > 2,\, 0 \leq \beta \leq 1 - 2/p) \tag{13.104}$$

holds. Thus, for $p \geq 2/(1-\alpha)$,

$$|T_2(\phi)| \leq C_p(\Delta x)^{\alpha - 1/2} \|\phi\|_{W_0^{1,p}(\Omega)}, \tag{13.105}$$

and hence

$$\|T_2\|_{W^{-1,q}(\Omega)} \leq C_p(\Delta x)^{\alpha-1/2} \to 0 \quad \text{as } \Delta x \downarrow 0, \tag{13.106}$$

for $q = p/(p-1) \in (1,2)$. Therefore, T_2 is in a compact set of $W^{-1,q}(\Omega)$ for a $q \in (1,2)$. And T_1 is in a compact set of $W^{-1,q}(\Omega)$ by Theorem 22. Therefore $\eta_t^{\Delta} + q_x^{\Delta}$ is so. On the other hand, for $1 < p < 2$,

$$\left| \iint \{\eta(U^{\Delta})\phi_t + q(U^{\Delta})\phi_x\} \right| \leq C(\Omega, p, \eta, \Sigma_1) \|\phi\|_{W_0^{1,p}(\Omega)} \tag{13.107}$$

since U^{Δ} is bounded. This shows $\eta_t^{\Delta} + q_x^{\Delta}$ is in a bounded set of $W^{-1,r}(\Omega)$ for $r = p/(p-1) > 2$. Therefore, $\eta_t^{\Delta} + q_x^{\Delta}$ is in a compact set of $H^{-1}_{loc}(\Omega)$ by the Murat's theorem 23.

PROPOSITION 24 *Let Ω be a bounded set of $R_t \times R_x$ and let (η, q) be a smooth weak entropy pair satisfying Lemma 17. Then, for the approximate solution $U^{\Delta}(t,x)$ for Problem* (I) *constructed in the section 10.1,*

$$\{\eta(U^{\Delta})_t + q(U^{\Delta})_x\}_{\Delta x \downarrow 0} \tag{13.108}$$

is in a compact set of $H^{-1}_{loc}(\Omega)$.

If $\eta_t^{\Delta} + q_x^{\Delta}$ is divided to T_1, T_2 satisfying (13.100), then we can prove the compactness for Problems (II) and (III).

For Problem (II), U^{Δ} is not uniformly bounded globally, but is uniformly bounded for a bounded set Ω. Hence, we can calculate similarly except for the term on $x = x_2^{\Delta}(t)$. However, the term has the similar estimate to $x = x_1^{\Delta}(t)$. Thus, we obtain the following proposition.

PROPOSITION 25 *Let Ω be a bounded set of $R_t \times R_x$ and let (η, q) be a smooth weak entropy pair satisfying Lemma 17. Then, for the approximate solution $U^\Delta(t,x)$ for Problem* (II) *constructed in the section 10.2,*

$$\{\eta(U^\Delta)_t + q(U^\Delta)_x\}_{\Delta x \downarrow 0} \tag{13.109}$$

is in a compact set of $H^{-1}_{loc}(\Omega)$.

For Problem (III), we consider two part divided with the free piston

$$\begin{aligned} U_l^\Delta(t,x) &= \begin{cases} U^\Delta(t,x) & (x < x^\Delta(t), t > 0) \\ 0 & (x > x^\Delta(t), t > 0), \end{cases} \\ U_r^\Delta(t,x) &= \begin{cases} 0 & (x < x^\Delta(t), t > 0) \\ U^\Delta(t,x) & (x > x^\Delta(t), t > 0). \end{cases} \end{aligned} \tag{13.110}$$

The structures of U_l^Δ and U_r^Δ are the same as the approximate solution for Problem (I). Therefore, the compactness is led similarly.

PROPOSITION 26 *Let Ω be a bounded set of $R_t \times R_x$ and let $(\eta_l(U;P_1), q_l(U;P_1))$ and $(\eta_r(U;P_2), q_r(U;P_2))$ be smooth weak entropy pairs satisfying Lemma 17. Then, for the divided approximate solution for Problem* (III) *$U_l^\Delta(t,x)$, $U_r^\Delta(t,x)$ defined by (13.110)*

$$\begin{aligned} &\{\eta_l(U_l^\Delta;P_1)_t + q_l(U_l^\Delta;P_1)_x\}_{\{\Delta x \downarrow 0\}} \\ &\{\eta_r(U_r^\Delta;P_2)_t + q_r(U_r^\Delta;P_2)_x\}_{\{\Delta x \downarrow 0\}} \end{aligned} \tag{13.111}$$

is in a compact set of $H^{-1}_{loc}(\Omega)$.

Last, we consider for Problem (IV).

For the problem, we can only construct the approximate solution $0 < t < T_1$ for any $T_1 < T_0^\infty$. Then, for fixed $T_1 < T_0^\infty$, we consider such approximate solutions defined in $[0, T_1] \times (1, \infty)$ in this section.

Let Ω be a bounded open subset of $R_t \times R_r$ such that

$$\bar{\Omega} \cap [0,\infty) \times [1,\infty) \subset [0, T_1) \times [1, \infty), \tag{13.112}$$

and let T_0 and R_0 be values such that

$$\begin{aligned} &\bar{\Omega} \cap [0,\infty) \times [1,\infty) \subset [0, T_0) \times [1, R_0), \qquad 0 < T_0 < T_1, \quad 1 < R_0 < \infty \\ &\qquad (T_0 = T_0(\Omega),\ R_0 = R_0(\Omega)). \end{aligned} \tag{13.113}$$

We use the notation in the section 10.4. Suppose that δ is so small that

$$T_1 < T_1^\infty(\delta) < T_0^\infty \tag{13.114}$$

and we take B such that

$$B > \beta_1(T_1) = \beta_1(T_1;\delta). \tag{13.115}$$

And let $\Sigma_4 = \Sigma(B,-B)$. We constructed the approximate solution $U^\Delta(t,r)$ for so small mesh length Δr and Δt that

$$\Delta t < \frac{\delta}{2B}, \quad \frac{\Delta r}{\Delta t} \geq 2B = \Lambda(\Sigma_4). \tag{13.116}$$

In this section, we also suppose

$$\Delta t < T_1 - T_0, \ \frac{\Delta r}{\Delta t} \leq C(T_1). \tag{13.117}$$

Then, there exist a mesh point for t in the interval (T_0, T_1) and $U^\Delta(t,r)$ is in Σ_4 till the mesh point. Let l and m be integers such that

$$(m-1)\Delta t \leq T_0 < m\Delta t, \qquad 1+2(l-1)\Delta r < R_0 \leq 1+2l\Delta r. \tag{13.118}$$

Then, for $\Delta r \leq 1/2$ and $\Delta t \leq 1$,

$$\begin{aligned} &1+2l\Delta r < R_0 + 2\Delta r \leq 1 + R_0 = 1 + R_0(\Omega), \\ &m\Delta t \leq T_0 + \Delta t \leq 1 + T_0 = 1 + T_0(\Omega). \end{aligned} \tag{13.119}$$

Let $T = m\Delta t$ and $L = 1 + 2l\Delta r$.

First, we obtain the energy estimate by integrating the equality

$$\eta_*(U_0^\Delta)_t + q_*(U_0^\Delta)_r = 0 \qquad \text{a.e.} \tag{13.120}$$

Then,

$$\begin{aligned} 0 &= \sum_{n=1}^{m}\int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^{L+\varepsilon_n\Delta r} \{\eta_*(U_0^\Delta)_t + q_*(U_0^\Delta)_r\} dr \\ &= \sum_{n=1}^{m}\int_1^{L+\varepsilon_n\Delta r} \left\{\eta_*(U_0^\Delta(n\Delta t - 0, r)) - \eta_*(U_0^\Delta((n-1)\Delta t + 0, r))\right\} dr \\ &\quad + \int_0^T \sum_{\text{shock}} (\sigma[\eta_*(U_0^\Delta)] - [q_*(U_0^\Delta)]) dt - \int_0^T q_*(U_0^\Delta(t,1)) dt \\ &\quad + \sum_{n=1}^{m}\int_{(n-1)\Delta t}^{n\Delta t} q_*(U_0^\Delta(t, L+\varepsilon_n\Delta r)) dt \\ &= \sum_{n=1}^{m}\int_1^{L+\varepsilon_n\Delta r} [\eta_*(U_0^\Delta)]_{n\Delta t+0}^{n\Delta t-0} dr + \int_0^T \sum_{\text{shock}} (\sigma[\eta_*(U_0^\Delta)] - [q_*(U_0^\Delta)]) dt \\ &\quad - \int_0^T q_*(U_0^\Delta(t,1)) dt + \int_1^{L+\varepsilon_m\Delta r} \eta_*(U_0^\Delta(T+0,r)) dr - \int_1^{L+\Delta r} \eta_*(U_0^\Delta(+0,r)) dr \\ &\quad - \sum_{n=1}^{m-1}\int_{L+\varepsilon\Delta r}^{L+\varepsilon_{n+1}\Delta r} \eta_*(U_0^\Delta(n\Delta t + 0, r)) dr + \sum_{n=1}^{m} \Delta t q_*(\bar{U}_{2l+\varepsilon_n}^{n-1}. \end{aligned}$$

Since the boundary condition $m_0^\Delta(t,1)=0$ yields $q_*(U_0^\Delta(t,1))=0$, we have

$$\begin{aligned}
&\sum_{n=1}^{m}\int_1^{L+\varepsilon_n\Delta r}[\eta_*(U_0^\Delta)]_{n\Delta t+0}^{n\Delta t-0}dr+\int_0^T\sum_{\text{shock}}(\sigma[\eta_*(U_0^\Delta)]-[q_*(U_0^\Delta)])dt\\
=&\int_1^{L+\Delta r}\eta_*(U_0^\Delta)(0,r)dr-\int_1^{L+\varepsilon_m\Delta r}\eta_*(U_0^\Delta)(T+0,r)dr\\
&+\sum_{n=1}^{m}(-1)^{\varepsilon_n}\eta_*(\bar{U}_{0,2l+\varepsilon_n-1}^n)\Delta r\\
\le& C(\Sigma_4,\Omega,T_1).
\end{aligned}$$

We divide

$$\begin{aligned}
&\int_1^{L+\varepsilon_n\Delta r}[\eta_*(U_0^\Delta)]_{n\Delta t+0}^{n\Delta t-0}dr\\
=&\int_1^{L+\varepsilon_n\Delta r}\{\eta_*(U_0^\Delta(n\Delta t-0,r))-\eta_*(U^\Delta(n\Delta t-0,r))\}dr\\
&+\int_1^{L+\varepsilon_n\Delta r}\{\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(U_0^\Delta(n\Delta t+0,r))\}dr.
\end{aligned}$$

Then, since

$$\begin{cases}|\nabla\eta_*(U^\Delta+y(U_0^\Delta-U^\Delta))|\le C(\Sigma_4),\\ (U_0^\Delta-U^\Delta)(n\Delta t-0,r)|=|H(r,U_0^\Delta)\Delta t|\le C(\Sigma_4)\Delta t,\end{cases}\tag{13.121}$$

it follows

$$\begin{aligned}
&|\eta_*(U_0^\Delta(n\Delta t-0,r))-\eta_*(U^\Delta(n\Delta t-0,r))|\\
=&\left|\int_0^1\nabla\eta_*(U^\Delta+y(U_0^\Delta-U^\Delta))dy(U_0^\Delta-U^\Delta)(n\Delta t-0,r)\right|\\
\le& C(\Sigma_4)\Delta t.
\end{aligned}$$

On the other hand, in

$$\begin{aligned}
&\int_1^{L+\varepsilon_n\Delta r}\{\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(U_0^\Delta(n\Delta t+0,r))\}dr\\
=&\sum_{j\in\tilde{J}_4^n,j\le 2l}\int_{\tilde{E}_j^n}\{\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(\bar{U}_j^n)\}dr,
\end{aligned}$$

if $\bar{\rho}_j^n=0$, then $\rho^\Delta(n\Delta t-0,r)=0$ on $\tilde{E}_j^n$, and hence

$$\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(\bar{U}_j^n)=0,\tag{13.122}$$

if $\bar{\rho}_j^n>0$, then

$$\begin{aligned}
&\int_{\tilde{E}_j^n}\{\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(\bar{U}_j^n)\}dr\\
=&\int_{\tilde{E}_j^n}dr\int_0^1(1-y)^t(U^\Delta-\bar{U}_j^n)\nabla^2\eta_*(\bar{U}_j^n+y(U^\Delta-\bar{U}_j^n))(U^\Delta-\bar{U}_j^n)dy.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\int_1^{L+\varepsilon_n\Delta r}\{\eta_*(U^\Delta(n\Delta t-0,r))-\eta_*(U_0^\Delta(n\Delta t+0,r))\}dr\\
&= \sum_{j\in \bar{J}_4^n, j\le 2l, \bar{\rho}_j^n>0}\int_{\bar{E}_j^n}dr\int_0^1(1-y)^t(U^\Delta-\bar{U}_j^n)\nabla^2\eta_*(\bar{U}_j^n+y(U^\Delta-\bar{U}_j^n))(U^\Delta-\bar{U}_j^n)dy\\
&\ge C(\Sigma_4)^{-1}\sum_{j\in \bar{J}_4^n, j\le 2l}\int_{\bar{E}_j^n}|U^\Delta(n\Delta t-0,r)-\bar{U}_j^n|^2dr.
\end{aligned}$$

Therefore, we have the energy estimate

$$\begin{aligned}
&\sum_{n=1}^m\sum_{j\in \bar{J}_4^n, j\le 2l, \bar{\rho}_j^n>0}\int_{\bar{E}_j^n}dr\int_0^1(1-y)^t(U^\Delta-\bar{U}_j^n)\nabla^2\eta_*(\bar{U}_j^n+y(U^\Delta-\bar{U}_j^n))(U^\Delta-\bar{U}_j^n)dy\\
&\qquad+\int_0^1\sum_{\text{shock}}(\sigma([\eta_*(U_0^\Delta)]-[q_*(U_0^\Delta)])dt\le C(\Sigma_4,\Omega,T_1) \qquad (13.123)\\
&\sum_{n=1}^m\sum_{j\in \bar{J}_4^n, j\le 2l}\int_{\bar{E}_j^n}|U^\Delta(n\Delta t-0,r)-\bar{U}_j^n|^2dr\le C(\Sigma_4,\Omega,T_1) \qquad (13.124)
\end{aligned}$$

1

We consider for weak entropy pair (η,q). Let $\phi\in C_0^1(\Omega)$ and we set

$$I(\phi)=\iint\{\eta(U^\Delta)\phi_t+q(U^\Delta)\phi_r\}. \qquad (13.125)$$

We divide

$$I(\phi)=N(\phi)+\iint_{t>0,r>1}\{\eta(U_0^\Delta)\phi_t+q(U_0^\Delta)\phi_r\} \qquad (13.126)$$

where

$$N(\phi)=\iint_{t>0,r>1}\left(\{\eta(U^\Delta)-\eta(U_0^\Delta)\}\phi_t+\{q(U^\Delta)-q(U_0^\Delta)\}\phi_r\right). \qquad (13.127)$$

Since U_0^Δ satisfies $(U_0^\Delta)_t+F(U_0^\Delta)_r=0$ almost everywhere, we can write

$$\begin{aligned}
I(\phi) &= N(\phi)+\iint_{t>0,r>1}\{(\eta_0^\Delta\phi)_t+(q_0^\Delta\phi)_r\}\\
&= N(\phi)+\Sigma(\phi)+M(\phi)+L(\phi)+\Pi(\phi),
\end{aligned}$$

where

$$\begin{aligned}
\Sigma(\phi) &= \int_0^T\sum_{\text{shock}}\phi(\sigma[\eta_0^\Delta]-[q_0^\Delta])dt,\\
M(\phi) &= -\int_1^{L+\Delta r}\phi(0,r)\eta_0^\Delta(+0,r)dr,\\
L(\phi) &= \sum_{n=1}^{m-1}\int_1^{L+\varepsilon_n\Delta r}\phi(n\Delta t,r)[\eta_0^\Delta]_{n\Delta t+0}^{n\Delta t-0}dr,\\
\Pi(\phi) &= -\int_0^T(q_0^\Delta\phi)(t,1)dt.
\end{aligned}$$

We divide again

$$L(\phi) = L_1(\phi) + L_2(\phi) + L_3(\phi), \tag{13.128}$$

where

$$\begin{aligned} L_1(\phi) &= \sum_{n=1}^{m-1} \sum_{j\in \bar{J}_4^n, j\le 2l} \phi_j^n \int_{\tilde{E}_j^n} \{\eta^\Delta(n\Delta t - 0, r) - \eta(\bar{U}_j^n)\} dr \\ &\qquad (\phi_j^n = \phi(n\Delta t, 1 + j\Delta r)), \\ L_2(\phi) &= \sum_{n=1}^{m-1} \sum_{j\in \bar{J}_4^n, j\le 2l} \int_{\tilde{E}_j^n} \phi(n\Delta t, r)\{\eta_0^\Delta(n\Delta t - 0, r) - \eta^\Delta(n\Delta t - 0, r)\} dr, \\ L_3(\phi) &= \sum_{n=1}^{m-1} \sum_{j\in \bar{J}_4^n, j\le 2l} \int_{\tilde{E}_j^n} \{\phi(n\Delta t, r) - \phi_j^n\}\{\eta^\Delta(n\Delta t - 0, r) - \eta(\bar{U}_j^n)\} dr. \end{aligned}$$

Similarly for Problem (I), we can show

$$\begin{aligned} &|(M + L_1 + \Pi + \Sigma)(\phi)| \le C(\Sigma_4, \Omega, T_1, \eta)\|\phi\|_{C(\Omega)} \\ &|L_3(\phi)| \le C(\Sigma_4, \Omega, T_1, \eta, \alpha)(\Delta r)^{\alpha - 1/2}\|\phi\|_{C^\alpha(\bar{\Omega})}. \end{aligned}$$

For $L_2(\phi)$,

$$\begin{aligned} |L_2(\phi)| &\le C(\Sigma_4, \eta)\|\phi\|_{C(\Omega)} \sum_{n=1}^{m-1} \sum_{j\in \bar{J}_4^n, j\le 2l} \int_{\tilde{E}_j^n} |U_0^\Delta(n\Delta t - 0, r) - U^\Delta(n\Delta t - 0, r)| dr \\ &= C(\Sigma_4, \eta)\|\phi\|_{C(\Omega)} \Delta t \sum_{n=1}^{m-1} \sum_{j\in \bar{J}_4^n, j\le 2l} \int_{\tilde{E}_j^n} |H(r, U_0^\Delta(n\Delta t - 0, r))| dr \\ &\le C(\Sigma_4, \Omega, T_1, \eta)\|\phi\|_{C(\Omega)}. \end{aligned}$$

Thus, let

$$T_1(\phi) = (M + L_1 + L_2 + \Pi + \Sigma)(\phi), \quad T_2(\phi) = L_3(\phi), \tag{13.129}$$

then, it follows

$$\begin{aligned} |T_1(\phi)| &\le C(\Sigma_4, \eta, \Omega, T_1)\|\phi\|_{C(\Omega)}, \\ |T_2(\phi)| &\le C(\Sigma_4, \eta, \Omega, T_1, \alpha)(\Delta r)^{\alpha - 1/2}\|\phi\|_{C^\alpha(\bar{\Omega})}, \\ I(\phi) &= T_1(\phi) + T_2(\phi) + N(\phi). \end{aligned}$$

By the similar argument for Problem (I), we obtain that $T_1 + T_2$ is in a compact set of $H_{loc}^{-1}(\Omega)$. For $N(\phi)$, since

$$\begin{aligned} |N(\phi)| &\le \left| \iint_{r>1, t>0} \{(\eta^\Delta - \eta_0^\Delta)\phi_t + (q^\Delta - q_0^\Delta)\phi_r\} \right| \\ &\le C(\Sigma_4, \eta, q) \iint_{r>1, t>0} |U^\Delta - U_0^\Delta|\{|\phi_t| + |\phi_r|\} \\ &\le C(\Sigma_4, \eta, \Omega, T_1)\Delta r\|\phi\|_{H_0^1(\Omega)}, \end{aligned}$$

it follows

$$\|N\|_{H^{-1}(\Omega)} \leq C\Delta r \to 0, \tag{13.130}$$

that is, N is in a compact set of $H^{-1}(\Omega)$. Therefore, $I = \eta_t^\Delta + q_r^\Delta$ is in a compact set of $H_{loc}^{-1}(\Omega)$.

PROPOSITION 27 *For the approximate solution $U^\Delta(t,r)$ $(0 < t < T_1)$, under the assumption before,*

$$\{\eta(U^\Delta)_t + q(U^\Delta)_r\}_{\Delta r \downarrow 0} \tag{13.131}$$

is in a compact set of $H_{loc}^{-1}(\Omega)$ for any smooth weak entropy pair (η, q) satisfying Lemma 17.

13.3 Convergence to a weak solution

We can prove that each approximate solution constructed in Chapter 10 for Problem (I)-(IV) has a subsequence which converges almost everywhere by using the Theorem 18 from the boundedness shown in Chapter 10 and the weak compactness given in the section 13.2. We prove that the limit of the convergent subsequence is a weak solution. In order to obtain that, we may show the following proposition, for example, for Problem (I).

PROPOSITION 28 *The approximate solution $U^\Delta(t,x)$ constructed in the section 10.1 satisfies that*

$$\iint_{D_1} (\rho^\Delta \phi_t + m^\Delta \phi_x) dx dt + \int_0^\infty \rho_0(x)\phi(0,x)dx = O(\sqrt{\Delta x})$$
$$\iint_{D_1} \left\{m^\Delta \psi_t + \left((m^\Delta)^2/\rho^\Delta + P(\rho^\Delta)\right)\psi_x\right\} dx dt + \int_0^\infty m_0(x)\psi(0,x)dx = O(\sqrt{\Delta x})$$

for any ϕ, $\psi \in C_0^1(R_t \times R_x)$ such that $\psi = 0$ on $x = x_1(t)$.

Proof

We consider the calculation for (η, q) in the section 13.2 setting $(\eta, q) = (\rho, m)$. Then, $\sigma[\rho^\Delta] - [m^\Delta] = 0$ bye the Rankine-Hugoniot condition, $(x_1^\Delta(t)'\rho^\Delta - m^\Delta)(t, x_1^\Delta(t)) = 0$ by the boundary condition, and

$$\int_{\tilde{E}_j^n} [\rho^\Delta]_{n\Delta t+0}^{n\Delta t-0} dx = 0. \tag{13.132}$$

Thus, in this case, L_1, Σ, and II vanish, and we have

$$\begin{aligned} &\iint (\rho^\Delta \phi_t + m^\Delta \phi_x) \\ &= -\int_0^\infty \phi(0,x)\rho_0(x)dx + \sum_{n=0}^{m-1} \sum_{j \in \tilde{J}_j^n} \int_{\tilde{E}_j^n} \{\phi(n\Delta t, x) - \phi_j^n\}[\rho^\Delta] dx \end{aligned}$$

Let

$$D_1^\Delta = \{(t,x); x > x_1^\Delta(t), t > 0\}. \tag{13.133}$$

Then,

$$\begin{aligned}
&\left|\iint_{D_1^\Delta}(\rho^\Delta\phi_t + m^\Delta\phi_x) + \int_0^\infty \phi(0,x)\rho_0(x)dx\right| \\
&\leq \sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1^n}\int_{\bar{E}_j^n}|\phi(n\Delta t,x)-\phi_j^n||[\rho^\Delta]|dx \\
&\leq \left(\sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1^n}\int_{\bar{E}_j^n}|\phi(n\Delta t,x)-\phi_j^n|^2dx\right)^{1/2}\left(\sum_{n=0}^{m-1}\sum_{j\in\bar{J}_1^n}\int_{\bar{E}_j^n}|[U^\Delta]|^2dx\right)^{1/2} \\
&\leq C(\Omega,\Lambda_1)\sqrt{\Delta x}\, C(\Omega,\Sigma_1,x_1,\Lambda_1)\|\phi_x\|_{C(\Omega)} \\
&= C(\Omega,\Sigma_1,x_1,\Lambda_1,\phi_x)\sqrt{\Delta x}
\end{aligned}$$

On the other hand, it follows

$$\begin{aligned}
&\left|\iint_{D_1^\Delta}(\rho^\Delta\phi_t + m^\Delta\phi_x) - \iint_{D_1}(\rho^\Delta\phi_t + m^\Delta\phi_x)\right| \\
&\leq \iint_{\{(D_1^\Delta\setminus D_1)\cup(D_1\setminus D_1^\Delta)\}\cap\{t\leq T\}}|\rho^\Delta\phi_t + m^\Delta\phi_x|,
\end{aligned}$$

where the area of the domain of integration of the last integral is

$$m\left(\{(D_1^\Delta\setminus D_1)\cup(D_1\setminus D_1^\Delta)\}\cap([0,T]\times R)\right) = \int_0^T |x_1(t)-x_1^\Delta(t)|dt. \tag{13.134}$$

Since

$$\begin{aligned}
x_1^\Delta(t) &= x_1(n\Delta t) + \frac{t-n\Delta t}{\Delta t}\{x_1((n+1)\Delta t) - x_1(n\Delta t)\}, \\
&= x_1(n\Delta t) + \frac{t-n\Delta t}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} x_1'(s)ds \\
x_1(t) &= x_1(n\Delta t) + \int_{n\Delta t}^{t} x_1'(s)ds,
\end{aligned}$$

we have

$$\begin{aligned}
|x_1^\Delta(t) - x_1(t)| &\leq \frac{(n+1)\Delta t - t}{\Delta t}\int_{n\Delta t}^t |x_1'| + \frac{t-n\Delta t}{\Delta t}\int_t^{(n+1)\Delta t}|x_1'| \\
&\leq \frac{\Delta t}{2}\|x_1'\|_{L^\infty}.
\end{aligned}$$

Therefore

$$\left|\iint_{D_1^\Delta}(\rho^\Delta\phi_t + m^\Delta\phi_x) - \iint_{D_1}(\rho^\Delta\phi_t + m^\Delta\phi_x)\right| \leq C(\Sigma_1,\phi_t,\phi_x,x_1,\Lambda_1,\Omega)\Delta x. \tag{13.135}$$

For the conservation of momentum, the estimate is obtained similarly except for the term on the boundary

$$\int_0^T \{P(\rho^\Delta)\psi\}(t, x_1^\Delta(t))dt. \tag{13.136}$$

However, by $\psi(t, x_1(t)) = 0$,

$$\begin{aligned}
&\left|\int_0^T \{P(\rho^\Delta)\psi\}(t, x_1^\Delta(t))dt\right| \\
\leq\ & \int_0^T P(\rho^\Delta(t, x_1^\Delta(t)))|\psi(t, x_1^\Delta(t)) - \psi(t, x_1(t))|dt \\
\leq\ & C(\Sigma_1, \psi_x, \Omega, \Lambda_1, x_1)\Delta x.
\end{aligned}$$

This completes the proof. ∎

For Problem (II), we can obtain the result similarly.

PROPOSITION 29 *The approximate solution $U^\Delta(t,x)$ constructed in the section 10.2 satisfies that*

$$\iint_{D_2} (\rho^\Delta \phi_t + m^\Delta \phi_x)dxdt + \int_{x_1(0)}^{x_2(0)} \rho_0(x)\phi(0,x)dx = O(\sqrt{\Delta x})$$

$$\begin{aligned}
&\iint_{D_2} \left\{m^\Delta \psi_t + \left((m^\Delta)^2/\rho^\Delta + P(\rho^\Delta)\right)\psi_x\right\} dxdt + \int_{x_1(0)}^{x_2(0)} m_0(x)\psi(0,x)dx \\
&\quad = O(\sqrt{\Delta x})
\end{aligned}$$

for any ϕ, $\psi \in C_0^1(R_t \times R_x)$ such that $\psi = 0$ on $x = x_1(t)$ and on $x = x_2(t)$.

We consider for Problem (III).

PROPOSITION 30 *There is a continuously differentiable function $x(t)$ and there is a countable sequence $\{\Delta' x\}$ which tends to zero such that*

$$x^{\Delta'}(t) \to x(t), \quad \frac{d}{dt}x^{\Delta'}(t) \to x'(t) \qquad (\textit{uniformly in } [0, T_0]) \tag{13.137}$$

for any $T_0 > 0$, where $x^\Delta(t)$ is the approximate boundary constructed in 10.3, and

$$x^{\Delta'}(t) = [x^\Delta(t)]_{\Delta x = \Delta' x}. \tag{13.138}$$

Moreover, the derivative $x'(t)$ is Lipschitz continuous, that is $x''(t)$ is bounded.

Proof

We set u_n^Δ as the gradient of the polygonal line $x^\Delta(t)$

$$\begin{cases} u_n^\Delta = \left[\dfrac{d}{dt}x^\Delta(t)\right]_{t=n\Delta t - 0} & (n \geq 1), \\ u_0^\Delta = x_0', \end{cases} \tag{13.139}$$

then,

$$u_n^\Delta - u_{n-1}^\Delta = \Delta t g^\Delta(t) \qquad ((n-1)\Delta t < t < n\Delta t), \tag{13.140}$$

where $n = 1, 2, \ldots$ and

$$g^\Delta(t) = k\{P_1(\rho^\Delta(x^\Delta(t) - 0, t)) - P_2(\rho^\Delta(x^\Delta(t) + 0, t))\}. \tag{13.141}$$

from the definition of $x^\Delta(t)$. Let

$$u^\Delta(t) = \frac{d}{dt}x^\Delta(t), \tag{13.142}$$

then $u^\Delta(t)$ is uniformly bounded step function by Proposition 15. Hence, by Ascoli-Arzelà's theorem, it follows that there is a countable sequence $\{\Delta' x\}$ such that

$$\Delta' x = \Delta'_n x \to 0 \tag{13.143}$$

and there is a Lipschitz continuous function $x(t)$ such that

$$x^{\Delta'}(t) \longrightarrow x(t) \qquad \text{(uniformly in } [0, T_0]) \tag{13.144}$$

for any $T_0 > 0$.

We consider the derivative. Since $g^\Delta(t)$ is constant on the each interval $((n-1)\Delta t, n\Delta t)$ $(n = 1, 2, \ldots)$, we get

$$u_n^\Delta - u_{n-1}^\Delta = \Delta t g^\Delta(t) = \int_{(n-1)\Delta t}^{n\Delta t} g^\Delta(s) ds. \tag{13.145}$$

Hence,

$$u_n^\Delta = x'_0 + \int_0^{n\Delta t} g^\Delta(s) ds. \tag{13.146}$$

We define a Lipschitz continuous function $\tilde{u}^\Delta(t)$ by

$$\tilde{u}^\Delta(t) = x'_0 + \int_0^t g^\Delta(s) ds, \tag{13.147}$$

then, for $t \in ((n-1)\Delta t, n\Delta t)$,

$$|\tilde{u}^\Delta(t) - u^\Delta(t)| = |\tilde{u}^\Delta(t) - u_n^\Delta| = \left| \int_t^{n\Delta t} g^\Delta(s) ds \right| \leq M\Delta t, \tag{13.148}$$

where M is independent on Δx since $g^\Delta(t)$ is uniformly bounded by Proposition 15. We obtain by Ascoli-Arzelà's theorem that there is a Lipschitz continuous function $\bar{u}(t)$ such that

$$\tilde{u}^{\Delta'}(t) \longrightarrow \bar{u}(t) \qquad \text{(uniformly in } [0, T_0]) \tag{13.149}$$

for any $T_0 > 0$ by taking a subsequence of $\{\Delta' x\}$ if we need. We set

$$\tilde{x}^\Delta(t) = \int_0^t \tilde{u}^\Delta(s) ds, \tag{13.150}$$

then for $t \in [0, T_0]$,

$$|x^\Delta(t) - \tilde{x}^\Delta(t)| = \left|\int_0^t \{u^\Delta(s) - \tilde{u}^\Delta(s)\}ds\right| \le T_0 M \Delta t, \tag{13.151}$$

hence,

$$x^{\Delta'}(t) \longrightarrow x(t) \qquad \text{(uniformly in } [0, T_0]) \tag{13.152}$$

for any T_0. From these results, we obtain

$$u^{\Delta'}(t) \longrightarrow \bar{u}(t) \qquad \text{(uniformly in } [0, T_0]) \tag{13.153}$$

and

$$x(t) = \int_0^t \bar{u}(s)ds. \tag{13.154}$$

Therefore, it follows

$$\begin{aligned}
&x^{\Delta'}(t) \longrightarrow x(t) \qquad \text{(uniformly in } [0, T_0]) \\
&\frac{d}{dt}x^{\Delta'}(t) = u^{\Delta'}(t) \longrightarrow \bar{u}(t) = x'(t) \qquad \text{(uniformly in } [0, T_0]) \\
&x'(t) = \bar{u}(t)
\end{aligned} \tag{13.155}$$

This completes the proof. ∎

PROPOSITION 31 *Let ϕ, $\psi \in C_0^1$. Then, for the approximate solution $U^\Delta(t, x)$ constructed in the section 10.3 and for $x(t)$ and $\{\Delta' x\}$ which appear in Proposition 30, it follows that*

$$\begin{aligned}
&\iint_{x > x(t)} (\rho^{\Delta'}\phi_t + m^{\Delta'}\phi_x) + \int_0^\infty \rho_0(x)\phi(0, x)dx, \\
&\iint_{x < x(t)} (\rho^{\Delta'}\phi_t + m^{\Delta'}\phi_x) + \int_{-\infty}^0 \rho_0(x)\phi(0, x)dx, \\
&\iint_{t>0} \left\{m^{\Delta'}\psi_t + \left((m^{\Delta'})^2/\rho^{\Delta'} + P(\rho^{\Delta'})\right)\psi_x\right\} + \int_R m_0(x)\psi(0, x)dx \\
&\qquad + \frac{1}{k}\int_{x=x(t)} x'(t)\{\psi_t + x'(t)\psi_x\}dt + \frac{1}{k}x_0'\psi(0, 0)
\end{aligned}$$

tend to zero as $\Delta' x \downarrow 0$.

Proof

For simplicity, we denote Δ' by Δ. The proof that the first two expressions for the conservation of mass tend to zero is done similarly for Problem (I) since

$$x^\Delta(t) \longrightarrow x(t) \qquad \text{(uniformly in } [0, T]) \tag{13.156}$$

shows

$$\int_0^T |x^\Delta(t) - x(t)|dt \longrightarrow 0. \tag{13.157}$$

Thus, we consider the last expression. Let

$$D_3^\Delta = \{(t,x); x < x^\Delta(t), t > 0\}, \qquad D_4^\Delta = \{(t,x); x > x^\Delta(t), t > 0\} \tag{13.158}$$

corresponding to

$$D_3 = \{(t,x); x < x(t), t > 0\}, \qquad D_4 = \{(t,x); x > x(t), t > 0\}. \tag{13.159}$$

Taking (η, q) as $(m, m^2/\rho + P_2(\rho))$ in the calculation of $\eta_t + q_x$ for Problem (I) in D_4^Δ, we have,

$$\begin{aligned}
&\iint_{D_4^\Delta} \left\{ m^\Delta \psi_t + \left((m^\Delta)^2/\rho^\Delta + P_2(\rho^\Delta) \right) \psi_x \right\} dxdt \\
&= -\int_0^\infty m_0(x)\psi(0,x)dx + \sum_{n=1}^{m-1} \int_{x^\Delta(n\Delta t)}^L \psi(n\Delta t, x)[m^\Delta]_{n\Delta t+0}^{n\Delta t-0} dx \\
&\quad -\int_0^T \psi(t, x^\Delta(t)) P_2(\rho^\Delta(t, x^\Delta(t)+0))dt.
\end{aligned}$$

By the similar calculation in D_3^Δ, we have

$$\begin{aligned}
&\iint_{t>0} \left\{ m^\Delta \psi_t + \left((m^\Delta)^2/\rho^\Delta + P^\Delta \right) \psi_x \right\} dxdt + \int_R m_0(x)\psi(0,x)dx \\
&= \sum_{n=0}^{m-1} \int_{-L}^L [m^\Delta]\psi(n\Delta t, x)dx - \int_0^T [P^\Delta]_{x^\Delta(t)-0}^{x^\Delta(t)+0} \psi(t, x^\Delta(t))dt,
\end{aligned}$$

where

$$P^\Delta = P^\Delta(t,x) = \begin{cases} P_1(\rho^\Delta(t,x)) & ((t,x) \in D_3^\Delta), \\ P_2(\rho^\Delta(t,x)) & ((t,x) \in D_4^\Delta). \end{cases} \tag{13.160}$$

From the energy estimate, we obtain

$$\left| \sum_{n=1}^{m-1} \int_{-L}^L [m^\Delta]\psi(n\Delta t, x)dx \right| \le C\sqrt{\Delta x} \tag{13.161}$$

by the similar argument in proof for Proposition 28.

We consider the last term of (13.160). We set

$$u^\Delta(t) = \frac{d}{dt}x^\Delta(t) = x_0' \qquad \text{for } t < 0. \tag{13.162}$$

We use the same notations in the proof of Proposition 30. Then, since we may write

$$\begin{aligned}
[P^\Delta]_{x^\Delta(t)-0}^{x^\Delta(t)+0} &= P_2(\rho^\Delta(t, x^\Delta(t)+0)) - P_1(\rho^\Delta(t, x^\Delta(t)-0)) \\
&= -\frac{1}{k} g^\Delta(t) \\
&= -\frac{1}{k} \frac{u^\Delta(t) - u^\Delta(t-\Delta t)}{\Delta t} \qquad (t > 0),
\end{aligned}$$

it follows

$$
\begin{aligned}
& -\int_0^T [P^\Delta]\psi(t, x^\Delta(t))dt \\
= \;& \frac{1}{k}\int_0^T \psi(t, x^\Delta(t))\frac{u^\Delta(t) - u^\Delta(t-\Delta t)}{\Delta t}dt \\
= \;& \frac{1}{k}\int_0^T \frac{\psi(t, x^\Delta(t)) - \psi(t+\Delta t, x^\Delta(t+\Delta t))}{\Delta t}u^\Delta(t)dt \\
& -\frac{1}{k}\int_0^{\Delta t} \psi(t, x^\Delta(t))\frac{u^\Delta(t-\Delta t)}{\Delta t}dt.
\end{aligned}
$$

It is not difficult to show that

$$
\begin{aligned}
& \frac{\psi(t, x^\Delta(t)) - \psi(t+\Delta t, x^\Delta(t+\Delta t))}{\Delta t} \\
= \;& -\frac{1}{\Delta t}\int_t^{t+\Delta t}\{\psi_t + \frac{d}{ds}x^\Delta(s)\psi_x\}(s, x^\Delta(s))ds \\
\to \;& -(\psi_t + x'(t)\psi_x)(t, x(t)) \qquad \text{(uniformly in } [0,T]),
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^{\Delta t} \psi(t, x^\Delta(t))\frac{u^\Delta(t-\Delta t)}{\Delta t}dt \\
= \;& \frac{x_0'}{\Delta t}\int_0^{\Delta t}\psi(t, x^\Delta(t))dt \\
\to \;& x_0'\psi(0,0).
\end{aligned}
$$

Thus, we have

$$
-\int_0^T [P^\Delta]\psi(t, x^\Delta(t))dt \to -\frac{1}{k}\int_0^T x'(t)\{\psi_t + x'(t)\psi_x\}(t, x(t))dt - \frac{1}{k}x_0'\psi(0,0). \tag{13.163}
$$

There remains only the difference between the term for $P^\Delta(t,x)$ and for $P(\rho^\Delta(t,x))$. For this, since

$$
P(\rho^\Delta(t,x)) = \begin{cases} P_2(\rho^\Delta(t,x)) & (x \in D_4), \\ P_1(\rho^\Delta(t,x)) & (x \in D_3) \end{cases} \tag{13.164}
$$

it follows

$$
\begin{aligned}
& \left|\iint_{t>0} P^\Delta(t,x)\psi_x dxdt - \iint_{t>0} P(\rho^\Delta(t,x))\psi_x dxdt\right| \\
\le \;& \iint_{(D_3^\Delta \setminus D_3)\cup(D_3\setminus D_3^\Delta)} |\psi_x||P_2(\rho^\Delta(t,x)) - P_1(\rho^\Delta(t,x))|dt \\
\le \;& C(\Sigma_3, \psi)\int_0^T |x(t) - x^\Delta(t)|dt \\
\to \;& 0.
\end{aligned}
$$

This completes the proof. ∎

Last, we consider for Problem (IV). We use the notation in the section 10.4, 13.2. Let $\phi,\ \psi \in C_0^1(\Omega)$ such that $\psi = 0$ on $r = 1$, and let $\mu = (\phi, \psi)$. We set J and divide J to J_1 and J_2 by

$$\begin{aligned}
J &= \int_0^T dt \int_1^L \{\mu_t U^\Delta + \mu_r F(U^\Delta) + \mu H(r, U^\Delta)\} dr + \int_1^L \mu(0,r) U^\Delta(+0,r) dr \\
&= J_1 + J_2, \\
J_1 &= \int_0^T dt \int_1^L \{\mu_t U_0^\Delta + \mu_r F(U_0^\Delta) + \mu H(r, U_0^\Delta)\} dr + \int_1^L \mu(0,r) U_0^\Delta(+0,r) dr, \\
J_2 &= \int_0^T dt \int_1^L \Big(\{\mu_t (U^\Delta - U_0^\Delta) + \mu_r \{F(U^\Delta) - F(U_0^\Delta)\} \\
&\quad + \mu\{H(r, U^\Delta) - H(r, U_0^\Delta)\}\Big) dr \\
&= \int_0^T dt \int_1^L \Big(\{\mu_t + \mu_r \int_0^1 \nabla F(U_0^\Delta + y(U^\Delta - U_0^\Delta)) dy\}(U^\Delta - U_0^\Delta) \\
&\quad + \mu(H(r, U^\Delta) - H(r, U_0^\Delta))\Big) dr,
\end{aligned}$$

Since

$$\begin{aligned}
&|U^\Delta - U_0^\Delta| \le C(\Sigma_4)\Delta t, \\
&|\nabla F(U_0^\Delta + y(U^\Delta - U_0^\Delta))| \le C(\Sigma_4), \\
&|H(r, U^\Delta) - H(r, U_0^\Delta)| \le C(\Sigma_4)|U^\Delta - U_0^\Delta| \le C(\Sigma_4)\Delta t
\end{aligned} \tag{13.165}$$

We have

$$|J_2| \le C(\Sigma_4, \Omega, T_1, \mu)\Delta r. \tag{13.166}$$

We divide J_1

$$\begin{aligned}
J_1 &= \sum_{n=1}^{m} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^L \{(\mu U_0^\Delta)_t + (\mu F_0^\Delta)_r\} dr + \int_1^L \mu(0,r) U_0^\Delta(+0,r) dr \\
&\quad + \int_0^T dt \int_1^L \mu H(r, U_0^\Delta) dr \\
&= \sum_{n=1}^{m-1} \int_1^L \mu(n\Delta t, r)[U_0^\Delta]_{n\Delta t+0}^{n\Delta t-0} dr + \int_0^T dt \int_1^L \mu H(r, U_0^\Delta) dr \\
&= J_{11} + J_{12}, \\
J_{11} &= \sum_{n=1}^{m-1} \sum_{j \in \bar{J}_4^n} \int_{\bar{E}_j^n} \{\mu(n\Delta t, r) - \mu_j^n\}[U_0^\Delta]_{n\Delta t+0}^{n\Delta t-0} dr \\
J_{12} &= \sum_{n=1}^{m-1} \sum_{j \in \bar{J}_4^n} \mu_j^n \int_{\bar{E}_j^n} [U_0^\Delta]_{n\Delta t+0}^{n\Delta t-0} dr + \int_0^T dt \int_1^L \mu H(r, U_0^\Delta) dr \\
&\quad (\mu_j^n = \mu(n\Delta t, 1 + j\Delta r)),
\end{aligned}$$

Then, since

$$|U_0^\Delta(n\Delta t - 0, r) - \bar{U}_j^n|$$

$$
\begin{aligned}
&\le |U^\Delta(n\Delta t-0,r)-\bar U_j^n| + |U^\Delta(n\Delta t-0,r)-U_0^\Delta(n\Delta t-0,r)| \\
&\le |U^\Delta(n\Delta t-0,r)-\bar U_j^n| + |H(r,U_0^\Delta)|\Delta t \\
&\le |U^\Delta(n\Delta t-0,r)-\bar U_j^n| + C(\Sigma_4)\Delta t
\end{aligned}
$$

we have

$$
\begin{aligned}
|J_{11}| &\le \left(\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n}\int_{\bar E_j^n}|\mu(n\Delta t,r)-\mu_j^n|^2dr\right)^{1/2}\left(\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n, j\le 2l}\int_{\bar E_j^n}|[U_0^\Delta]|^2dr\right)^{1/2} \\
&\le C(\Omega,T_1)\sqrt{\Delta r}\left(\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n, j\le 2l}\int_{\bar E_j^n}|U_0^\Delta(n\Delta t-0,r)-\bar U_j^n|^2dr\right)^{1/2} \\
&\le C(\Sigma_4,\Omega,T_1)\sqrt{\Delta r},
\end{aligned}
$$

because the energy estimate (13.124) show

$$
\begin{aligned}
&\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n, j\le 2l}\int_{\bar E_j^n}|U_0^\Delta(n\Delta t-0,r)-\bar U_j^n|^2dr \\
&\le 2\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n, j\le 2l}\int_{\bar E_j^n}|U^\Delta(n\Delta t-0,r)-\bar U_j^n|^2dr + C(\Sigma_4,\Omega,T_1)\Delta r \\
&\le C(\Sigma_4,\Omega,T_1).
\end{aligned}
$$

Hence,

$$
|J_{11}| \le C(\Sigma_4,\Omega,T_1)\sqrt{\Delta r} \tag{13.167}
$$

For J_{12}, it follows

$$
\begin{aligned}
&\int_{\bar E_j^n}[U_0^\Delta]_{n\Delta t+0}^{n\Delta t-0}dr \\
&= \int_{\bar E_j^n}\{U_0^\Delta(n\Delta t-0,r)-\bar U_j^n\}dr \\
&= \int_{\bar E_j^n}U_0^\Delta(n\Delta t-0,r)dr - \int_{\bar E_j^n}U^\Delta(n\Delta t-0,r)dr \\
&= -\Delta t\int_{\bar E_j^n}H(r,U_0^\Delta(n\Delta t-0,r))dr.
\end{aligned}
$$

We divide

$$
\begin{aligned}
J_{12} &= -\sum_{n=1}^{m-1}\sum_{j\in \bar J_4^n}\mu_j^n\Delta t\int_{\bar E_j^n}H(r,U_0^\Delta(n\Delta t-0,r))dr + \int_0^T dt\int_1^L \mu H(r,U_0^\Delta(t,r))dr \\
&= J_{121}+J_{122} \\
J_{121} &= \sum_{n=1}^{m}\sum_{j\in \bar J_4^n}\int_{(n-1)\Delta t}^{n\Delta t}dt\int_{\bar E_j^n}(\mu-\mu_j^n)H(r,U_0^\Delta(t,r))dr \\
J_{122} &= \sum_{n=1}^{m}\sum_{j\in \bar J_4^n}\mu_j^n\int_{(n-1)\Delta t}^{n\Delta t}dt\int_{\bar E_j^n}\{H(r,U_0^\Delta(t,r))-H(r,U_0^\Delta(n\Delta t-0,r))\}dr,
\end{aligned}
$$

then we obtain

$$\begin{aligned}|J_{121}| &\leq \|\mu\|_{C^1(\Omega)}C(\Omega,\Sigma_4,T_1)\Delta r,\\ |J_{122}| &\leq C(\Sigma_4)\|\mu\|_{C(\Omega)}\sum_{n=1}^{m}\sum_{j\in\bar{J}_4^n, j\leq 2l}\int_{(n-1)\Delta t}^{n\Delta t}dt\int_{\bar{E}_j^n}|U_0^\Delta(t,r)-U_0^\Delta(n\Delta t-0,r)|dr.\end{aligned}$$

Therefore, we may show that

$$\sum_{n=1}^{m}\sum_{j\in\bar{J}_4^n, j\leq 2l}\int_{(n-1)\Delta t}^{n\Delta t}dt\int_{\bar{E}_j^n}|U_0^\Delta(t,r)-U_0^\Delta(n\Delta t-0,r)|dr \tag{13.168}$$

tends to zero. Ding, Chen and Luo show this by too difficult calculation to understand ([6]). However, by using the following lemma, the proof becomes simple from energy estimate.

LEMMA 32 *Let $U(t,x)$ be a solution of the Riemann problem*

$$\begin{cases} U_t + F(U)_x = 0 \quad (x\in R,\, t>0),\\ U(0,x) = \begin{cases} U_l & (x<0),\\ U_r & (x>0)\end{cases}\end{cases} \tag{13.169}$$

constructed in the section 8.4. If the absolute values of the velocity for the simple waves is not greater than

$$\frac{1}{2}\frac{\Delta x}{\Delta t} \tag{13.170}$$

in the solution $U(t,x)$, then,

$$\int_{-\Delta x}^{\Delta x}|U(\Delta t,x)-U(t,x)|^2dx \leq 6\int_{-\Delta x}^{\Delta x}|U(\Delta t,x)-\bar{U}|^2dx \qquad (0<t<\Delta t), \tag{13.171}$$

where

$$\bar{U} = \frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x}U(t,x)dx. \tag{13.172}$$

Proof

It follows

$$\begin{aligned}&\int_{-\Delta x}^{\Delta x}|U(\Delta t,x)-U(t,x)|^2dx\\ &\leq 2\int_{-\Delta x}^{\Delta x}|U(\Delta t,x)-\bar{U}|^2dx + 2\int_{-\Delta x}^{\Delta x}|U(t,x)-\bar{U}|^2dx.\end{aligned}$$

Since $U(t,x)$ is depend only on x/t, we set

$$U(t,x) = V\left(\frac{x}{t}\right). \tag{13.173}$$

Then,

$$
\begin{aligned}
&\int_{-\Delta x}^{\Delta x} |U(t,x) - \bar{U}|^2 dx \\
&= \int_{-\Delta x}^{\Delta x} |V(x/t) - \bar{U}|^2 dx \\
&= \frac{t}{\Delta t} \int_{-\Delta x \Delta t/t}^{\Delta x \Delta t/t} |V(y/\Delta t) - \bar{U}|^2 dy \\
&= \frac{t}{\Delta t} \int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy + \frac{t}{\Delta t} \int_{\Delta x}^{\Delta x \Delta t/t} |V(y/\Delta t) - \bar{U}|^2 dy \\
&\quad + \frac{t}{\Delta t} \int_{-\Delta x \Delta t/t}^{-\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy.
\end{aligned}
$$

By the assumption for $U(t,x)$,

$$
V(y/\Delta t) = \begin{cases} U_r & (y \geq \Delta x/2), \\ U_l & (y \leq -\Delta x/2). \end{cases} \tag{13.174}
$$

This shows

$$
\begin{aligned}
&\int_{-\Delta x}^{\Delta x} |U(t,x) - \bar{U}|^2 dx \\
&= \frac{t}{\Delta t} \int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy + \frac{t}{\Delta t} \left(\Delta x \frac{\Delta t}{t} - \Delta x \right) |U_r - \bar{U}|^2 \\
&\quad + \frac{t}{\Delta t} \left(-\Delta x + \Delta x \frac{\Delta t}{t} \right) |U_l - \bar{U}|^2
\end{aligned}
$$

.

On the other hand, since

$$
\begin{aligned}
&\int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy \\
&= \int_{-\Delta x/2}^{\Delta x/2} |V(y/\Delta t) - \bar{U}|^2 dy + \int_{-\Delta x}^{-\Delta x/2} |V(y/\Delta t) - \bar{U}|^2 dy \\
&\quad + \int_{\Delta x/2}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy \\
&\geq (|U_r - \bar{U}|^2 + |U_l - \bar{U}|^2) \Delta x/2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\int_{-\Delta x}^{\Delta x} |U(t,x) - \bar{U}|^2 dx \\
&= \frac{t}{\Delta t} \int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy + \Delta x \left(1 - \frac{t}{\Delta t} \right) |U_r - \bar{U}|^2 + \Delta x \left(1 - \frac{t}{\Delta t} \right) |U_l - \bar{U}|^2 \\
&\leq \left\{ \frac{t}{\Delta t} + 2 \left(1 - \frac{t}{\Delta t} \right) \right\} \int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \bar{U}|^2 dy \\
&\leq 2 \int_{-\Delta x}^{\Delta x} |U(\Delta t, x) - \bar{U}|^2 dx.
\end{aligned}
$$

Therefore, it follows

$$\int_{-\Delta x}^{\Delta x} |U(\Delta t, x) - U(t, x)|^2 dx \le 6 \int_{-\Delta x}^{\Delta x} |U(\Delta t, x) - \bar{U}|^2 dx. \tag{13.175}$$

∎

By assumptions,

$$\frac{\Delta r}{\Delta t} \ge 2B = 2\Lambda(\Sigma_4), \tag{13.176}$$

a function $U_0^\Delta(t, r)$ satisfies the condition of Lemma 32. Thus,

$$\int_{\tilde{E}_j^n} |U_0^\Delta(n\Delta t - 0, r) - U_0^\Delta(t, r)|^2 dr \le 6 \int_{\tilde{E}_j^n} |U_0^\Delta(n\Delta t - 0, r) - \bar{U}_{0,j}^n|^2 dr, \tag{13.177}$$

where

$$\bar{U}_{0,j}^n = \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U_0^\Delta(n\Delta t - 0, y) dy \qquad (j \in \tilde{J}_4^n). \tag{13.178}$$

This shows

$$\begin{aligned}
&\sum_{n=1}^{m} \sum_{j \in \tilde{J}_4^n, j \le 2l} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_{\tilde{E}_j^n} |U_0^\Delta(t, r) - U_0^\Delta(n\Delta t - 0, r)| dr \\
&\le \left(\sum_{n=1}^{m} \sum_{j \in \tilde{J}_4^n, j \le 2l} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_{\tilde{E}_j^n} dr \right)^{1/2} \\
&\quad \times \left(\sum_{n=1}^{m} \sum_{j \in \tilde{J}_4^n, j \le 2l} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_{\tilde{E}_j^n} |U_0^\Delta(t, r) - U_0^\Delta(n\Delta t - 0, r)|^2 dr \right)^{1/2} \\
&\le C(\Omega) \left(\sum_{n=1}^{m} \sum_{j \in \tilde{J}_4^n, j \le 2l} \Delta t \int_{\tilde{E}_j^n} |U_0^\Delta(n\Delta t - 0, r) - \bar{U}_{0,j}^n|^2 dr \right)^{1/2}.
\end{aligned}$$

Now, since

$$\begin{aligned}
\bar{U}_j^n &= \frac{1}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} U^\Delta(n\Delta t - 0, r) dr \\
&= \bar{U}_{0,j}^n + \frac{\Delta t}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} H(r, U_0^\Delta(n\Delta t - 0, r)) dr,
\end{aligned}$$

it follows

$$\begin{aligned}
&U_0^\Delta(n\Delta t - 0, r) - \bar{U}_{0,j}^n \\
&= U^\Delta(n\Delta t - 0, r) - \bar{U}_j^n - \Delta t H(r, U_0^\Delta(n\Delta t - 0, r)) \\
&\quad + \frac{\Delta t}{m(\tilde{E}_j^n)} \int_{\tilde{E}_j^n} H(r, U_0^\Delta(n\Delta t - 0, r)) dr.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sum_{n=1}^{m}\sum_{j\in \bar{J}_4^n, j\le 2l}\int_{\bar{E}_j^n}|U_0^\Delta(n\Delta t-0,r)-\bar{U}_{0,j}^n|^2dr\\
&\le 2\sum_{n=1}^{m}\sum_{j\in \bar{J}_4^n, j\le 2l}\int_{\bar{E}_j^n}|U^\Delta(n\Delta t-0,r)-\bar{U}_j^n|^2dr+C(\Sigma_4,\Omega,T_1)\Delta r.
\end{aligned}$$

Hence, it follows

$$|J_{122}|\le C(\Sigma_4,\Omega,T_1)\sqrt{\Delta r}. \tag{13.179}$$

Therefore, we obtain

$$J=O(\sqrt{\Delta r}) \qquad (\Delta r\downarrow 0). \tag{13.180}$$

For the difference between $U^\Delta(+0,r)$ and $U_0(r)$,

$$\begin{aligned}
&\left|\int_1^L \mu(0,r)U^\Delta(+0,r)dr-\int_1^L\mu(0,r)U_0(r)dr\right|\\
&= \left|\sum_{j\in\bar{J}_4}\int_{\bar{E}_j^n}\mu(0,r)\{U^\Delta(+0,r)-U_0(r)\}dr\right|\\
&\le \sum_{j\in\bar{J}_4}\int_{\bar{E}_j^n}|\mu(0,r)-\mu(0,1+j\Delta r)||U^\Delta(+0,r)-U_0(r)|dr\\
&\qquad\left(\int_{\bar{E}_j^n}\{U^\Delta(+0,r)-U_0(r)\}dr=0\right)\\
&\le \Delta r\|\mu\|_{C^1(\Omega)}C(U_0,\Omega)\\
&\to 0.
\end{aligned}$$

Thus, we have the following Proposition.

PROPOSITION 33 *Let $U^\Delta(t,r)$ be an approximate solution constructed in the section 10.4 for $0<t<T$ $(T<T_0^\infty)$, and let Ω be an open set in $R_t\times R_r$ such that*

$$\bar{\Omega}\cap[0,\infty)\times[1,\infty)\subset[0,T)\times[1,\infty). \tag{13.181}$$

Let $\phi,\psi\in C_0^1(\Omega)$ such that $\psi=0$ on $r=1$ and let $\mu=(\phi,\psi)$. Then,

$$\int_0^T dt\int_1^\infty\{\mu_tU^\Delta+\mu_rF(U^\Delta)+\mu H(r,U^\Delta)\}dr+\int_1^\infty\mu(0,r)U_0(r)dr=O(\sqrt{\Delta r}) \tag{13.182}$$

From Propositions 28, 29, 31, 33 and Theorem 18, it follows immediately that there exist weak solutions and satisfy the estimates for the boundedness

13.4 Entropy conditions

In this section we show that the weak solutions given in this paper satisfy the entropy conditions.

PROPOSITION 34 *The weak solution $U(t,x)$ for Problem (I) given in this paper satisfies the following entropy condition. Let (η, q) be a smooth weak entropy pair such that*

$$\begin{cases} [\eta(\rho,\rho u)]_{\rho=0} = [q(\rho,\rho u)]_{\rho=0} = 0, \\ \eta \text{ is convex, that is } \nabla^2\eta \geq 0, \\ \eta \text{ satisfies Lemma 17.} \end{cases} \tag{13.183}$$

Then,

$$\eta(U(t,x))_t + q(U(t,x))_x \leq 0 \qquad (x > x_1(t), t > 0) \tag{13.184}$$

holds in the distribution sense.

We remark that the assumption for (η, q) is satisfied for the smooth weak pair by made from Darboux's formula, in particular, for (η_*, q_*).
Proof
We show

$$\iint_{D_1} \{\eta(U)\phi_t + q(U)\phi_x\} \geq 0 \tag{13.185}$$

for $\phi \in C_0^\infty(\{x > x(t), t > 0\})$ such that $\phi \geq 0$. Using the notation in the section 13.2, we may write

$$\iint_{D_1^\Delta} \{\eta(U^\Delta)\phi_t + q(U^\Delta)\phi_x\} = L_1(\phi) + L_2(\phi) + \Sigma(\phi) + \Pi(\phi). \tag{13.186}$$

Since $\phi(t, x_1(t)) = 0$ and ϕ is smooth, we have

$$\begin{aligned} |\Pi(\phi)| &= \left| \int_0^T \{x_1^\Delta(t)'\eta^\Delta - q^\Delta\}\phi(t, x_1^\Delta(t)dt) \right| \\ &\leq \int_0^T |x_1^\Delta(t)'\eta^\Delta - q^\Delta||\phi(t, x_1^\Delta(t)) - \phi(t, x_1(t))|dt \\ &\leq \Delta t C(\Sigma_1, \eta, \Omega, x_1, \phi_x) \\ &\leq \Delta x C(\Sigma_1, \eta, \Omega, x_1, \phi_x, \Lambda_1) \to 0. \end{aligned}$$

From $\nabla^2\eta \geq 0$ and $\phi \geq 0$,

$$\Sigma(\phi) = \int_0^T \sum_{\text{shock}} \phi(\sigma[\eta^\Delta] - [q^\Delta])dt \geq 0 \tag{13.187}$$

(cf. [5]), and $L_1(\phi) \geq 0$. For $L_2(\phi)$,

$$
\begin{aligned}
|L_2(\phi)| &= \left| \sum_{n=0}^{m-1} \sum_{j \in \bar{J}_1} \int_{\bar{E}_j^n} \{\phi(n\Delta t, x) - \phi_j^n\}[\eta^\Delta] dx \right| \\
&\leq C(\eta, \Sigma_1, \Omega, \phi)\sqrt{\Delta x} \left(\sum_{n=0}^{m-1} \sum_{j \in \bar{J}_1, j \leq 2l} \int_{\bar{E}_j^n} |[U^\Delta]|^2 dx \right)^{1/2} \\
&\leq C(\eta, \Sigma_1, \Omega, \phi, \Lambda_1, x_1)\sqrt{\Delta x} \to 0
\end{aligned}
$$

Thus we obtain

$$
\iint_{D_1^{\Delta'}} \{\eta(U^{\Delta'})\phi_t + q(U^{\Delta'})\phi_x\} \geq -C\sqrt{\Delta' x}. \tag{13.188}
$$

For some subsequences $\{\Delta' x\}$ of $\{\Delta x\}$,

$$
U^{\Delta'} \to U \text{ a.e.} \tag{13.189}
$$

Therefore, it follows

$$
\iint_{D_1} \{\eta(U)\phi_t + q(U)\phi_x\} \geq 0. \tag{13.190}
$$

∎

For Problems (II), (III), it is easy to show the entropy condition is satisfied, because the condition is the argument on inner domain.

PROPOSITION 35 *The weak solution $U(t,x)$ for Problem (II) given in this paper satisfies the entropy condition*

$$
\eta(U(t,x))_t + q(U(t,x))_x \leq 0 \qquad (x_1(t) < x < x_2(t), t > 0) \tag{13.191}
$$

in the distribution sense for the entropy pair (η, q) satisfying the assumption in Proposition 34.

PROPOSITION 36 *The weak solution $U(t,x)$ for Problem (III) given in this paper satisfies the entropy condition*

$$
\begin{cases}
\eta_l(U(t,x); P_1)_t + q_l(U(t,x); P_1)_x \leq 0 & (x < x(t), t > 0), \\
\eta_r(U(t,x); P_2)_t + q_r(U(t,x); P_2)_x \leq 0 & (x > x(t), t > 0)
\end{cases} \tag{13.192}
$$

in the distribution sense for the entropy pairs $(\eta_l(U; P_1), q_l(U; P_1))$, $(\eta_r(U; P_2), q_r(U; P_2))$ satisfying the assumptions in Proposition 34.

We consider for Problem (IV).

PROPOSITION 37 *The weak solution $U(t,r)$ $(0 < t < T_1, T_1 < T_0^\infty)$ given in this paper satisfies the entropy condition*

$$\eta(U(t,r))_t + q(U(t,r))_r \le \nabla\eta(U(t,r))H(r,U(t,r)) \qquad (0 < t < T_1,\ r > 1) \tag{13.193}$$

in the distribution sense for the entropy pair (η, q) *satisfying the assumptions in Proposition 34.*

Proof

Let $G(r,U) = \nabla\eta(U)H(r,U)$ Then, for $U \in \Sigma_4$,

$$\begin{aligned}&|\nabla\eta(U)| \le C(\Sigma_4), \quad |\nabla q| = |\nabla\eta\nabla F| \le C(\Sigma_4)\\ &|\nabla G| \le C(\Sigma_4)\end{aligned} \tag{13.194}$$

since

$$\begin{aligned}&\nabla G = {}^t H\nabla^2\eta + \nabla\eta\nabla H,\\ &\rho\nabla^2\eta \le C(\Sigma_4,\eta)\rho\nabla^2\eta_* \le C(\Sigma_4,\eta),\\ &|H/\rho| \le C(\Sigma_4),\\ &|\nabla H| = \left|\begin{pmatrix} 0 & -2/r \\ 2u^2/r - M/r^2 & -4u/r\end{pmatrix}\right| \le C(\Sigma_4).\end{aligned}$$

Let $\phi \in C_0^1((0,T_1)\times(1,\infty))$ such that $\phi \ge 0$. Then, we may show

$$\begin{aligned} I(\phi) &= \iint_{r>1,t>0} \{\phi_t\eta(U^\Delta) + \phi_r q(U^\Delta) + \phi G(r,U^\Delta)\}dtdr \\ &= \int_0^T dt \int_1^L \{\phi_t\eta(U^\Delta) + \phi_r q(U^\Delta) + \phi G(r,U^\Delta)\}dtdr \\ &\ge 0. \end{aligned}$$

We divide

$$I(\phi) = \int_0^T dt \int_1^L \{\phi_t\eta_0^\Delta + \phi_r q_0^\Delta + \phi G(r,U_0^\Delta)\}dr + N(\phi). \tag{13.195}$$

Then,

$$N(\phi) = \int_0^T dt \int_1^L \{\phi_t(\eta^\Delta - \eta_0^\Delta) + \phi_r(q^\Delta - q_0^\Delta) + \phi(G^\Delta - G_0^\Delta)\}dr \tag{13.196}$$

tends to zero since

$$\begin{aligned}&|\nabla\eta|, |\nabla q|, |\nabla G| \le C(\Sigma_4),\\ &|U^\Delta - U_0^\Delta| \le C(\Sigma_4)\Delta t \le C(\Sigma_4, T_1)\Delta r.\end{aligned} \tag{13.197}$$

We divide

$$\begin{aligned} I(\phi) - N(\phi) &= L(\phi) + \Sigma(\phi) + \Phi(\phi), \\ L(\phi) &= \sum_{n=1}^{m-1} \int_1^L \phi(n\Delta t, r)[\eta_0^\Delta]_{n\Delta t+0}^{n\Delta t-0} dr \\ \Sigma(\phi) &= \int_0^T \sum_{\text{shock}} \phi(\sigma[\eta_0^\Delta] - [q_0^\Delta])dt \ge 0, \\ \Phi(\phi) &= \int_0^T dt \int_1^L \phi G_0^\Delta dr, \end{aligned}$$

and divide

$$
\begin{aligned}
L(\phi) &= L_1(\phi) + L_2(\phi) + L_2(\phi) \\
L_1(\phi) &= \sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n} \phi_j^n \int_{\tilde{E}_j^n} \{\eta^\Delta(n\Delta t - 0, r) - \eta(\bar{U}_j^n)\} dr, \\
L_2(\phi) &= \sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n} \int_{\tilde{E}_j^n} \phi(n\Delta t, r)\{\eta_0^\Delta(n\Delta t - 0, r) - \eta^\Delta(n\Delta t - 0, r)\} dr, \\
L_3(\phi) &= \sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n} \int_{\tilde{E}_j^n} \{\phi(n\Delta t, r) - \phi_j^n\}\{\eta^\Delta(n\Delta t - 0, r) - \eta(\bar{U}_j^n)\} dr \\
&\quad (\phi_j^n = \phi(n\Delta t, 1 + j\Delta r)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
L_1(\phi) &= \sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n, \bar{\rho}_j^n > 0} \phi_j^n \int_{\tilde{E}_j^n} dr \int_0^1 (1-y)^t [U^\Delta] \nabla^2 \eta(\bar{U}_j^n + y[U^\Delta])[U^\Delta] dy \\
&\geq 0 \quad ([U^\Delta] = U^\Delta(n\Delta t - 0, r) - \bar{U}_j^n), \\
|L_3(\phi)| &\leq C(\Sigma_4, \phi, T_1)\sqrt{\Delta r} \left(\sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n, j \leq 2l} \int_{\tilde{E}_j^n} |\eta^\Delta(n\Delta t - 0, r) - \eta(\bar{U}_j^n)|^2 dr \right)^{1/2} \\
&\leq C(\Sigma_4, \eta, \phi, T_1)\sqrt{\Delta r} \to 0.
\end{aligned}
$$

We divide

$$
\begin{aligned}
\Phi(\phi) &= \Phi_1(\phi) + \Phi_2(\phi), \\
\Phi_1(\phi) &= \sum_{n=1}^{m-1} \Delta t \int_1^L \phi(n\Delta t, r) G_0^\Delta(n\Delta t - 0, r) dr \\
\Phi_2(\phi) &= \sum_{n=1}^{m-1} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^L \{\phi(t, r) G_0^\Delta(t, r) - \phi(n\Delta t, r) G_0^\Delta(n\Delta t - 0, r)\} dr,
\end{aligned}
$$

Then, since

$$
\begin{aligned}
&\phi(t, r) G_0^\Delta(t, r) - \phi(n\Delta t, r) G_0^\Delta(n\Delta t - 0, r) \\
&= \phi(n\Delta t, r)\{G_0^\Delta(t, r) - G_0^\Delta(n\Delta t - 0, r)\} + G_0^\Delta(t, r)\{\phi(t, r) - \phi(n\Delta t, r)\},
\end{aligned}
$$

it follows

$$
\begin{aligned}
|\Phi_2(\phi)| &\leq \sum_{n=1}^{m-1} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^L \phi(n\Delta t, r) |G_0^\Delta(t, r) - G_0^\Delta(n\Delta t - 0, r)| dr \\
&\quad + \sum_{n=1}^{m-1} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^L |G_0^\Delta(t, r)| |\phi(t, r) - \phi(n\Delta t, r)| dr \\
&\leq C(\Sigma_4, \eta, \phi) \left(\sum_{n=1}^{m-1} \sum_{j \in \tilde{J}_4^n, j \leq 2l} \int_{(n-1)\Delta t}^{n\Delta t} dt \right.
\end{aligned}
$$

$$\times \int_{\bar{E}_j^n} |U_0^\Delta(t,r) - U_0^\Delta(n\Delta t - 0, r)|^2 dr\Bigg)^{1/2} + C(\Sigma_4)\Delta t$$
$$\leq C(\Sigma_4, \eta, \phi, T_1)\sqrt{\Delta r} \to 0$$

similarly in the section 13.3. Thus, we may show

$$L_2(\phi) + \Phi_1(\phi) \to 0. \tag{13.198}$$

This sum may be written

$$\begin{aligned} & L_2(\phi) + \Phi_1(\phi) \\ = & \sum_{n=1}^{m-1} \sum_{j \in \bar{J}_4^n} \int_{\bar{E}_j^n} \phi(n\Delta t, r)\{\eta_0^\Delta(n\Delta t - 0, r) - \eta^\Delta(n\Delta t - 0, r) + \Delta t G_0^\Delta(n\Delta t - 0, r)\} dr. \end{aligned}$$

Now, If $\rho_0^\Delta(n\Delta t - 0, r) > 0$, then

$$\begin{aligned} & \eta^\Delta(n\Delta t - 0, r) \\ = & \eta(U_0^\Delta(n\Delta t - 0, r)) + \nabla\eta(U_0^\Delta(n\Delta t - 0, r))(U^\Delta - U_0^\Delta)(n\Delta t - 0, r) \\ & + \int_0^1 (1-y)^t (U^\Delta - U_0^\Delta)\nabla^2\eta(U_0^\Delta + y(U^\Delta - U_0^\Delta))(U^\Delta - U_0^\Delta) dy. \\ = & \eta_0^\Delta(n\Delta t - 0, r) + \Delta t G_0^\Delta(n\Delta t - 0, r) \\ & + (\Delta t)^2 \int_0^1 (1-y)^t H(r, U_0^\Delta)\nabla^2\eta(U_0^\Delta + y(U^\Delta - U_0^\Delta))H(r, U_0^\Delta) dy. \end{aligned}$$

since

$$\begin{aligned} & \nabla\eta(U_0^\Delta(n\Delta t - 0, r))(U^\Delta - U_0^\Delta)(n\Delta t - 0, r) \\ = & \Delta t \nabla\eta(U_0^\Delta(n\Delta t - 0, r))H(r, U_0^\Delta(n\Delta t - 0, r)) \\ = & \Delta t G_0^\Delta(n\Delta t - 0, r). \end{aligned}$$

And if $\rho_0^\Delta(n\Delta t - 0, r) = 0$, then $\rho^\Delta(n\Delta t - 0, r) = 0$

$$\eta_0^\Delta(n\Delta t - 0, r) = \eta^\Delta(n\Delta t - 0, r) = G_0^\Delta(n\Delta t - 0, r) = 0. \tag{13.199}$$

Thus, we have

$$\begin{aligned} & L_2(\phi) + \Phi_1(\phi) \\ = & -\sum_{n=1}^{m-1} \sum_{j \in \bar{J}_4^n} \Delta t^2 \int_{\bar{E}_j^n \cap \{r; \rho_0^\Delta(n\Delta t - 0, r) > 0\}} dr \\ & \times \int_0^1 (1-y)^t H(r, U_0^\Delta)\nabla^2\eta(U_0^\Delta + y(U^\Delta - U_0^\Delta))H(r, U_0^\Delta) dy. \end{aligned}$$

For the Hessian for η, it follows

$$\begin{aligned} \nabla^2\eta(U_0^\Delta + y(U^\Delta - U_0^\Delta)) & \leq C(\eta, \Sigma_4)\nabla^2\eta_*(U_0^\Delta + y(U^\Delta - U_0^\Delta)) \\ & \leq C(\eta, \Sigma_4)\frac{1}{\rho_0^\Delta + y(\rho^\Delta - \rho_0^\Delta)}. \end{aligned}$$

Since

$$|\rho^\Delta - \rho_0^\Delta| = |2m_0^\Delta \Delta t/r| \leq \rho_0^\Delta \delta, \tag{13.200}$$

we have

$$\frac{1}{\rho_0^\Delta + y(\rho^\Delta - \rho_0^\Delta)} \leq \frac{1}{\rho_0^\Delta}\frac{1}{1-\delta}. \tag{13.201}$$

Thus, we obtain

$$\left|\int_0^1 (1-y)^t H(r, U_0^\Delta)\nabla^2\eta(U_0^\Delta + y(U^\Delta - U_0^\Delta))H(r, U_0^\Delta)dy\right| \leq \frac{1}{1-\delta}C(\Sigma_4, \eta) \tag{13.202}$$

Therefore,

$$|L_2(\phi) + \Phi_1(\phi)| \leq C(\Sigma_4, \eta, T_1, \phi)\frac{\Delta r}{1-\delta} \to 0. \tag{13.203}$$

This completes the proof. ∎

13.5 Improvements of approximations

In this section we consider the convergence of the simple approximation for Problem (IV) mentioned last in Chapter 12. In order to that, we show that boundedness, compactness of $\eta_t + q_x$, convergence to a weak solution, and entropy condition.

We omit the proofs of the Godunov scheme, for the simple approximations for Problems (I) and (II), and for the another form approximation of the outer force for Problem (IV), because they are done similarly below.

First, we consider the boundedness. For $0 < t < \Delta t$, $r > 1 + \Delta r$, it follows

$$w(U^\Delta(t,r)) \leq \sup_{r>1} w(U_0(r)), \quad z(U^\Delta(t,r)) \geq \sup_{r>1} z(U_0(r)). \tag{13.204}$$

For $1 \leq r \leq 1 + \Delta r$, these estimate hold since the solution of the Riemann problem crossing the boundary is equal to the solution of the boundary value problem (8.67) for $r > 1$.

For the average $\bar{U}_{0,j}^1$ by the integration, the same estimates valid for $j \geq 2$. we consider for $\bar{U}_{0,0}^1$.

By the definition of $\bar{U}_{-1}^0$, $\bar{U}_{-1}^0 \in \Sigma(B_0, -B_0)$ if $\bar{U}_1^0 \in \Sigma(B_0, -B_0)$. Then, $\bar{U}_{0,0}^1 \in \Sigma(B_0, -B_0)$. Let

$$\sigma(U) = \max\{w(U), -z(U)\}. \tag{13.205}$$

Then

$$\begin{aligned} \sigma(U^\Delta(t,r)) &\leq \sigma_0 = \sup_{r>1}\sigma(U_0(r)) \qquad (0 < t < \Delta t), \\ \sigma(\bar{U}_{0,j}^1) &\leq \sigma_0 \qquad (j = 0, 2, \ldots). \end{aligned}$$

Now, since

$$\bar{U}_j^1 = \bar{U}_{0,j}^1 + H(1+j\Delta r, \bar{U}_{0,j}^1)\Delta t \qquad (j \geq 2), \tag{13.206}$$

$w(\bar{U}_j^1)$ becomes

$$\begin{aligned} w(\bar{U}_j^1) &= w(\bar{U}_{0,j}^1) - \frac{M\Delta t/(1+j\Delta r)^2}{1-2\bar{u}_{0,j}^2\Delta t/(1+j\Delta r)} \\ &\quad + \frac{\sqrt{a\gamma}}{\theta}(\bar{\rho}_{0,j}^1)^\theta\{(1-2\bar{u}_{0,j}^1\Delta t/(1+j\Delta r))^\theta - 1\}. \end{aligned}$$

Thus, the bounded estimates are the same as the one in the section 10.4.

Next, we consider $\eta_t + q_r$. Let $T = 2m\Delta t$.

$$\begin{aligned} 0 &= \sum_{n=1}^{2m}\int_{(n-1)\Delta t}^{n\Delta t} dt \int_1^{L+\varepsilon_n\Delta r}\{\eta_*(U^\Delta)_t + q_*(U^\Delta)_r\}dr \\ &= \sum_{n=1}^{2m}\int_1^{L+\varepsilon_n\Delta r}[\eta_*(U^\Delta)]_{n\Delta t+0}^{n\Delta t-0}dr + \int_0^T \sum_{\text{shock}}(\sigma[\eta_*(U^\Delta)] - [q_*(U^\Delta)])dt \\ &\quad + \int_1^L \eta_*(U^\Delta(T+0,r))dr - \int_1^{L+\Delta r}\eta_*(U^\Delta(0,r))dr \\ &\quad - \sum_{n=1}^{2m-1}\int_{L+\varepsilon_n\Delta r}^{L+\varepsilon_{n+1}\Delta r}\eta_*(U^\Delta(n\Delta t+0,r))dr + \sum_{n=1}^{m}\Delta t q_*(\bar{U}_{2l+\varepsilon_n}^{n-1}) \end{aligned}$$

Consider the term including $[\eta_*(U^\Delta)]$. If $\bar{\rho}_{0,j}^n = 0$, then $[\eta_*(U^\Delta)] = 0$. If $\bar{\rho}_{0,j}^n > 0$, then

$$\begin{aligned} [\eta_*(U^\Delta)] &= \eta_*(U^\Delta(n\Delta t-0,r)) - \eta_*(\bar{U}_{0,j}^n) + \eta_*(\bar{U}_{0,j}^n) - \eta_*(\bar{U}_j^n) \\ &= \nabla\eta_*(\bar{U}_{0,j}^n)(U^\Delta(n\Delta t-0,r) - \bar{U}_{0,j}^n) \\ &\quad + \int_0^1 (1-y)^t[U_0^\Delta]\nabla^2\eta_*(\bar{U}_{0,j}^n + y[U_0^\Delta])[U_0^\Delta]dy \\ &\quad + \int_0^1 \nabla\eta_*(\bar{U}_j^n + y(\bar{U}_{0,j}^n - \bar{U}_j^n))dy(\bar{U}_{0,j}^n - \bar{U}_j^n) \end{aligned}$$

for $j \geq 2$, where $[U_0^\Delta] = U^\Delta(n\Delta t - 0, r) - \bar{U}_{0,j}^n$. In the case $j = 0$, since $\bar{U}_{0,0}^n = \bar{U}_0^n$, the last term does not appear.

The first term vanishes by the integration on E_j^n for $j \geq 1$. We consider for $j = 0$. In this case, n is odd, and we saw in Chapter 12 that

$$\int_1^{1+\Delta r}\rho^\Delta(n\Delta t-0,r)dr = \frac{1}{2}\int_{1-\Delta r}^{1+\Delta r}\rho^\Delta(n\Delta t-0,r)dr = \Delta r\bar{\rho}_{0,0}^n. \tag{13.207}$$

and $\bar{m}_{0,0}^n = 0$. On the other hand,

$$\nabla\eta_*(\bar{U}_{0,0}^n) = (*, m/\rho)(\bar{U}_{0,0}^n) = (*, 0). \tag{13.208}$$

Thus, we have

$$\int_1^{1+\Delta r}\nabla\eta_*(\bar{U}_{0,0}^n)\{U^\Delta(n\Delta t-0,r) - \bar{U}_{0,0}^n\}dr = (*\ \ 0)^t(0\ \ *) = 0, \tag{13.209}$$

that is, the first term vanishes by the integration for any j. Therefore, it follows

$$\begin{aligned}
&\sum_{n=1}^{2m}\int_1^{L+\varepsilon_n\Delta r}[\eta_*^\Delta]dr - \int_1^{L+\Delta r}\eta_*(U^\Delta(0,r))dr \\
&= \sum_{n=0}^{2m}\sum_{j\in I_n,\bar\rho_{0,j}^n>0}\int_{F_j^n}dr\int_0^1(1-y)^t[U_0^\Delta]\nabla^2\eta_*(\bar U_{0,j}^n+y[U_0^\Delta])[U_0^\Delta]dy \\
&\quad+\sum_{n=1}^{2m}\sum_{j\in I_n,j>0,\bar\rho_{0,j}^n>0}2\Delta r\int_0^1\nabla\eta_*(\bar U_j^n+y(\bar U_{0,j}^n-\bar U_j^n))dy(\bar U_{0,j}^n-\bar U_j^n) \\
&\quad-\int_1^{L+\Delta r}\eta_*(U_0(r))dr
\end{aligned}$$

where

$$I_n=\{j;j\in J_4^n,0\le j\le 2l\},\qquad F_j^n=E_j^n\,(j\ge1),\ \ F_0^n=(1,1+\Delta r)). \tag{13.210}$$

Since the second term is bounded

$$\begin{aligned}
&\left|\sum_{n=1}^{2m}\sum_{j\in I_n,j>0,\bar\rho_{0,j}^n>0}2\Delta r\int_0^1\nabla\eta_*(\bar U_j^n+y(\bar U_{0,j}^n-\bar U_j^n))dy(\bar U_{0,j}^n-\bar U_j^n)\right| \\
&\le C(\Sigma_4,T,L),
\end{aligned}$$

we have energy estimates

$$\begin{aligned}
&\sum_{n=0}^{2m}\sum_{j\in I_n,\bar\rho_{0,j}^n>0}\int_{F_j^n}dr\int_0^1(1-y)^t[U_0^\Delta]\nabla^2\eta_*(\bar U_{0,j}^n+y[U_0^\Delta])[U_0^\Delta]dy \\
&\qquad+\int_0^T\sum_{\text{shock}}(\sigma[\eta_*(U^\Delta)]-[q_*(U^\Delta)])dt\le C(\Sigma_4,T,L,T_1) \\
&\sum_{n=0}^{2m}\sum_{j\in I_n}\int_{F_j^n}|\bar U_{0,j}^n-U^\Delta(n\Delta t-0,r)|^2dr\le C(\Sigma_4,T,L,T_1)
\end{aligned}$$

Let $\phi\in C_0^1([0,T)\times[1,L))$. we divide

$$\begin{aligned}
\int_0^T dt\int_1^L(\eta^\Delta\phi_t+q^\Delta\phi_r)dr &= L(\phi)+M(\phi)+\Pi(\phi)+\Sigma(\phi) \\
L(\phi) &= \sum_{n=0}^{2m-1}\int_1^L\phi(n\Delta t,r)[\eta^\Delta]_{n\Delta t+0}^{n\Delta t-0}dr, \\
M(\phi) &= -\int_1^L\phi(0,r)\eta(U_0(r))dr, \\
\Pi(\phi) &= -\int_0^T\phi(t,1)q^\Delta(t,1)dt, \\
\Sigma(\phi) &= \int_0^T\sum_{\text{shock}}\phi(\sigma[\eta^\Delta]-[q^\Delta])dt, \\
L(\phi) &= L_1(\phi)+L_2(\phi),
\end{aligned}$$

$$
\begin{aligned}
L_1(\phi) &= \sum_{n=0}^{2m-1}\sum_{j\in I_n}\phi_j^n\int_{F_j^n}[\eta^\Delta]dr,\\
L_2(\phi) &= \sum_{n=0}^{2m-1}\sum_{j\in I_n}\int_{F_j^n}\{\phi(n\Delta t,r)-\phi_j^n\}[\eta^\Delta]dr.
\end{aligned}
$$

Then, by the similar argument in the section 13.2 and from the energy estimates, we have

$$
\begin{cases}
|(M+\Sigma+\Pi+L_1)(\phi)|\le C(\Sigma_4,T,L,T_1,\eta)\|\phi\|_{C(\Omega)},\\
|L_2(\phi)|\le C(\Sigma_4,T,L,T_1,\eta,\alpha)(\Delta r)^{\alpha-1/2}\|\phi\|_{C^\alpha(\bar\Omega)}.
\end{cases}
\tag{13.211}
$$

Thus, we obtain the compactness of η_t+q_r.

Next, we consider the convergence to a weak solution. By Theorem 18, it follows that $\{U^\Delta(t,r)\}_{\Delta r\downarrow 0}$ has convergent subsequence $\{U^{\Delta'}(t,r)\}_{\Delta' r\downarrow 0}$.

Hence, we may show that

$$
\begin{aligned}
I(\phi) &= \int_0^T dt\int_1^L\{\mu_t U^\Delta+\mu_r F(U^\Delta)+\mu H(r,U^\Delta)\}dr+\int_1^L\mu(0,r)U_0(r)dr\\
&= \sum_{n=0}^{2m-1}\int_1^L\mu(n\Delta t,r)[U^\Delta]dr+\int_0^T dt\int_1^L\mu H(r,U^\Delta)dr\\
&= \sum_{n=0}^{2m-1}\sum_{j\in I_n}\int_{F_j^n}\mu(n\Delta t,r)\{U^\Delta(n\Delta t-0,r)-\bar U_{0,j}^n\}dr\\
&\quad -\sum_{n=1}^{2m-1}\sum_{j\in I_n,j>0}\Delta t\int_{F_j^n}\mu(n\Delta t,r)drH(1+j\Delta r,\bar U_{0,j}^n)+\int_0^T dt\int_1^L\mu H(r,U^\Delta)dr
\end{aligned}
$$

tends to zero as $\Delta r\downarrow 0$ for any $\mu=(\phi,\psi)\in C_0^1(\Omega)$ such that $\psi=0$ on $r=1$. Here, since

$$
\begin{aligned}
&\int_{F_0^{2n-1}}\{U^\Delta((2n-1)\Delta t-0,r)-\bar U_{0,0}^{2n-1}\}dr={}^t(0\ \ *)\\
&\mu_0^n=(*\ \ 0)
\end{aligned}
$$

for $j=0$, it follows

$$
\mu_j^n\int_{F_j^n}\{U^\Delta(n\Delta t-0,r)-\bar U_{0,j}^n\}dr=0 \tag{13.212}
$$

for $n\ge 0, j\in I_n$. Hence, if we divide

$$
\begin{aligned}
I &= \sum_{n=0}^{2m-1}\sum_{j\in I_n}\int_{F_j^n}\{\mu(n\Delta t,r)-\mu_j^n\}\{U^\Delta(n\Delta t-0,r)-\bar U_{0,j}^n\}dr\\
&\quad -\sum_{n=1}^{2m-1}\sum_{j\in I_n,j>0}\Delta t\int_{F_j^n}\mu(n\Delta t,r)drH(1+j\Delta r,\bar U_{0,j}^n)+\int_0^T dt\int_1^L\mu H(r,U^\Delta)dr\\
&= I_1+I_2,\\
I_1 &= \sum_{n=0}^{2m-1}\sum_{j\in I_n}\int_{F_j^n}\{\mu(n\Delta t,r)-\mu_j^n\}\{U^\Delta(n\Delta t-0,r)-\bar U_{0,j}^n\}dr,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{2m} \sum_{j \in I_n} \int_{(n-1)\Delta t}^{n\Delta t} dt \int_{F_j^n} \{\mu(t,r)H(r,U^\Delta(t,r)) - \mu(n\Delta t, r)H(1+j\Delta r, \bar{U}_{0,j}^n)\}dr \\
&\quad + \sum_{n=1}^{m} \int_1^{1+\Delta r} \mu((2n-1)\Delta t, r)dr H(1, \bar{U}_{0,0}^{2n-1})\Delta t, \\
|I_1| &\leq C(\Sigma_4, \mu, T, L, T_1)\sqrt{\Delta r}
\end{aligned}
$$

hold from the energy estimate. Thus, we obtain

$$
I_2 = O(\sqrt{\Delta x}). \tag{13.213}
$$

Therefore, it follows the limit of the convergent subsequence is a weak solution.

We also obtain similarly with the proof in the section 13.4 that the weak solution satisfies the entropy condition. Thus, such simple approximations have convergent subsequences to admissible weak solutions.

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