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Curvatures of Almost Hermitian Manifolds

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Contents

Introduction

Almost complex manifolds are defined as a generalization of complex manifolds. Let M be a complex manifold of dimension n and $(U; z^1, \cdots, z^n)$ a local coordinate neighborhood of M. We put $z^i = x^i + \sqrt{-1}y^i$ $(i = 1, \cdots, n)$, then $(U; x^1, \cdots, x^n, y^1, \cdots, y^n)$ can be considered as a real coordinate neighborhood, that is, M can be considered as a real differentiable manifold of dimension $2n$. On each coordinate neighborhood U, a tensor field J_U of type (1,1) can be defined by

$$J_U \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad J_U \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \quad (i = 1, \cdots, n).$$

Then the family $\{J_U\}$ induces a tensor field J on M of type (1,1) such that $J^2 = -I$, where I denotes the identity. This tensor field J is called an almost complex structure corresponding to the complex structure of M.

In general, if a real differential manifold M admits a tensor field J of type (1,1) such that

$$J^2 = -I,$$

then (M, J) is called an *almost complex manifold*, and J is called an *almost complex structure* on M. By the definition, a complex manifold is an almost complex manifold. The almost complex structure J of an almost complex manifold M is said to be *integrable* if there exists a complex structure on M and corresponding almost complex structure coincides with J.

If an almost complex manifold (M, J) admits a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y),$$

for any vector fields X, Y on M, then (M, J, g) is called an *almost Hermitian manifold.*

In the present thesis, we shall study the geometry of almost Hermitian manifolds. Since an almost Hermitian manifold is equipped with an almost complex structure and a Riemannian structure, it is important to study the relation between the geometric properties with respect to the Riemannian structure and the properties of the almost complex structure. According to the words of R. Osserman [29], "the notion of curvature is one of the central concepts of differential geometry." It is obvious that curvatures play an essential role in Riemannian geometry. So, it would be natural to investigate how curvature and almost complex structure influence to each other in an almost Hermitian manifold.

Among almost Hermitian manifolds Kähler manifolds have particularly rich geometric (and other) properties. In order to generalize Kähler geometry, various authors have studied certain kinds of almost Hermitian manifolds, e.g., Hermitian manifolds, nearly Kähler manifolds, almost Kähler manifolds, and so on. We shall focus our attention to these kinds of almost Hermitian manifolds mainly.

As is well-known, the sectional curvature is defined by using the curvature tensor. The holomorphic sectional curvature is the sectional curvature determined by the holomorphic section (i.e., the J-invariant plane in a tangent space) in an almost Hermitian manifold. An almost Hermitian manifold with constant holomorphic sectional curvature may be regarded as a so-called "space form" in almost Hermitian manifolds. It is well-known that there is a classification of Kähler manifolds of constant holomorphic sectional curvature. We shall consider almost Hermitian manifolds of constant holomorphic sectional curvature which are not necessarily Kählerian. The study of such manifolds is much more complicated and interesting than in the Kählerian case. We first obtain an expression for the curvature tensor in an almost Hermitian manifold of (pointwise) constant holomorphic sectional curvature which is a generalization of the Kählerian case. As an application of this expression, we can discuss the problem : "When does the theorem of Schur type hold ?" Moreover, we can consider the classification problem for almost Hermitian manifolds of constant holomorphic sectional curvature. We obtain the following:

1. A non-Kähler nearly Kähler manifold of constant holomorphic sectional curvature is (locally) isometric to a 6-dimensional sphere S^6.

2. A compact almost Kähler manifold of constant non-negative holomorphic sectional curvature satisfying certain condition is a Kähler manifold.

3. A compact Hermitian surface of constant non-positive holomorphic sectional curvature is a Kähler surface.

4. A compact Hermitian surface of constant positive holomorphic sectional curvature is biholomorphically equivalent to a complex projective surface $P^2(\mathbf{C})$.

We can investigate further properties on nearly Kähler manifolds. Many examples of nearly Kähler manifolds are constructed by means of the Riemannian 3-symmetric spaces. We find that one of them has positive sectional curvature which gives a counter example to the conjecture proposed by A. Gray. Furthermore, we shall discuss the pinching problem on the holomorphic sectional curvature of a compact non-Kähler nearly Kähler manifold. And we obtain some generalizations of the result 1. mentioned above.

The present thesis consists of four chapters. In Chapter I, we prepare some fundamental notations and definitions in an almost Hermitian manifold. We give the curvature expression in an almost Hermitian manifold of constant holomorphic sectional curvature. We also consider the theorems of Schur type. Chapter II is devoted to the study of nearly Kähler manifolds. After recalling some identities in nearly Kähler manifolds, we shall make a brief review on the Riemannian 3-symmetric spaces. Then we shall prove a theorem on the spectrum of Laplacian for the 6-dimensional nearly Kähler manifolds. Furthermore, we shall investigate the nearly Kähler manifolds of constant holomorphic sectional curvature and the ones with positive holomorphic sectional curvature. In Chapter III, we shall consider the almost Kähler manifolds. We first observe the fundamental facts on almost Kähler manifolds. Then we shall deal with the almost Kähler manifolds of constant holomorphic sectional curvature. In Chapter IV, we shall study the Hermitian surfaces of constant holomorphic sectional curvature. We shall prove the results related to A. Balas and P. Gauduchon [4]. The Miyaoka's inequality on Chern numbers plays a key role in our arguments.

Throughout this thesis, all manifolds are assumed to be connected and of class C^∞ unless otherwise specified.

Chapter I

Almost Hermitian manifolds

1 Preliminaries

In this section, we prepare some fundamental notations and definitions in an almost Hermitian manifold.

Let $M = (M, J, g)$ be an almost Hermitian manifold. More precisely, M is a C^∞ differentiable manifold of dimension $m(= 2n)$, J is an almost complex structure on M, i.e., it is a tensor field of type (1,1) such that

$$J^2 = -I,$$

and g is a Riemannian metric compatible with J, i.e.,

$$g(JX, JY) = g(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M).

We denote by ∇ and $R = (R_{ijk}{}^l)$ the Riemannian connection and the curvature tensor of M, respectively. We assume that the curvature tensor R is defined by

(1.1) $$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

and

(1.2) $$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. The Ricci tensor $\rho = (\rho_{ij})$ is a symmetric tensor of type (0,2) defined by

$$\rho(x, y) = trace\ of\ [z \mapsto R(z, x)y], \tag{1.3}$$

and the Ricci tensor ρ^1 of type (1,1) is given by $\rho(x, y) = g(\rho^1 x, y)$, for $x, y, z \in T_pM$ $(p \in M)$. The scalar curvature τ is given by

$$\tau = trace\ of\ \rho^1. \tag{1.4}$$

To describe the geometry of an almost Hermitian manifold M, it is useful to consider the following special tensors. The Kähler form $\Omega = (\Omega_{ij})$ is a 2-form defined by

$$\Omega(X, Y) = g(X, JY), \tag{1.5}$$

for $X, Y \in \mathfrak{X}(M)$. In terms of local components,

$$\Omega_{ij} = g_{ik} J_j{}^k = J_{ji}. \tag{1.6}$$

We assume that M is oriented by the volume form $dM = \dfrac{(-1)^n}{n!}\Omega^n$. The Nijenhuis tensor N of almost complex structure J is a tensor field of type (1,2) defined by

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \tag{1.7}$$

for $X, Y \in \mathfrak{X}(M)$. It is easy to see that

$$N(X, Y) = -N(Y, X), \quad N(JX, Y) = N(X, JY) = -JN(X, Y). \tag{1.8}$$

It is well known that the almost complex structure J of an almost complex manifold M is integrable if and only if the Nijenhuis tensor N of J vanishes identically on M ([27, 23]).

The Ricci $*$-tensor ρ^* of type (0,2), resp. ρ^{*1} of type (1,1), is defined by

$$\begin{aligned} \rho^*(x, y) &= g(\rho^{*1} x, y) = trace\ of\ [z \mapsto R(x, Jz)Jy] \\ &= \sum_{a=1}^{m} R(e_a, x, Jy, Je_a), \end{aligned} \tag{1.9}$$

where $\{e_1, ..., e_m\}$ is an arbitrary orthonormal basis of T_pM ([55]). In terms of local components, $\rho^* = (\rho^*_{ij})$ is given by

$$\rho^*_{ij} = J_j{}^s R_{aisb} J^{ab} = \frac{1}{2} J_j{}^s R_{isa}{}^b J_b{}^a. \tag{1.10}$$

The $*$-scalar curvature τ^* is given by

$$\tau^* = trace\ of\ \rho^{*1} = J^{ab} J^{st} R_{astb}. \tag{1.11}$$

It is easy to see that

$$\rho^*(Jx, Jy) = \rho^*(y, x). \tag{1.12}$$

By using the Ricci $*$-tensor ρ^*, the first Chern form $\gamma = (\gamma_{ij})$ is given by

$$8\pi\gamma_{ij} = -4J_j{}^k \rho^*_{ik} - J^{kl}(\nabla_j J_k{}^h)\nabla_i J_{lh}, \tag{1.13}$$

([45]). It is well known that the 2-form γ represents the first Chern class $c_1(M)$ of M in the de Rham cohomology group.

The sectional curvature, the holomorphic bisectional curvature and the holomorphic sectional curvature are defined respectively by

$$K(x, y) = -\frac{R(x, y, x, y)}{\|x\|^2\|y\|^2}, \tag{1.14}$$

for $x, y \in T_pM$ $(p \in M)$ with $x \neq 0, y \neq 0, g(x, y) = 0$,

$$B(x, y) = -\frac{R(x, Jx, y, Jy)}{\|x\|^2\|y\|^2}, \tag{1.15}$$

for $x, y \in T_pM$ $(p \in M)$ with $x \neq 0, y \neq 0, g(x, y) = 0$, and

$$\begin{aligned} H(x) &= K(x, Jx) = B(x, x) \\ &= -\frac{R(x, Jx, x, Jx)}{\|x\|^4}, \end{aligned} \tag{1.16}$$

for $x \in T_pM$ $(p \in M)$ with $x \neq 0$.

It is well known that the sectional curvature $K(x, y)$ is determined by the plane $\Pi\{x, y\}$ spanned by x, y, and is independent of the choice of a basis $\{x, y\}$ for the plane $\Pi\{x, y\}$. In particular, the holomorphic sectional curvature $H(x)$

is determined by the holomorphic plane $\Pi\{x, Jx\}$. If $K(x,y)$ is a constant for all planes $\Pi\{x,y\} \subset T_pM$ and for all points $p \in M$, then M is called a *space of constant curvature.* The following theorem due to F. Schur is well known (cf. [23]).

Theorem 1.1 *Let M be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature $K(x,y)$ $(x, y \in T_pM)$ depends only on p, then M is a space of constant curvature.*

Similarly, we use the following terminology. If the holomorphic sectional curvature $H(x)$ is constant for all non-zero $x \in T_pM$ at each point $p \in M$, then M is called a *space of pointwise constant holomorphic sectional curvature.* Moreover, if $H(x)$ is constant for all non-zero $x \in T_pM$ and for all points $p \in M$, then M is called a *space of constant holomorphic sectional curvature.* We also note that the holomorphic sectional curvature $H(x)$ can be regarded as a differentiable function on the unit tangent bundle $U(M)$ over M.

Now, we introduce the tensor field $G = (G_{ijkl})$ by

$$G(X,Y,Z,W) = R(X,Y,Z,W) - R(X,Y,JZ,JW). \tag{1.17}$$

It seems that the tensor field G is important in the study of almost Hermitian manifolds (cf. A. Gray [18, 20], L. Vanhecke [58], F. Tricerri–L. Vanhecke [55], etc.). For $X, Y, Z, W \in \mathfrak{X}(M)$, G satisfies the following relations;

$$G(X,Y,Z,W) = -G(Y,X,Z,W) = -G(X,Y,W,Z), \tag{1.18}$$

$$G(X,Y,JZ,JW) = -G(X,Y,Z,W), \tag{1.19}$$

$$G(X,Y,JZ,W) = G(X,Y,Z,JW), \tag{1.20}$$

$$G(X,Y,Z,JZ) = 0, \tag{1.21}$$

$$\begin{aligned} &G(X,Y,Z,W) + G(JX,JY,JZ,JW) \\ &\quad = G(Z,W,X,Y) + G(JZ,JW,JX,JY), \end{aligned} \tag{1.22}$$

$$\begin{aligned} &G(X,JY,Z,JW) + G(JX,Y,JZ,W) \\ &\quad = G(Z,JW,X,JY) + G(JZ,W,JX,Y), \end{aligned} \tag{1.23}$$

$$\sum_{i=1}^{m} G(E_i, X, Y, E_i) = \rho(X,Y) - \rho^*(X,Y), \tag{1.24}$$

where $\{E_i\}_{i=1,\dots,m}$ is a local orthonormal frame of M.

2 Classes of almost Hermitian manifolds

In this section, we shall define some special classes of almost Hermitian manifolds which we shall deal with in this thesis.

An almost Hermitian manifold $M = (M, J, g)$ is called a *Kähler manifold* provided that

$$\nabla J = 0. \tag{2.1}$$

This condition is equivalent to $N = 0$ and $d\Omega = 0$.

By generalizing the notion of Kähler manifolds, we shall define the following classes: $M = (M, J, g)$ is called a *Hermitian manifold* if

$$N = 0, \tag{2.2}$$

or equivalently,

$$(\nabla_X J)Y - (\nabla_{JX} J)(JY) = 0, \quad \text{for all } X, Y \in \mathfrak{X}(M). \tag{2.3}$$

M is called an *almost Kähler manifold* if

$$d\Omega = 0, \tag{2.4}$$

or equivalently,

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0, \tag{2.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

M is called a *nearly Kähler manifold* (also known as *K-space* or *almost Tachibana space*) if

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad \text{for all } X, Y \in \mathfrak{X}(M). \tag{2.6}$$

M is called a *quasi-Kähler manifold* (or *$^*O$-space*) if

$$(2.7) \qquad (\nabla_X J)Y + (\nabla_{JX} J)(JY) = 0, \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Let $\mathcal{K}$, $\mathcal{H}$, $\mathcal{AK}$, $\mathcal{NK}$ and $\mathcal{QK}$ denote the classes of Kähler, Hermitian, almost Kähler, nearly Kähler and quasi-Kähler manifolds respectively. Then the following relations are well-known:

$$\mathcal{K} \subset \mathcal{AK} \subset \mathcal{QK}, \quad \mathcal{K} \subset \mathcal{NK} \subset \mathcal{QK}, \quad \mathcal{K} \subset \mathcal{H},$$

$$\mathcal{AK} \cap \mathcal{NK} = \mathcal{K}, \quad \mathcal{H} \cap \mathcal{QK} = \mathcal{K}.$$

Curvature identities are important to study the geometry of these classes of almost Hermitian manifolds. A. Gray has obtained the following

Lemma 2.1 ([20]) *Let M be a Hermitian manifold. Then*

$$(2.8) \qquad \begin{aligned} &G(X,Y,Z,W) + G(JX,JY,JZ,JW) \\ &\quad - G(JX,Y,JZ,W) - G(X,JY,Z,JW) = 0. \end{aligned}$$

Lemma 2.2 ([20]) *Let M be a quasi-Kähler manifold. Then*

$$(2.9) \qquad \begin{aligned} &G(X,Y,Z,W) + G(JX,JY,JZ,JW) \\ &\quad + G(JX,Y,JZ,W) + G(X,JY,Z,JW) \\ &= -2g((\nabla_{(\nabla_X J)Y} J)Z, W) - 2g((\nabla_{(\nabla_Y J)X} J)W, Z). \end{aligned}$$

Lemma 2.3 ([20]) *Let M be an almost Kähler manifold. Then*

$$(2.10) \qquad \begin{aligned} &G(X,Y,Z,W) + G(JX,JY,JZ,JW) \\ &\quad + G(JX,Y,JZ,W) + G(X,JY,Z,JW) \\ &= 2g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z). \end{aligned}$$

Lemma 2.4 ([18]) *Let M be a nearly Kähler manifold. Then*

$$(2.11) \qquad G(X,Y,Z,W) = -g((\nabla_X J)Y, (\nabla_Z J)W).$$

Next, we shall consider the following conditions on the tensor field G for an almost Hermitian manifold M :

$$(a) \quad G = 0,$$

$$(b) \quad G(JX, Y, JZ, W) = G(X, Y, Z, W), \quad \text{for } X, Y, Z, W \in \mathfrak{X}(M),$$

$$(c) \quad G(X, Y, Z, W) = G(Z, W, X, Y), \quad \text{for } X, Y, Z, W \in \mathfrak{X}(M).$$

It is easily shown that $(a) \Rightarrow (b) \Rightarrow (c)$. We remark that the condition (a) is equivalent to

$$R(X, Y) \circ J = J \circ R(X, Y), \tag{2.12}$$

for $X, Y \in \mathfrak{X}(M)$. An almost Hermitian manifold M satisfying (2.12) (and so, the condition (a)) is called a *para-Kähler manifold* or an *F-space*([39],[58]). The condition (b) is equivalent to

$$\begin{aligned} \text{(2.13)} \quad & R(X, Y, Z, W) \\ & = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW), \end{aligned}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. The condition (c) is equivalent to

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W), \tag{2.14}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. An almost Hermitian manifold M satisfying (2.14) (and so, the condition (c)) is called an *RK-manifold* ([58]).

The identities (2.12), (2.13) and (2.14) in almost Hermitian manifolds are studied by A. Gray, F. Tricerri, L. Vanhecke, etc.. It is well known that Kähler manifolds satisfy the condition (a). Nearly Kähler manifolds and Riemannian locally 3-symmetric spaces satisfy the condition (b) ([19]). It is proved in [20], some quasi-Kähler manifolds satisfy the condition (c). In Hermitian manifolds, the condition (b) is equivalent to (c). S. Tanno gave in [52] a condition for an RK-manifold to be of constant holomorphic sectional curvature.

In an RK-manifold M, we have easily the following

$$G(JX, JY, JZ, JW) = G(X, Y, Z, W), \tag{2.15}$$

$$(2.16) \qquad \rho(JX, JY) = \rho(X, Y),$$

$$(2.17) \qquad \rho^*(X, Y) = \rho^*(Y, X),$$

$$(2.18) \qquad \rho^*(JX, JY) = \rho^*(X, Y),$$

for $X, Y, Z, W \in \mathfrak{X}(M)$.

3 Almost Hermitian manifolds of constant holomorphic sectional curvature

In an almost Hermitian manifold $M = (M, J, g)$, we put

$$\lambda(x, y) = G(x, y, x, y),$$

and

$$Q(x) = R(x, Jx, x, Jx) = -H(x)\|x\|^4,$$

for $x, y \in T_pM$. L. Vanhecke proved the following

Proposition 3.1 ([59]) *Let M be an almost Hermitian manifold and $x, y \in T_pM$. Then*

$$\begin{aligned}(3.1) \qquad & R(x, y, x, y) \\ &= \frac{1}{32}\{3Q(x + Jy) + 3Q(x - Jy) - Q(x + y) - Q(x - y) - 4Q(x) - 4Q(y)\} \\ &+ \frac{1}{16}\{13\lambda(x, y) - 3\lambda(Jx, Jy) + \lambda(x, Jy) + \lambda(Jx, y)\}.\end{aligned}$$

Now, we assume that the holomorphic sectional curvature $H(x)$ is constant $c(p)$ for all $x \in T_pM$ at each point p of M. For a Kähler manifold$M$ of pointwise constant holomorphic sectional curvature $c(p)$, it is well known that the curvature tensor of M at p is given by

$$(3.2) \qquad R(x, y, z, w) = \frac{c(p)}{4} H(x, y, z, w),$$

where

$$H(x,y,z,w) = g(x,w)g(y,z) - g(x,z)g(y,w) \tag{3.3}$$
$$+g(x,Jw)g(y,Jz) - g(x,Jz)g(y,Jw) - 2g(x,Jy)g(z,Jw),$$

and c is a constant function on M, if M is connected ([23]).

As a generalization of (3.2), we have the following

Theorem 3.2 *Let M be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then*

$$R(x,y,z,w) = \frac{c(p)}{4}H(x,y,z,w) + P(x,y,z,w), \tag{3.4}$$

where

$$\begin{aligned}
&P(x,y,z,w)\\
&\quad = \frac{1}{96}[26\{G(x,y,z,w) + G(z,w,x,y)\}\\
&\quad - 6\{G(Jx,Jy,Jz,Jw) + G(Jz,Jw,Jx,Jy)\}\\
&\quad + 13\{G(x,z,y,w) + G(y,w,x,z) - G(x,w,y,z) - G(y,z,x,w)\}\\
&\quad - 3\{G(Jx,Jz,Jy,Jw) + G(Jy,Jw,Jx,Jz)\\
&\qquad\qquad - G(Jx,Jw,Jy,Jz) - G(Jy,Jz,Jx,Jw)\}\\
&\quad + 4\{G(x,Jy,z,Jw) + G(Jx,y,Jz,w)\}\\
&\quad + 2\{G(x,Jz,y,Jw) + G(Jx,z,Jy,w)\\
&\qquad\qquad - G(x,Jw,y,Jz) - G(Jx,w,Jy,z)\}].
\end{aligned}$$

Proof. Since $Q(x) = -c(p)\|x\|^4$, we have from (3.1)

$$R(x,y,x,y) = \frac{c(p)}{4}\{g(x,y)^2 - \|x\|^2\|y\|^2 - 3g(x,Jy)^2\} \tag{3.5}$$
$$+\frac{1}{16}\{13\lambda(x,y) - 3\lambda(Jx,Jy) + \lambda(x,Jy) + \lambda(Jx,y)\}.$$

By linearizing (3.5), we get

$$R(x,y,z,w) + R(z,y,x,w) \tag{3.6}$$

$$
\begin{aligned}
&= \frac{c(p)}{4}\{g(x,y)g(z,w) + g(x,w)g(z,y) - 2g(x,z)g(y,w) \\
&\qquad - 3g(x,Jy)g(z,Jw) - 3g(x,Jw)g(z,Jy)\} \\
&+ \frac{1}{32}[13\{G(x,y,z,w) + G(z,y,x,w) + G(x,w,z,y) + G(z,w,x,y)\} \\
&\quad - 3\{G(Jx,Jy,Jz,Jw) + G(Jz,Jy,Jx,Jw) \\
&\qquad + G(Jx,Jw,Jz,Jy) + G(Jz,Jw,Jx,Jy)\} \\
&\quad + 2\{G(x,Jy,z,Jw) + G(z,Jy,x,Jw) \\
&\qquad + G(Jx,y,Jz,w) + G(Jz,y,Jx,w)\}].
\end{aligned}
$$

Interchanging x and y in (3.6) and subtracting the resulting equation from (3.6), and taking account of (1.20), (1.22), we have finally (3.4). ∎

From this theorem and Lemma 2.1, we have easily the following

Corollary 3.3 *Let M be a Hermitian manifold of pointwise constant holomorphic sectional curvature. Then the curvature tensor of M is given by* (3.4), *where*

$$
\begin{aligned}
&P(x,y,z,w) \\
&\quad = \frac{1}{48}[14\{G(x,y,z,w) + G(z,w,x,y)\} \\
&\quad - 2\{G(Jx,Jy,Jz,Jw) + G(Jz,Jw,Jx,Jy)\} \\
&\quad + 7\{G(x,z,y,w) + G(y,w,x,z) - G(x,w,y,z) - G(y,z,x,w)\} \\
&\quad - \{G(Jx,Jz,Jy,Jw) + G(Jy,Jw,Jx,Jz) \\
&\qquad - G(Jx,Jw,Jy,Jz) - G(Jy,Jz,Jx,Jw)\}].
\end{aligned}
$$

Corollary 3.4 *Let M be an RK-manifold of pointwise constant holomorphic sectional curvature. Then the curvature tensor of M is given by* (3.4), *where*

$$
\begin{aligned}
&P(x,y,z,w) \\
&\quad = \frac{1}{24}\{10G(x,y,z,w) + 5G(x,z,y,w) - 5G(x,w,y,z) \\
&\quad + 2G(x,Jy,z,Jw) + G(x,Jz,y,Jw) - G(x,Jw,y,Jz)\}.
\end{aligned}
$$

Corollary 3.5 ([22]) *Let M be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature. If M satisfies the condition (b), then the curvature tensor of M is given by (3.4), where*

$$P(x,y,z,w) = \frac{1}{4}\{2G(x,y,z,w) + G(x,z,y,w) - G(x,w,y,z)\}.$$

Corollary 3.6 ([39]) *Let M be a para-Kähler manifold of pointwise constant holomorphic sectional curvature. Then the curvature tensor of M is given by (3.2).*

By (3.4), (1.24), (2.15), (2.16), (2.17) and (2.18), we have

Corollary 3.7 *Let M be an m-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then*

$$\rho(x,y) + \rho(Jx,Jy) + 3(\rho^*(x,y) + \rho^*(y,x)) = 2(m+2)c(p)g(x,y), \tag{3.7}$$

$$\tau + 3\tau^* = m(m+2)c(p). \tag{3.8}$$

In particular, if M is an RK-manifold, then

$$\rho(x,y) + 3\rho^*(x,y) = (m+2)c(p)g(x,y). \tag{3.9}$$

From Lemma 2.2 and Corollary 3.5, we have

Theorem 3.8 *Let M be a quasi-Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$. If M satisfies the condition (b), then the curvature tensor of M is given by*

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c(p)}{4}H(X,Y,Z,W) \\ &+\frac{1}{8}\{g((\nabla_{(\nabla_X J)W-(\nabla_W J)X}J)Y,Z) - g((\nabla_{(\nabla_X J)Z-(\nabla_Z J)X}J)Y,W) \\ &- 2g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W)\}. \end{aligned} \tag{3.10}$$

From Lemma 2.3 and Corollary 3.5, we have

Theorem 3.9 *Let M be an almost Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$. If M satisfies the condition (b), then the curvature tensor of M is given by*

$$
\begin{aligned}
(3.11)\quad R(X,Y,Z,W) = {} & \frac{c(p)}{4} H(X,Y,Z,W) \\
& -\frac{1}{8}\{g((\nabla_X J)W - (\nabla_W J)X, (\nabla_Y J)Z - (\nabla_Z J)Y) \\
& - g((\nabla_X J)Z - (\nabla_Z J)X, (\nabla_Y J)W - (\nabla_W J)Y) \\
& - 2g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z)\}.
\end{aligned}
$$

From Lemma 2.4 and Corollary 3.5, we have

Theorem 3.10 ([40]) *Let M be a nearly Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then the curvature tensor of M is given by*

$$
\begin{aligned}
(3.12)\quad R(X,Y,Z,W) = {} & \frac{c(p)}{4} H(X,Y,Z,W) \\
& +\frac{1}{4}\{g((\nabla_X J)W, (\nabla_Y J)Z) - g((\nabla_X J)Z, (\nabla_Y J)W) \\
& - 2g((\nabla_X J)Y, (\nabla_Z J)W)\}.
\end{aligned}
$$

4 Theorems of Schur type

In this section, we shall consider the following problem:

> *Let M be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c(p)$. When is c a constant function?*

Concerning this problem, A. Gray and L. Vanhecke [22] have proved an interesting theorem (see Theorem 4.7 below). Moreover, they have shown that this problem does not hold for the class of Hermitian manifolds, that is, there exists a Hermitian manifold of pointwise constant holomorphic sectional curvature which is not globally constant. We shall make a slightly different approach to this problem.

Let M be a para-Kähler manifold of pointwise constant holomorphic sectional curvature. Then, by Corollary 3.6, we have

$$R(X,Y,Z,W)=\frac{c(p)}{4}H(X,Y,Z,W). \tag{4.1}$$

Contracting (4.1), we have

$$\rho(X,Y)=\frac{m+2}{4}c(p)g(X,Y),$$

which shows that M is an Einstein manifold and c is constant. Thus we have

Theorem 4.1 *Let M be a connected para-Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$ with $\dim M=m\geq 4$. Then c is a constant function on M.*

The Bochner curvature tensor B on an $m=2n$-dimensional almost Hermitian manifold M is defined by

$$B=R-\frac{1}{2(n+2)}S+\frac{\tau}{4(n+1)(n+2)}H,$$

where S is given by

$$\begin{aligned}
&S(X,Y,Z,W)\\
&\quad=\rho(X,W)g(Y,Z)-\rho(Y,W)g(X,Z)\\
&\quad\quad+g(X,W)\rho(Y,Z)-g(Y,W)\rho(X,Z)\\
&\quad\quad+\rho(JX,W)g(JY,Z)-\rho(JY,W)g(JX,Z)\\
&\quad\quad+g(JX,W)\rho(JY,Z)-g(JY,W)\rho(JX,Z)\\
&\quad\quad-2\rho(JX,Y)g(JZ,W)-2g(JX,Y)\rho(JZ,W).
\end{aligned}$$

Suppose that $B=0$ and $\rho(JX,JY)=\rho(X,Y)$ for all $X,Y\in\mathfrak{X}(M)$, then we have directly $G=0$, that is, M is para-Kählerian. Thus we have from Theorem 4.1,

Theorem 4.2 *Let M be a connected RK-manifold of pointwise constant holomorphic sectional curvature $c(p)$ with $\dim M \geq 4$. If the Bochner curvature tensor vanishes, then c is a constant function.*

Let $\{E_i\}$ be a local orthonormal frame of an almost Hermitian manifold M. We prepare the following

Lemma 4.3 *Let M be a quasi-Kähler manifold satisfying the condition (c). Then*

$$(4.2) \qquad 2\sum_j (\nabla_{E_j}\rho^*)(X, E_j) - \nabla_X \tau^* = 2\sum_{i,j} R(E_i, X, JE_j, (\nabla_{E_j}J)E_i),$$

for $X \in \mathfrak{X}(M)$.

Proof. By the definition, we have

$$\rho^*(X, Y) = \sum R(E_i, X, JY, JE_i),$$

and

$$\tau^* = \sum \rho^*(E_j, E_j) = \sum R(E_i, E_j, JE_j, JE_i).$$

By the standard calculation,

$$(4.3) \quad (\nabla_V \rho^*)(X, Y) = \sum\{(\nabla_V R)(E_i, X, JY, JE_i) + R(E_i, X, (\nabla_V J)Y, JE_i) + R(E_i, X, JY, (\nabla_V J)E_i)\},$$

and making use of Bianchi identity,

$$\begin{aligned}(4.4) \quad \nabla_X \tau^* &= \sum\{(\nabla_X R)(E_i, E_j, JE_j, JE_i) + 2R(E_i, E_j, JE_j, (\nabla_X J)E_i)\} \\ &= 2\sum\{(\nabla_{E_j} R)(E_i, X, JE_j, JE_i) + R(E_i, E_j, JE_j, (\nabla_X J)E_i)\}.\end{aligned}$$

From (4.3) and (4.4), we have

$$(4.5) \quad 2\sum(\nabla_{E_j}\rho^*)(X, E_j) - \nabla_X \tau^* = 2\sum\{R(X, E_i, (\nabla_{E_j}J)E_j, JE_i) + R(E_i, X, JE_j, (\nabla_{E_j}J)E_i) - R(E_i, E_j, JE_j, (\nabla_X J)E_i)\}.$$

It is clear that in quasi-Kähler manifolds,

$$\sum(\nabla_{E_j}J)E_j = 0,$$

therefore the first term of the right hand side of (4.5) is equal to zero. By making use of (2.14),

$$\begin{aligned}\sum R(E_i, E_j, JE_j, (\nabla_X J)E_i) &= \sum R(JE_i, JE_j, E_j, (\nabla_X J)JE_i) \\ &= -\sum R(E_i, E_j, JE_j, (\nabla_X J)E_i),\end{aligned}$$

so the last term is also zero. Thus, we obtain (4.2). ∎

Lemma 4.4 *Let M be a quasi-Kähler manifold satisfying the condition (c). Then*

$$\begin{aligned}(4.6)\qquad & 2\sum_{i,j} R(E_i, X, JE_j, (\nabla_{E_j}J)E_i) \\ &= \sum_{i,j}\{(\nabla_{E_j}R)(X, E_i, JE_j, JE_i) - (\nabla_{E_j}R)(JX, JE_i, E_j, E_i)\},\end{aligned}$$

for $X \in \mathfrak{X}(M)$.

Proof. By the condition (c),

$$R(X, Y, JZ, JW) = R(JX, JY, Z, W).$$

Differentiating this

$$\begin{aligned}&(\nabla_V R)(X, Y, JZ, JW) + R(X, Y, (\nabla_V J)Z, JW) + R(X, Y, JZ, (\nabla_V J)W) \\ &= (\nabla_V R)(JX, JY, Z, W) + R((\nabla_V J)X, JY, Z, W) + R(JX, (\nabla_V J)Y, Z, W).\end{aligned}$$

Putting $Y = W = E_i, Z = V = E_j$ and summing up with respect to i, j,

$$\begin{aligned}(4.7)\qquad & \sum_{i,j}\{(\nabla_{E_j}R)(X, E_i, JE_j, JE_i) - (\nabla_{E_j}R)(JX, JE_i, E_j, E_i)\} \\ &= \sum_{i,j}\{R((\nabla_{E_j}J)X, JE_i, E_j, E_i) + R(JX, (\nabla_{E_j}J)E_i, E_j, E_i) \\ &\qquad -R(X, E_i, (\nabla_{E_j}J)E_j, JE_i) - R(X, E_i, JE_j, (\nabla_{E_j}J)E_i)\}.\end{aligned}$$

By the same reason as in the proof of Lemma 4.3, we see that the first and third term of the right hand side of (4.7) vanish. Taking account of

$$\begin{aligned}
\sum R(JX, (\nabla_{E_j}J)E_i, E_j, E_i) &= -\sum R(JX, E_i, E_j, (\nabla_{E_j}J)E_i) \\
&= \sum R(X, JE_i, JE_j, J(\nabla_{E_j}J)E_i) \\
&= -\sum R(X, JE_i, JE_j, (\nabla_{E_j}J)JE_i) \\
&= -\sum R(X, E_i, JE_j, (\nabla_{E_j}J)E_i) \\
&= \sum R(E_i, X, JE_j, (\nabla_{E_j}J)E_i),
\end{aligned}$$

we obtain (4.6). ∎

Now, we prove the following "Schur's theorem" for quasi-Kähler manifolds under some conditions.

Theorem 4.5 *Let M be a connected quasi-Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$ with $\dim M \geq 4$. If M satisfies the condition (c) and*

$$(4.8) \qquad (\nabla_V R)(X, Y, Z, W) + (\nabla_V R)(JX, JY, JZ, JW) = 0,$$

for $X, Y, Z, W \in \mathfrak{X}(M)$, then c is a constant function.

In particular, we have

Corollary 4.6 *Let M be a connected locally symmetric quasi-Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$ with $\dim M \geq 4$. If M satisfies the condition (c), then c is a constant function.*

Proof of Theorem 4.5. By the condition (4.8) and Bianchi identity, we have

$$\begin{aligned}
&\sum_{i,j}\{(\nabla_{E_j}R)(X, E_i, JE_j, JE_i) - (\nabla_{E_j}R)(JX, JE_i, E_j, E_i)\} \\
&\qquad = 2\sum_{i,j}(\nabla_{E_j}R)(X, E_i, JE_j, JE_i) \\
&\qquad = \sum_{i,j}(\nabla_X R)(E_j, E_i, JE_j, JE_i).
\end{aligned}$$

By using (4.8) again,

$$\begin{aligned}(\nabla_X R)(E_j, E_i, JE_j, JE_i) &= -(\nabla_X R)(JE_j, JE_i, E_j, E_i) \\ &= -(\nabla_X R)(E_j, E_i, JE_j, JE_i),\end{aligned}$$

from which we see that the right hand side of (4.6) vanishes. From Lemma 4.3 and Lemma 4.4, we have

$$2\sum(\nabla_{E_j}\rho^*)(X, E_j) = \nabla_X \tau^*. \tag{4.9}$$

On the other hand, it is well known that

$$2\sum(\nabla_{E_j}\rho)(X, E_j) = \nabla_X \tau. \tag{4.10}$$

Differentiating (3.8) with respect to arbitrary E_i, we have

$$\nabla_{E_i}\tau + 3\nabla_{E_i}\tau^* = m(m+2)\nabla_{E_i}c. \tag{4.11}$$

On one hand, by making use of (4.9), (4.10) and (3.9),

$$\begin{aligned}&\nabla_{E_i}\tau + 3\nabla_{E_i}\tau^* \\ &= 2\sum_j\{(\nabla_{E_j}\rho)(E_i, E_j) + 3(\nabla_{E_j}\rho^*)(E_i, E_j)\} \\ &= 2(m+2)\nabla_{E_i}c.\end{aligned} \tag{4.12}$$

By (4.11) and (4.12),

$$(m-2)(m+2)\nabla_{E_i}c = 0,$$

from which it follows that c is a constant function. ∎

The following theorem is due to A. Gray and L. Vanhecke. We shall give another proof of the theorem.

Theorem 4.7 ([22]) *Let M be a connected quasi-Kähler manifold of pointwise constant holomorphic sectional curvature $c(p)$ with $\dim M \geq 4$. If M satisfies the condition (b), then c is a constant function.*

Proof. Observing the proof of Theorem 4.5, it is sufficient to prove (4.9). By Corollary 3.5, the curvature tensor of M is given by

$$(4.13) \quad R(X,Y,Z,W) = \frac{c}{4}H(X,Y,Z,W) + \frac{1}{4}\{2G(X,Y,Z,W) + G(X,Z,Y,W) - G(X,W,Y,Z)\}.$$

It is easy to check

$$\sum H(E_i, X, JE_j, (\nabla_{E_j}J)E_i) = 0.$$

By using the condition (*b*),

$$\begin{aligned} \sum G(E_i, X, JE_j, (\nabla_{E_j}J)E_i) &= \sum G(JE_i, X, JE_j, J(\nabla_{E_j}J)E_i) \\ &= -\sum G(JE_i, X, JE_j, (\nabla_{E_j}J)JE_i) \\ &= -\sum G(E_i, X, JE_j, (\nabla_{E_j}J)E_i), \end{aligned}$$

from which we have

$$\sum G(E_i, X, JE_j, (\nabla_{E_j}J)E_i) = 0.$$

Similarly by using the condition (*b*) and the definition of quasi-Kähler manifold, we get

$$\sum G(E_i, JE_j, X, (\nabla_{E_j}J)E_i) = 0,$$

and

$$\sum G(E_i, (\nabla_{E_j}J)E_i, X, JE_j) = 0.$$

Hence we have from (4.13),

$$\sum R(E_i, X, JE_j, (\nabla_{E_j}J)E_i) = 0.$$

This proves (4.9) by virtue of Lemma 4.3. Thus we obtain the theorem. ∎

Chapter II

Nearly Kähler manifolds

1 Identities in nearly Kähler manifolds

Let $M = (M, J, g)$ be an $m(= 2n)$-dimensional nearly Kähler manifold. We shall recall some useful identities in M in terms of local components for the later use.

By the definition,

$$\nabla_i J_{jk} + \nabla_j J_{ik} = 0. \tag{1.1}$$

Since M is also quasi-Kählerian and RK-manifold, we have ([25, 42, 60])

$$\nabla_i J_{jk} + J_i{}^a J_j{}^b \nabla_a J_{bk} = 0, \tag{1.2}$$

$$\rho_{ij} = J_i{}^a J_j{}^b \rho_{ab}, \tag{1.3}$$

$$\rho^*_{ij} = J_i{}^a J_j{}^b \rho^*_{ab}, \tag{1.4}$$

$$\rho^*_{ij} = \rho^*_{ji}. \tag{1.5}$$

By Lemma 2.4 of Chapter I, we have

$$G_{ijkl} = R_{ijkl} - J_k{}^a J_l{}^b R_{ijab} = -(\nabla_i J_j{}^r)\nabla_k J_{lr}, \tag{1.6}$$

$$(\nabla_i J_{rs})\nabla_j J^{rs} = \rho_{ij} - \rho^*_{ij}, \tag{1.7}$$

$$\|\nabla J\|^2 = \tau - \tau^* = \text{const.} \geq 0. \tag{1.8}$$

From (1.6) and (1.7), we have

$$\|G\|^2 = \|\rho - \rho^*\|^2, \tag{1.9}$$

(1.10) $$R^{ijkl}R_{ijab}J_k{}^aJ_l{}^b = \|R\|^2 - \frac{1}{2}\|\rho - \rho^*\|^2.$$

Moreover, it is also known ([60, 40])

(1.11) $$2(\nabla^s J^{ab})R_{abij} = -(\rho^{st} - \rho^{*st})\nabla_t J_{ij},$$

(1.12) $$(\nabla^i J^j{}_a)(\nabla^k J^{la})R_{ijkl} = -\frac{1}{2}\|\rho - \rho^*\|^2,$$

(1.13) $$(\nabla^i J^l{}_a)(\nabla^j J^{ka})R_{ijkl} = \frac{1}{4}\|\rho - \rho^*\|^2,$$

(1.14) $$\|\nabla(\rho - \rho^*)\|^2 = \frac{1}{8}(\rho_{ij} - 5\rho^*_{ij})(\rho^{ik} - \rho^{*ik})(\rho^j{}_k - \rho^{*j}{}_k).$$

By (1.1) and the Ricci identity, we have

(1.15) $$\nabla^i\nabla_i J_{jk} = (\rho_{ja} - \rho^*_{ja})J_k{}^a.$$

From (I.1.13) and (1.6), the first Chern form γ is given by

(1.16) $$8\pi\gamma_{ij} = J_j{}^a(\rho_{ia} - 5\rho^*_{ia}),$$

it follows that the first Chern form γ of M vanishes if and only if $\rho = 5\rho^*$.

By (I.1.14), (I.1.15), (I.1.16), (1.6) and Proposition 3.1 of Chapter I, we have

(1.17) $$\begin{aligned} &K(x,y)\\ &= \frac{1}{8}\{3H(x+Jy) + 3H(x-Jy) - H(x+y) - H(x-y) - H(x) - H(y)\}\\ &\quad + \frac{3}{4}\|(\nabla_x J)y\|^2, \end{aligned}$$

(1.18) $$B(x,y) = K(x,y) + K(x,Jy) - 2\|(\nabla_x J)y\|^2,$$

for $x, y \in T_pM$ with $\|x\| = \|y\| = 1, g(x,y) = g(x,Jy) = 0$.

It is well known that 2 or 4-dimensional nearly Kähler manifolds are Kählerian ([17, 50]). So the 6-dimensional one is particularly interesting. M. Matsumoto [25] has proved that any 6-dimensional non-Kähler nearly Kähler manifold is an Einstein manifold with positive scalar curvature. Furthermore, the following equalities hold in a 6-dimensional non-Kähler nearly Kähler manifold M ([25, 42]):

(1.19) $$(\nabla_i J_j{}^r)\nabla_k J_{lr} = -\frac{\tau}{30}\{g_{il}g_{jk} - g_{ik}g_{jl} - J_{il}J_{jk} + J_{ik}J_{jl}\},$$

$$\nabla_i \nabla_j J_{kl} = -\frac{\tau}{30}\{g_{ij}J_{kl} - g_{ik}J_{jl} + g_{il}J_{jk}\}, \tag{1.20}$$

$$\rho_{ij} = 5\rho^*_{ij} \quad (= \frac{\tau}{6}g_{ij}). \tag{1.21}$$

By (1.21), the first Chern form of 6-dimensional non-Kähler nearly Kähler manifold M vanishes.

Finally, we shall recall the following "Lichnerowicz's formula". Let (M, g) be an m-dimensional Riemannian manifold and $\Delta = -\delta d$ be the Laplace–Beltrami operator acting on differentiable functions on M. Then we have

$$\frac{1}{2}\Delta(\|R\|^2) = \|\nabla R\|^2 + 4R^{ijkl}\nabla_i\nabla_l\rho_{jk} + 2\rho^{ab}R_{ajkl}R_b{}^{jkl} + \widehat{R} + 4\overset{\circ}{R}, \tag{1.22}$$

where

$$\widehat{R} = R^{ijkl}R_{ij}{}^{ab}R_{abkl}, \tag{1.23}$$

$$\overset{\circ}{R} = R^{ijkl}R_i{}^a{}_k{}^b R_{ajbl}. \tag{1.24}$$

If (M, g) is Einsteinian, (1.22) reduces to

$$\frac{1}{2}\Delta(\|R\|^2) = \|\nabla R\|^2 + \frac{2\tau}{m}\|R\|^2 + \widehat{R} + 4\overset{\circ}{R}. \tag{1.25}$$

2 Riemannian 3-symmetric spaces

A typical example of nearly Kähler manifolds is a 6-dimensional unit sphere S^6. A nearly Kähler structure on S^6 is constructed as follows [13, 14, 23]:

Let $\mathcal{C}$ be the Cayley algebra and E be a set of all pure imaginary Cayley numbers. Then E can be identified with $\mathbb{R}^7$. For any two points $x, y \in E$, the inner product (x, y) and the vector product $x \times y$ are defined by

$$\begin{aligned} -(x, y) &= \text{the real part of } xy, \\ x \times y &= \text{the imaginary part of } xy, \end{aligned}$$

where xy is the product of x and y in $\mathcal{C}$. The 6-dimensional unit sphere S^6 is defined by $S^6 = \{x \in E \mid (x,x) = 1\}$. We can identify the tangent space T_pS^6 $(p \in S^6)$ with the subspace $V_p = \{x \in E \mid (x,p) = 0\}$ of E. Define a linear endomorphism J_p of V_p such that $J_p^2 = -1$ by

$$J_p(x) = p \times x, \quad (x \in V_p,\ p \in S^6).$$

Then J_p $(p \in S^6)$ defines a non-integrable almost complex structure J on S^6 ([13]). Let g_0 denotes a canonical Riemannian metric of S^6 induced from $\mathbb{R}^7$. It is known that (J, g_0) is almost Hermitian structure satisfying

$$(\nabla_x J)y + (\nabla_y J)x = 0,$$

for $x, y \in V_p$ $(p \in S^6)$ ([14]). In this way, (S^6, J, g_0) is a nearly Kähler manifold.

Now, for any point $p \in S^6$, we define a map $\theta_p : S^6 \longrightarrow S^6$ by

$$\theta_p(x) = \frac{3}{2}(p,x)p - \frac{1}{2}x + \frac{\sqrt{3}}{2}p \times x,$$

for $x \in S^6$. Then we can see by straightforward computation that $\theta_p^3 = 1$ and p is an isolated fixed point of θ_p. This fact suggests the notion of Riemannian 3-symmetric spaces introduced by A. Gray [19] .

Let (M, g) be a Riemannian manifold. We now suppose that M admits a local isometry $\theta_p : U_p \longrightarrow U_p$ (U_p is a neighborhood p in M) for each $p \in M$ such that

(i) $\theta_p^3 = 1$,

(ii) p is an isolated fixed point of θ_p,

(iii) the tensor field Θ defined by $\Theta_p = (d\theta_p)_p$ is C^∞.

Then there is an almost complex structure J naturally associated with the family $\{\theta_p\}_{p \in M}$ and the Riemannian metric g is almost Hermitian with respect to J. The almost complex structure J is given by

$$\frac{\sqrt{3}}{2}J = \Theta + \frac{1}{2}I,$$

and called the *canonical almost complex structure* of $\{\theta_p\}_{p\in M}$.

A Riemannian manifold (M,g) is called a *Riemannian locally 3-symmetric space* if it admits a family of local isometries $\{\theta_p\}_{p\in M}$ of (M,g) satisfying the conditions (i), (ii), (iii) and furthermore

(iv) $d\theta_p \circ J = J \circ d\theta_p, \qquad$ for all $p \in M$,

where J is the canonical almost complex structure of $\{\theta_p\}_{p\in M}$.

In particular, a Riemannian locally 3-symmetric space (M,g) is called a *Riemannian 3-symmetric space* if each θ_p can be extended to a global holomorphic isometry of M.

A. Gray [19] showed that a Riemannian 3-symmetric space is characterized by a triple (G,K,σ) satisfying the following conditions (1), (2) and (3):

(1) G is a connected Lie group and σ is an automorphism of G of order 3,

(2) K is a closed subgroup of G such that

$$G_0^\sigma \subset K \subset G^\sigma,$$

where $G^\sigma = \{x \in G \mid \sigma(x) = x\}$ and G_0^σ denotes the identity component of G^σ.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K, respectively, and

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\sigma^2 + \sigma + I)X = 0\}.$$

Then we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad Ad(K)\mathfrak{m} = \mathfrak{m}. \tag{2.1}$$

(3) There exists a positive-definite inner product $<\ ,\ >$ on $\mathfrak{m}$ which is both $Ad(K)$-invariant and σ-invariant.

The inner product $<\ ,\ >$ in (3) induces a G-invariant Riemannian metric g on the homogeneous space $M = G/K$, and (M,g) is a Riemannian 3-symmetric space. The canonical almost complex structure J on (M,g) is given by

$$\frac{\sqrt{3}}{2}J = \sigma_{|\mathfrak{m}} + \frac{1}{2}I, \qquad \text{at the origin } eK \in G/K.$$

A. Gray also showed that the corresponding almost Hermitian manifold (M, J, g) is a quasi-Kähler manifold, and that (M, J, g) is a nearly Kähler manifold if and only if (M, g) is a naturally reductive Riemannian homogeneous space with respect to the decomposition (2.1). It is well known that the Riemannian connection and the curvature tensor of naturally reductive Riemannian homogeneous space are given at $eK \in G/K$, by

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}, \tag{2.2}$$

$$\begin{aligned} R(X,Y)Z &= -[[X,Y]_{\mathfrak{k}}, Z] - \frac{1}{2}[[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} \\ &\quad -\frac{1}{4}[[Y,Z]_{\mathfrak{m}}, X]_{\mathfrak{m}} - \frac{1}{4}[[Z,X]_{\mathfrak{m}}, Y]_{\mathfrak{m}}, \end{aligned} \tag{2.3}$$

for $X, Y, Z \in \mathfrak{m}$ ([23]). Moreover it is also known that

$$(\nabla_X J)Y = -J[X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}, \tag{2.4}$$

$$K(X,Y) = \frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2 + \|[X,Y]_{\mathfrak{k}}\|^2, \tag{2.5}$$

where $X, Y \in \mathfrak{m}$ with $\|X\| = \|Y\| = 1$ and $< X, Y >= 0$ ([23, 62]).

J. A. Wolf and A. Gray [62] have obtained the complete classification of indecomposable Riemannian 3-symmetric spaces. For example, the following homogeneous spaces possess nearly Kähler structures:

$SU(n+1)/S(U(s) \times U(r-s) \times U(n-r+1)), \quad (1 \leq s < r \leq n)$

$SO(2n+1)/SO(2n-2r+1) \times U(r), \quad (1 < r \leq n)$

$Sp(n)/Sp(n-r) \times U(r), \quad (1 \leq r \leq n-1)$

$SO(2n)/SO(2n-2r) \times U(r), \quad (n \geq 4,\ 2 \leq r \leq n-2)$

$SO(2n)/SO(2) \times U(n-1), \quad (n \geq 5)$

$G_2/SU(3) \simeq S^6,$

$Spin(8)/G_2 \simeq S^7 \times S^7,$

$L \times L \times L/diag(L \times L \times L),$

where L is a compact connected simple Lie group.

We don't know any examples of non-locally homogeneous non-Kähler nearly Kähler manifolds.

In connection with the homogeneous almost Hermitian manifolds, K. Sekigawa proved the following

Theorem 2.1 ([41]) *Let (M, J, g) be a homogeneous almost Hermitian manifold. Then there exists a tensor field T of type $(1,2)$ on M satisfying the following conditions:*

$$(\nabla_X R)(Y, Z) = [T(X), R(Y, Z)] - R(T(X)Y, Z) - R(Y, T(X)Z), \tag{2.6}$$

$$(\nabla_X T)Y = [T(X), T(Y)] - T(T(X)Y), \tag{2.7}$$

$$\nabla_X J = [T(X), J], \tag{2.8}$$

$$g(T(X)Y, Z) + g(Y, T(X)Z) = 0, \tag{2.9}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Conversely, if a connected, simply connected, complete almost Hermitian manifold (M, J, g) admits a tensor field T of type $(1,2)$ satisfying the conditions (2.6), (2.7), (2.8), and (2.9), then (M, J, g) is a homogeneous almost Hermitian manifold.

By virtue of the above theorem, we can consider that the tensor field T characterizes the homogeneous almost Hermitian manifold. So we shall call a tensor field T of type $(1,2)$ a *homogeneous structure* of (M, J, g) if it satisfies (2.6), (2.7), (2.8), and (2.9) in Theorem 2.1 (cf. [56]).

Now, let (M, J, g) be a nearly Kähler manifold. We define a tensor field $\widehat{T}$ of type (1,2) by

$$\widehat{T}(X)Y = \frac{1}{2}J(\nabla_X J)Y. \tag{2.10}$$

It has been shown that $\widehat{T}$ always satisfies the conditions (2.7), (2.8), and (2.9) for the homogeneous structure [42]. Hence we shall consider only the condition

(2.6). For this, we define a tensor field L of type (1,4) by

$$
\begin{aligned}
(2.11)\quad & L(X,Y,Z) \\
&= (\nabla_X R)(Y,Z) - [\widehat{T}(X), R(Y,Z)] + R(\widehat{T}(X)Y, Z) + R(Y, \widehat{T}(X)Z),
\end{aligned}
$$

Obviously, $L = 0$ means that the tensor field $\widehat{T}$ satisfies (2.6). In the case of $\dim M = 6$, K. Sekigawa [42] has calculated the square of the length of L.

Theorem 2.2 ([42]) *Let (M, J, g) be a 6-dimensional complete non-Kähler nearly Kähler manifold. Then we have*

$$
\int_M \|L\|^2 dM = \int_M \{\|\nabla R\|^2 - \frac{\tau}{15}(\|R\|^2 - \frac{\tau^2}{15})\} dM. \tag{2.12}
$$

It should be noted that the assumption of compactness of M is not necessary in the above theorem. In fact, the 6-dimensional complete non-Kähler nearly Kähler manifold M is an Einstein manifold with $\tau > 0$, it follows that M is compact by virtue of Myers' theorem ([23]).

The relationship between Riemannian 3-symmetric spaces and homogeneous nearly Kähler manifolds is interesting. The author [32] proved the following

Theorem 2.3 *Let (M, J, g) be a complete and connected nearly Kähler manifold. Then the tensor field $\widehat{T}$ defined by (2.10) is a homogeneous structure of (M, J, g) if and only if (M, g) is a Riemannian locally 3-symmetric space with canonical almost complex structure J.*

3 Spectrum of Laplacian

Let $M = (M, g)$ be a compact orientable Riemannian manifold. By Spec(M, g) we denote the set of eigenvalues of the Laplacian Δ which is called the *spectrum* of (M, g), that is,

$$
\mathrm{Spec}(M, g) = \{0 \geq \lambda_1 \geq \lambda_2 \geq \cdots\},
$$

where each eigenvalue is repeated as many times as its multiplicities. The spectrum $\mathrm{Spec}(M,g)$ exerts an influence on the geometry of (M,g). It is interesting to see how $\mathrm{Spec}(M,g)$ determine the structure of (M,g). For the study of this, M. Berger [5] and T. Sakai [30] used the coefficients of the asymptotic expansion of Minakshisundaram-Pleijel. As an analogue of the theorem of Sakai [30], we shall prove the following

Theorem 3.1 ([33]) *Let (M,J,g) and (M',J',g') be 6-dimensional complete, connected non-Kähler nearly Kähler manifolds. We assume that $\chi(M)=\chi(M')$ and $\mathrm{Spec}(M,g)=\mathrm{Spec}(M',g')$. Then (M,g) is Riemannian locally 3-symmetric if and only if (M',g') is Riemannian locally 3-symmetric.*

First of all, we prove the following lemma due to Sakai [30].

Lemma 3.2 *Let (M,g) be a compact orientable Einstein manifold of dimension 6. The Euler-Poincaré characteristic $\chi(M)$ is given by*

$$\chi(M)=\frac{1}{384\pi^3}\int_M\{-6\widehat{R}-2\|\nabla R\|^2+\frac{\tau^3}{9}-\frac{5\tau}{3}\|R\|^2\}dM, \tag{3.1}$$

where $\widehat{R}$ is a function on M defined by (1.23).

Proof. In a 6-dimensional compact orientable Riemannian manifold M, it is well known that $\chi(M)$ is given by

$$\chi(M)=\frac{1}{384\pi^3}\int_M\{\tau^3-12\tau\|\rho\|^2+3\tau\|R\|^2+16\rho^{ij}\rho_i{}^k\rho_{jk}+24\rho^{ij}\rho^{kl}R_{ikjl}$$
$$-24\rho^{ab}R_{ajkl}R_b{}^{jkl}-2\widehat{R}+8R^{ijkl}R_i{}^a{}_k{}^bR_{jbla}\}dM.$$

By using Bianchi identity repeatedly, we have

$$R^{ijkl}R_i{}^a{}_k{}^bR_{jbla}=\mathring{R}-\frac{1}{4}\widehat{R},$$

where $\mathring{R}$ is given by (1.24). Thus, in Einstein case, we have

$$\chi(M)=\frac{1}{384\pi^3}\int_M\{\,\frac{\tau^3}{9}-\tau\|R\|^2+8\,\mathring{R}-4\widehat{R}\,\}dM. \tag{3.2}$$

On the other hand, applying the Green's theorem to the Lichnerowicz's formula (1.25), we have

$$\int_M \mathring{R}\, dM = -\frac{1}{4}\int_M \{ \ \|\nabla R\|^2 + \frac{\tau}{3}\|R\|^2 + \widehat{R}\ \} dM. \tag{3.3}$$

Therefore (3.1) is obtained from (3.2) and (3.3). ∎

Proof of Theorem 3.1. We recall the asymptotic expansion of Minakshisundaram-Pleijel for $\mathrm{Spec}(M, g)$ given by

$$\sum_{k=0}^{\infty} \exp(\lambda_k t) \underset{t\to 0+}{\sim} \frac{1}{(4\pi t)^{m/2}}[\ a_0 + a_1 t + a_2 t^2 + \cdots\ ],$$

where $m = \dim M$. The coefficients a_0, a_1, a_2 and a_3 have been computed by Berger [5] and Sakai [30] :

$$a_0 = \mathrm{Vol}(M), \tag{3.4}$$

$$a_1 = \frac{1}{6}\int_M \tau\ dM, \tag{3.5}$$

$$a_2 = \frac{1}{360}\int_M \{5\tau^2 - 2\|\rho\|^2 + 2\|R\|^2\}\ dM, \tag{3.6}$$

$$\begin{aligned} a_3 \ = \ & \frac{1}{6!}\int_M \{-\frac{142}{63}\|\nabla\tau\|^2 - \frac{26}{63}\|\nabla\rho\|^2 - \frac{1}{9}\|\nabla R\|^2 + \frac{5}{9}\tau^3 \\ & -\frac{2}{3}\tau\|\rho\|^2 + \frac{2}{3}\tau\|R\|^2 - \frac{4}{7}\rho^{ij}\rho_j{}^k\rho_{ik} + \frac{20}{63}\rho^{ij}\rho^{kl}R_{ikjl} \\ & -\frac{8}{63}\rho^{ab}R_{ajkl}R_b{}^{jkl} - \frac{8}{21}\widehat{R}\}\ dM. \end{aligned} \tag{3.7}$$

It may be noticed that we mainly use $a_i = a'_i$ $(i = 0, 1, 2, 3)$, instead of $\mathrm{Spec}(M, g) = \mathrm{Spec}(M', g')$.

Since the 6-dimensional non-Kähler nearly Kähler manifold (M, J, g) is an Einsteinian, the coefficients a_i are rewritten

$$a_0 = \mathrm{Vol}(M), \tag{3.8}$$

$$a_1 = \frac{1}{6}\tau \mathrm{Vol}(M), \tag{3.9}$$

$$a_2 = \frac{7}{3\cdot 180}\tau^2 \mathrm{Vol}(M) + \frac{1}{180}\int_M \|R\|^2\ dM, \tag{3.10}$$

$$(3.11) \qquad a_3 = \frac{248}{7! \cdot 3^4}\tau^3 \mathrm{Vol}(M) + \frac{122}{7! \cdot 3^3}\tau \int_M \|R\|^2 \, dM$$
$$- \frac{1}{5! \cdot 3^2} \int_M \{\frac{4}{7}\widehat{R} + \frac{1}{6}\|\nabla R\|^2\} \, dM.$$

From these, $a_i = a_i'$ $(i = 0, 1, 2, 3)$ imply

$$(3.12) \qquad \mathrm{Vol}(M) = \mathrm{Vol}(M'),$$

$$(3.13) \qquad \tau = \tau',$$

$$(3.14) \qquad \int_M \|R\|^2 dM = \int_{M'} \|R'\|^2 dM',$$

$$(3.15) \qquad \int_M \{\frac{24}{7}\widehat{R} + \|\nabla R\|^2\} dM = \int_{M'} \{\frac{24}{7}\widehat{R'} + \|\nabla' R'\|^2\} dM'.$$

By Lemma 3.2, we have

$$384\pi^3 \chi(M) = -\frac{7}{4}\int_M \{\frac{24}{7}\widehat{R} + \|\nabla R\|^2 dM$$
$$-\frac{1}{4}\int_M \{\|\nabla R\|^2 - \frac{\tau}{15}(\|R\|^2 - \frac{\tau^2}{15})\} dM$$
$$-\frac{101}{60}\tau \int_M \|R\|^2 dM - \frac{101}{900}\tau^3 \mathrm{Vol}(M).$$

Taking account of (3.12)~(3.15) and Theorem 2.2, $\chi(M) = \chi(M')$ implies

$$(3.16) \qquad \int_M \|L\|^2 dM = \int_{M'} \|L'\|^2 dM'.$$

Theorem 3.1 now follows from Theorem 2.3. ∎

In the course of the proof, we obtained the following

Corollary 3.3 ([33]) *Let (M, J, g) and (M', J', g') be 6-dimensional complete, connected non-Kähler nearly Kähler manifolds. We assume that* $\mathrm{Spec}(M, g) = \mathrm{Spec}(M', g')$. *If*

$$\int_M \widehat{R} dM = \int_{M'} \widehat{R'} dM', \quad \text{or} \quad \int_M \mathring{R} \, dM = \int_{M'} \mathring{R'} \, dM'$$

is satisfied, then (M, g) is Riemannian locally 3-symmetric if and only if (M', g') is Riemannian locally 3-symmetric.

4 Nearly Kähler manifolds of constant holomorphic sectional curvature

Let $M = (M, J, g)$ be an m-dimensional nearly Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. It is well-known that a simply connected complete Kähler manifold of constant holomorphic sectional curvature c can be identified with the complex projective space $P^n(\mathbb{C})$, the open unit ball D^n in $\mathbb{C}^n$ or $\mathbb{C}^n$ according as $c > 0, c < 0$ or $c = 0$ (cf. [23]). Here, we shall deal only with the non-Kählerian case. Hence, it is sufficient to consider $\dim M = m \geq 6$.

By Theorem 3.10 of Chapter I, the curvature tensor of M is given by

$$(4.1) \quad R_{ijkl} = \frac{c(p)}{4} H_{ijkl} + \frac{1}{4}\{(\nabla_i J_l{}^r)\nabla_j J_{kr} - (\nabla_i J_k{}^r)\nabla_j J_{lr} - 2(\nabla_i J_j{}^r)\nabla_k J_{lr}\},$$

where

$$H_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}.$$

From (4.1), (1.7), we have

$$(4.2) \qquad \rho_{ij} + 3\rho^*_{ij} = (m+2)c(p)g_{ij},$$

$$(4.3) \qquad \tau + 3\tau^* = m(m+2)c(p),$$

(cf. Corollary 3.7 of Chapter I). Moreover, we know [60]

$$(4.4) \qquad \rho_{ij} = 5\rho^*_{ij}, \qquad \tau = 5\tau^*,$$

$$(4.5) \qquad \rho_{ij} = \frac{\tau}{m} g_{ij},$$

which shows that M is Einstein. Taking account of (4.3), (4.4) and (4.5), we obtain

$$(4.6) \qquad c(p) = \frac{8\tau}{5m(m+2)}.$$

It follows that $c = c(p)$ is positive constant ([50]). This is another proof of Schur's theorem for nearly Kähler manifolds (cf. Theorem 4.7 of Chapter I).

By (4.1), we have immediately the following

Proposition 4.1 ([40]) *Let M be a non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature. Then we have*

$$\|R\|^2 = \frac{6m+44}{25m(m+2)}\tau^2 = constant. \tag{4.7}$$

Proof. Taking account of (I.1.10) and (4.4), we have

$$H_{ijkl}R^{ijkl} = 2(\tau + 3\tau^*) = \frac{16}{5}\tau. \tag{4.8}$$

By (1.12), (1.13), (4.4) and (4.5), we have

$$\begin{aligned}&\{(\nabla_i J_l{}^r)\nabla_j J_{kr} - (\nabla_i J_k{}^r)\nabla_j J_{lr} - 2(\nabla_i J_j{}^r)\nabla_k J_{lr}\}R^{ijkl} \\ &= \frac{3}{2}\|\rho - \rho^*\|^2 = \frac{24}{25m}\tau^2.\end{aligned} \tag{4.9}$$

From (4.1), (4.8), (4.9) and (4.6), we get (4.7). ∎

Now, we prove the following

Theorem 4.2 ([51]) *There does not exist any dimensional, except 6-dimensional, non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature.*

Proof. By Proposition 4.1, (1.25) can be written as

$$\|\nabla R\|^2 + \frac{4(3m+22)}{25m^2(m+2)}\tau^3 + \widehat{R} + 4\,\mathring{R} = 0. \tag{4.10}$$

Now, we shall compute $\widehat{R}$ and $\mathring{R}$. By (1.23) and (4.1), we have

$$\begin{aligned}\widehat{R} &= R^{ijkl}R_{ij}{}^{ab}R_{abkl} \\ &= \frac{c}{2}R^{ijkl}R_{ij}{}^{ab}\{g_{al}g_{bk} + J_{al}J_{bk} - J_{ab}J_{kl}\} \\ &\quad + \frac{1}{2}R^{ijkl}R_{ij}{}^{ab}\{(\nabla_a J_l{}^r)\nabla_b J_{kr} - (\nabla_a J_b{}^r)\nabla_k J_{lr}\}.\end{aligned} \tag{4.11}$$

We shall compute the right hand side of (4.11) term by term. Taking account of (4.7), (1.10), (4.4) and (4.5), we have

$$R^{ijkl}R_{ij}{}^{ab}g_{al}g_{bk} = -\|R\|^2 = -\frac{6m+44}{25m(m+2)}\tau^2, \tag{a}$$

$$(b) \qquad R^{ijkl}R_{ij}{}^{ab}J_{al}J_{bk} = -\|R\|^2 + \frac{1}{2}\|\rho - \rho^*\|^2 = -\frac{2m-28}{25m(m+2)}\tau^2,$$

$$(c) \qquad R^{ijkl}R_{ij}{}^{ab}J_{ab}J_{kl} = 4\|\rho^*\|^2 = \frac{4}{25m}\tau^2.$$

Making use of (4.1) again, and taking account of $(\nabla^a J_l{}^r)(\nabla^b J_{kr})\nabla_a J_{bs} = 0$, we have

$$\begin{aligned} R^{ijkl}R_{ij}{}^{ab}(\nabla_a J_l{}^r)\nabla_b J_{kr} &= \frac{c}{2}R^{ijkl}(\nabla^a J_l{}^r)\nabla^b J_{kr}\{g_{ib}g_{ja} + J_{ib}J_{ja} - J_{ij}J_{ab}\} \\ &\quad + \frac{1}{2}R^{ijkl}(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}. \end{aligned}$$

Calculating term by term in the above equation, we have

$$(d1) \qquad R^{ijkl}(\nabla^a J_l{}^r)(\nabla^b J_{kr})g_{ib}g_{ja} = -\frac{1}{4}\|\rho - \rho^*\|^2 = -\frac{4}{25m}\tau^2,$$

$$(d2) \qquad R^{ijkl}(\nabla^a J_l{}^r)(\nabla^b J_{kr})J_{ib}J_{ja} = -\frac{1}{4}\|\rho - \rho^*\|^2 = -\frac{4}{25m}\tau^2,$$

by virtue of (1.13). By (1.11) and (1.2),

$$(d3) \qquad R^{ijkl}(\nabla^a J_l{}^r)(\nabla^b J_{kr})J_{ij}J_{ab} = 2J^{ls}\rho^{*k}{}_s(\nabla^a J_l{}^r)(\nabla^b J_{kr})J_{ab} = -\frac{8}{25m}\tau^2.$$

Further using (4.1), we have

$$\begin{aligned} &R^{ijkl}(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as} \\ &= \frac{c}{4}(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}\{g^{il}g^{jk} - g^{ik}g^{jl} + J^{il}J^{jk} - J^{ik}J^{jl} - 2J^{ij}J^{kl}\} \\ &\quad + \frac{1}{4}(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}\{(\nabla^i J^{lt})\nabla^j J^k{}_t - (\nabla^i J^{kt})\nabla^j J^l{}_t - 2(\nabla^i J^{jt})\nabla^k J^l{}_t\}. \end{aligned}$$

Calculating each term in the above equation, we have the following:

$$(d4.1) \quad (\nabla^a J_l{}^r)(\nabla^b J_{kr})(\nabla_i J_b{}^s)(\nabla_j J_{as})g^{il}g^{jk} = (\nabla^a J^{ir})\nabla^b J^j{}_r(\nabla_i J_b{}^s)\nabla_j J_{as} = 0,$$

$$(d4.2) \qquad (\nabla^a J_l{}^r)(\nabla^b J_{kr})(\nabla_i J_b{}^s)(\nabla_j J_{as})g^{ik}g^{jl} = \|\rho - \rho^*\|^2 = \frac{16}{25m}\tau^2,$$

$$(d4.3) \qquad (\nabla^a J_l{}^r)(\nabla^b J_{kr})(\nabla_i J_b{}^s)(\nabla_j J_{as})J^{il}J^{jk} = 0,$$

$$(d4.4)\qquad (\nabla^a J_l{}^r)(\nabla^b J_{kr})(\nabla_i J_b{}^s)(\nabla_j J_{as}) J^{ik} J^{jl} = \|\rho - \rho^*\|^2 = \frac{16}{25m}\tau^2,$$

$$(d4.5)\qquad (\nabla^a J_l{}^r)(\nabla^b J_{kr})(\nabla_i J_b{}^s)(\nabla_j J_{as}) J^{ij} J^{kl} = \|\rho - \rho^*\|^2 = \frac{16}{25m}\tau^2.$$

Since $(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_j J_{as})\nabla^j J^k{}_t$ is symmetric with respect to indices l, t and $\nabla^j J^{lt}$ is skew-symmetric with l, t,

$$(d4.6)\qquad (\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}(\nabla^i J^{lt})\nabla^j J^k{}_t = 0.$$

By virtue of (1.2),

$$(d4.7)\qquad (\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}(\nabla^i J^{kt})\nabla^j J^l{}_t = 0,$$

$$(d4.8)\qquad (\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as}(\nabla^i J^{jt})\nabla^k J^l{}_t = 0.$$

From $(d4.1) \sim (d4.8)$, we have

$$(d4)\qquad R^{ijkl}(\nabla^a J_l{}^r)\nabla^b J_{kr}(\nabla_i J_b{}^s)\nabla_j J_{as} = -\frac{128}{125m^2(m+2)}\tau^3.$$

Moreover, from $(d1) \sim (d4)$, we have

$$(d)\qquad R^{ijkl}R_{ij}{}^{ab}(\nabla_a J_l{}^r)\nabla_b J_{kr} = -\frac{64}{125m^2(m+2)}\tau^3.$$

By making use of (1.11), we have

$$(e)\qquad R^{ijkl}R_{ij}{}^{ab}(\nabla_a J_b{}^r)\nabla_k J_{lr} = \frac{16}{125m^2}\tau^3.$$

Consequently, substituting $(a) \sim (e)$ into (4.11), we obtain

$$(4.12)\qquad \widehat{R} = -\frac{8m^2 + 96m + 416}{125m^2(m+2)^2}\tau^3.$$

Similarly, we obtain

$$(4.13)\qquad \mathring{R} = -\frac{m^2 + 164m + 260}{125m^2(m+2)^2}\tau^3.$$

Substituting (4.12), (4.13) into (4.10), we finally obtain

$$(4.14)\qquad \|\nabla R\|^2 + \frac{48(m-6)}{125m^2(m+2)}\tau^3 = 0.$$

Taking account of $\tau > 0$ and $m \geq 6$, we can conclude that $m = 6$ and $\nabla R = 0$. ∎

By the above theorem, we shall consider the case of $\dim M = 6$. Then, by Proposition 4.1, we have

$$\|R\|^2 = \frac{\tau^2}{15}, \tag{4.15}$$

which is equivalent to

$$\|Z\|^2 = 0, \tag{4.16}$$

where

$$Z_{ijkl} = R_{ijkl} - \frac{\tau}{30}(g_{il}g_{jk} - g_{ik}g_{jl}).$$

Thus, we have the following

Theorem 4.3 ([52, 40]) *Let M be a 6-dimensional non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is a space of constant curvature.*

By $S^6(c)$ we denote a 6-dimensional sphere of constant curvature $c > 0$, which has the natural nearly Kähler structure. Summing up the above theorems, we have the following

Theorem 4.4 *Let M be a non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is locally isometric to a 6-dimensional sphere $S^6(c)$.*

Let M be a *complete*, non-Kähler nearly Kähler manifold of constant holomorphic sectional curvature. Then M is compact, as we remarked in §2 (p.30). Furthermore we see that M is simply connected by the following Proposition 4.5 due to A. Gray.

Proposition 4.5 ([18]) *Let M be a compact nearly Kähler manifold of positive holomorphic sectional curvature. Then M is simply connected.*

Combining above results, we obtain the following

Theorem 4.6 *Let M be a complete non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is isometric to a 6-dimensional sphere $S^6(c)$.*

5 Positively curved nearly Kähler manifolds

In [18], A. Gray studied the structure of positively curved compact nearly Kähler manifolds and proposed the following conjecture:

> *Let $M = (M, J, g)$ be a compact nearly Kähler manifold with positive sectional curvature. If the scalar curvature of M is constant, then M is isometric to a complex projective space $P^n(\mathbb{C})$ with a Kähler metric of constant holomorphic sectional curvature or a 6-dimensional sphere S^6 with a Riemannian metric of constant sectional curvature.*

For Kähler manifolds, this conjecture is positive (cf. [9, 21], etc.). However, for non-Kähler nearly Kähler manifolds, this conjecture is negative. Namely, we shall give the following counter example to this conjecture ([46]).

We consider the 6-dimensional compact Riemannian 3-symmetric space $(Sp(2)/Sp(1) \times U(1), g)$ appeared in §2 (p.28). We put $G = Sp(2)$ and $K = Sp(1) \times U(1)$. Let $\mathbb{H}$ be the algebra of quaternions, that is, any element q in $\mathbb{H}$ is of the form $q = a_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$ ($a_i \in \mathbb{R}$), where

$$\boldsymbol{e}_i{}^2 = -1 \quad (i = 1, 2, 3),$$

$$\boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_3, \quad \boldsymbol{e}_2\boldsymbol{e}_3 = -\boldsymbol{e}_3\boldsymbol{e}_2 = \boldsymbol{e}_1, \quad \boldsymbol{e}_3\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_3 = \boldsymbol{e}_2.$$

Then it is well known that the Lie algebra $\mathfrak{sp}(2)$ of $Sp(2)$ is given by

$$\mathfrak{sp}(2) = \left\{ x \in \mathfrak{gl}(2, \mathbb{H}) \mid {}^t x = -\bar{x} \right\},$$

and the Riemannian metric g on G/K is induced from the inner product

$$< x, y >= -\text{the real part of } (\text{trace } xy), \quad x, y \in \mathfrak{sp}(2).$$

We put

$$\begin{aligned}
(5.1) \qquad x_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & y_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \boldsymbol{e}_1 \\ \boldsymbol{e}_1 & 0 \end{bmatrix}, \\
x_2 &= \begin{bmatrix} -\boldsymbol{e}_2 & 0 \\ 0 & 0 \end{bmatrix}, & y_2 &= \begin{bmatrix} \boldsymbol{e}_3 & 0 \\ 0 & 0 \end{bmatrix}, \\
x_3 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \boldsymbol{e}_2 \\ \boldsymbol{e}_2 & 0 \end{bmatrix}, & y_3 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \boldsymbol{e}_3 \\ \boldsymbol{e}_3 & 0 \end{bmatrix}, \\
s_1 &= \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{e}_2 \end{bmatrix}, & s_2 &= \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{e}_3 \end{bmatrix}, \\
t_1 &= \begin{bmatrix} \boldsymbol{e}_1 & 0 \\ 0 & 0 \end{bmatrix}, & t_2 &= \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{e}_1 \end{bmatrix}.
\end{aligned}$$

Then we see that $\{x_1, x_2, x_3, y_1, y_2, y_3, s_1, s_2, t_1, t_2\}$ is an orthonormal basis of $\mathfrak{g} = \mathfrak{sp}(2)$, the Lie algebra $\mathfrak{k}$ of K is linear spanned by $\{s_1, s_2, t_1, t_2\}$ and the subspace $\mathfrak{m}$ of $\mathfrak{g}$ in the decomposition (2.1) is spanned by $\{x_1, x_2, x_3, y_1, y_2, y_3, \}$ over $\mathbb{R}$ (cf. [47]).

The canonical almost complex structure J of G/K is given by

$$(5.2) \qquad Jx_i = y_i, \quad Jy_i = -x_i \qquad (i = 1, 2, 3).$$

By (2.4), we have

$$(5.3) \qquad \|(\nabla_x J)y\|^2 = 1,$$

for $x, y \in \mathfrak{m}$ with $\|x\| = \|y\| = 1, < x, y >=< x, Jy >= 0$. By (5.3), we see that $(Sp(2)/Sp(1) \times U(1), J, g)$ is non-Kählerian.

By (2.5), (5.1) and (5.2), by direct computation, we get

$$\begin{aligned}
(5.4) \qquad H(x) &= \|[x, Jx]_{\mathfrak{k}}\|^2 \\
&= 10\left({a_1}^2 + {b_1}^2 + {a_3}^2 + {b_3}^2 - \frac{3}{5}\right)^2 + \frac{2}{5},
\end{aligned}$$

for any unit vector $x = \sum_{i=1}^{3}(a_i x_i + b_i y_i) \in \mathfrak{m}$. By (5.4), we have easily

$$(5.5) \qquad \frac{2}{5} \le H(x) \le 4.$$

Let x be any unit vector in $\mathfrak{m}$ and y any unit vector in $\mathfrak{m}$ which is orthogonal to x. Then we may put

$$y = aJx + bz, \tag{5.6}$$

where z is a unit vector in $\mathfrak{m}$ with $< x, z >= 0, < Jx, z >= 0$ and $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$. By (2.5), taking account of (2.4), (5.3) and (5.6), we have

$$\begin{aligned} K(x, y) &= \frac{1}{4}\|b[x, z]_{\mathfrak{m}}\|^2 + \|a[x, Jx]_{\mathfrak{k}} + b[x, z]_{\mathfrak{k}}\|^2 \\ &= \frac{b^2}{4} + \|a[x, Jx]_{\mathfrak{k}} + b[x, z]_{\mathfrak{k}}\|^2. \end{aligned} \tag{5.7}$$

Therefore, by (5.4), (5.5) and (5.7), we may easily see that the nearly Kähler manifold $(Sp(2)/Sp(1) \times U(1), J, g)$ has strictly positive sectional curvature.

We remark that the space $Sp(2)/Sp(1)\times U(1)$ is diffeomorphic to $P^3(\mathbb{C})$ ([47]).

It is therefore important to study the pinching problem on the holomorphic sectional curvature of a compact non-Kähler nearly Kähler manifold.

A nearly Kähler manifold M is said to be of *holomorphically δ-pinched* ($0 \leq \delta \leq 1$) if there exists a positive constant l such that

$$\delta l \leq H(x) \leq l. \tag{5.8}$$

for all non-zero $x \in T_pM$, for all $p \in M$. Since we are dealing with nearly Kähler manifolds, the size $\|(\nabla_x J)y\|^2$ will be important in the pinching estimates. A nearly Kähler manifold M is said to satisfy the condition $T(\lambda, \mu)$ if

$$\lambda H(x) \leq \|(\nabla_x J)y\|^2 \leq \mu H(x), \tag{5.9}$$

for $x, y \in T_pM$ with $\|x\| = \|y\| = 1, g(x, y) = g(x, Jy) = 0$ for all $p \in M$ ([18]).

By (5.3) and (5.5), we see that the preceding nearly Kähler manifold $(Sp(2)/Sp(1) \times U(1), J, g)$ is holomorphically $\frac{1}{10}$-pinched and satisfies the condition $T(\frac{1}{4}, \frac{5}{2})$.

From now on we shall establish an integral formula on the unit tangent bundle $U(M)$ over a compact Einstein nearly Kähler manifold M.

Let $M = (M, J, g)$ be an m-dimensional nearly Kähler manifold. We denote by $T(M)$ and $U(M)$ the tangent bundle and the unit tangent bundle over M, respectively. For each point $p \in M$, we put

$$U_p = U_p(M) = \{x \in T_pM \mid \|x\| = 1\}.$$

Then U_p is isomorphic to S^{m-1}. We now recall the Sasaki metric g^s on $T(M)$ ([31]). Let X^h and X^v be the horizontal lift and the vertical lift of $X \in \mathfrak{X}(M)$, respectively. Then the Sasaki metric g^s on $T(M)$ is defined by

$$g^s(X^h, Y^h) = g(X,Y), \quad g^s(X^v, Y^v) = g(X,Y), \quad g^s(X^h, Y^v) = 0, \tag{5.10}$$

for $X, Y \in \mathfrak{X}(M)$. From (5.10), we get easily

$$(\nabla^s_{X^h} Y^h)_{(p,x)} = (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)x)^v, \tag{5.11}$$

where ∇^s denotes the Riemannian connection on $T(M)$ with respect to g^s.

We denote by using the same notation g^s the induced metric on $U(M)$ which is induced from the Sasaki metric g^s on $T(M)$. By $\sigma, \sigma_1 (= dM)$ and σ_2, we denote the volume element of $(U(M), g^s)$, (M, g) and (S^{m-1}, g_0) $(\cong U_p)$, respectively. Then we have easily

$$\sigma(p,x) = \sigma_1(p) \wedge \sigma_2(x), \qquad (p,x) \in U(M). \tag{5.12}$$

If M is compact, for any smooth function φ on $U(M)$, we have

$$\int_{U(M)} \varphi\sigma = \int_M \left\{ \int_{U_p} \varphi(p,x)\sigma_2 \right\} \sigma_1. \tag{5.13}$$

Let (p, x) be any point of $U(M)$. We choose an orthonormal basis $\{e_i\} = \{e_1, \cdots, e_m\}$ of T_pM such that $x = e_1$. Then $\left\{e_1^h, \cdots, e_m^h, e_2^v, \cdots, e_m^v\right\}$ is an orthonormal basis of the tangent space $T_{(p,x)}(U(M))$. For each $y \in T_pM$, the tangent space $T_y(T_pM)$ (i.e., the vertical subspace of $T_{(p,y)}(T(M))$) is identified with T_pM by means of parallel translation. Under this identification, e_i^v corresponds to e_i $(i = 1, \cdots, m)$. We denote by $(u_1, \cdots, u_m, v_2, \cdots, v_m)$ the normal coordinate system on a neighborhood of (p,x) in $U(M)$ with respect to the orthonormal basis $\left\{e_1^h, \cdots, e_m^h, e_2^v, \cdots, e_m^v\right\}$. In [21], A. Gray has introduced a second order linear differential operator $\boldsymbol{L}$ by

$$\boldsymbol{L}_{(p,x)} = \left\{\sum_{i=1}^{m} \frac{\partial^2}{\partial {u_i}^2} + \frac{1}{2}\sum_{i,j=2}^{m} h_{ij}\frac{\partial^2}{\partial v_i \partial v_j}\right\}_{(p,x)}, \tag{5.14}$$

where $h_{ij}(p,x) = -R(e_i, x, e_j, x)$.

For a smooth function φ on $U(M)$, we denote by $\mathrm{grad}^h\varphi$ (resp. $\mathrm{grad}^v\varphi$) the horizontal (resp. vertical) component of $\mathrm{grad}\,\varphi$. We can regard the holomorphic sectional curvature $H = H(x)$ as a smooth function on $U(M)$. Then we have

$$(\mathrm{grad}^h H)_{(p,x)} = -\sum_{i=1}^{m}\{(\nabla_{e_i}R)(x, Jx, x, Jx) + 2R(x, Jx, x, (\nabla_{e_i}J)x)\}\, e_i^h, \tag{5.15}$$

$$(\mathrm{grad}^v H)_{(p,x)} = -4\sum_{i=2}^{m} R(x, Jx, x, Je_i)\, e_i^v. \tag{5.16}$$

By (5.16), we see that

$$g^s((\mathrm{grad}^v H)_{(p,x)}, x^v) = g^s((\mathrm{grad}^v H)_{(p,x)}, (Jx)^v) = 0.$$

From the result due to Tanno[52] and (5.16), we may note the following

Proposition 5.1 *Let $M = (M, J, g)$ be a nearly Kähler manifold. Then M is a space of constant holomorphic sectional curvature if and only if* $\mathrm{grad}^v H = 0$ *on $U(M)$.*

Now, we assume that $M = (M, J, g)$ is a connected compact Einstein nearly Kähler manifold. We shall evaluate the value $\boldsymbol{L}(H)$ at any point $(p,x) \in U(M)$. By (5.15), we have

$$\begin{aligned}\sum_{i=1}^{m}\frac{\partial^2 H}{\partial {u_i}^2}(p,x) &= \sum_{i=1}^{m} g^s(\nabla^s_{e_i^h}(\mathrm{grad}^h H), e_i^h) \\ &= -\sum_{i=1}^{m}\{(\nabla_{e_i}\nabla_{e_i}R)(x, Jx, x, Jx) + 4(\nabla_{e_i}R)(x, Jx, x, (\nabla_{e_i}J)x) \\ &\quad +2R(x, (\nabla_{e_i}J)x, x, (\nabla_{e_i}J)x) + 2R(x, Jx, x, (\nabla_{e_i}\nabla_{e_i}J)x)\}.\end{aligned} \tag{5.17}$$

Taking account of the first Bianchi, the second Bianchi, the Ricci identities and (1.6), after long complicated computations, we get

$$\begin{aligned}
(5.18)\qquad & \sum_{i=1}^{m}(\nabla_{e_i}\nabla_{e_i}R)(x,Jx,x,Jx) \\
&= 2\sum_{i,j=1}^{m} R(e_i,x,e_j,x)\{H(x)\delta_{ij}+R(e_i,x,e_j,x) \\
&\qquad +3R(e_i,Jx,e_j,Jx)+3g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x)\} \\
&\quad +6\sum_{i,j=1}^{m} R(e_i,x,Je_j,Jx)g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x).
\end{aligned}$$

Thus, by (5.17) and (5.18), we have

$$\begin{aligned}
(5.19)\qquad & \sum_{i=1}^{m}\frac{\partial^2 H}{\partial {u_i}^2}(p,x) \\
&= 2\sum_{i,j=2}^{m} h_{ij}(p,x)\{H(x)\delta_{ij}+R(e_i,x,e_j,x) \\
&\qquad +3R(e_i,Jx,e_j,Jx)+3g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x)\} \\
&\quad -6\sum_{i,j=1}^{m} R(e_i,x,Je_j,Jx)g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x) \\
&\quad -\sum_{i=1}^{m}\{4(\nabla_{e_i}R)(x,Jx,x,(\nabla_{e_i}J)x)+2R(x,(\nabla_{e_i}J)x,x,(\nabla_{e_i}J)x) \\
&\qquad +2R(x,Jx,x,(\nabla_{e_i}\nabla_{e_i}J)x)\}.
\end{aligned}$$

By similar computation as in [21], we have

$$\begin{aligned}
(5.20)\qquad & \frac{\partial^2 H}{\partial v_i \partial v_j}(p,x) \\
&= -4\left\{H(x)\delta_{ij}+R(e_i,x,e_j,x)+3R(e_i,Jx,e_j,Jx)+3g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x)\right\}.
\end{aligned}$$

Now, we define smooth functions φ_α $(\alpha=1,2,3,4)$ on $U(M)$ by

$$\begin{aligned}
(5.21)\qquad \varphi_1(p,x) &= -\sum_{i,j=1}^{m} R(e_i,x,Je_j,Jx)g((\nabla_{e_i}J)x,(\nabla_{e_j}J)x), \\
\varphi_2(p,x) &= -\sum_{i=1}^{m}(\nabla_{e_i}R)(x,Jx,x,(\nabla_{e_i}J)x), \\
\varphi_3(p,x) &= -\sum_{i=1}^{m} R(x,(\nabla_{e_i}J)x,x,(\nabla_{e_i}J)x),
\end{aligned}$$

$$\begin{aligned}\varphi_4(p,x) &= -\sum_{i=1}^{m} R(x,Jx,x,(\nabla_{e_i}\nabla_{e_i}J)x)\\ &= R(x,Jx,x,(\rho^1-\rho^{*1})Jx).\end{aligned}$$

From (5.14), (5.19), (5.20) and (5.21),we have

$$(5.22) \qquad \boldsymbol{L}(H)(p,x) = 6\varphi_1(p,x) + 4\varphi_2(p,x) + 2\varphi_3(p,x) + 2\varphi_4(p,x),$$

for all $(p,x) \in U(M)$. Since M is an Einstein manifold, it follows that the operator $\boldsymbol{L}$ is self-adjoint ([21]). Thus we have the following equality:

$$\begin{aligned}(5.23)\quad 0 &= \int_{U(M)} \boldsymbol{L}(H^2)\,\sigma\\ &= \int_{U(M)} \left\{2H\boldsymbol{L}(H) + 2\|\mathrm{grad}^h H\|^2 - R(x,\mathrm{grad}^v H, x, \mathrm{grad}^v H)\right\}\,\sigma.\end{aligned}$$

We shall evaluate the integral $\int_{U(M)} \|\mathrm{grad}^h H\|^2\,\sigma$. We define smooth functions ψ_β $(\beta = 1,2,3)$ on $U(M)$ by

$$\begin{aligned}(5.24)\qquad \psi_1(p,x) &= \sum_{i=1}^{m}((\nabla_{e_i}R)(x,Jx,x,Jx))^2,\\ \psi_2(p,x) &= \sum_{i=1}^{m}(\nabla_{e_i}R)(x,Jx,x,Jx)R(x,Jx,x,(\nabla_{e_i}J)x),\\ \psi_3(p,x) &= \sum_{i=1}^{m} R(x,Jx,x,(\nabla_{e_i}J)x)^2.\end{aligned}$$

Then, by (5.15), we have

$$(5.25)\qquad \int_{U(M)} \|\mathrm{grad}^h H\|^2\,\sigma = \int_{U(M)}(\psi_1 + 4\psi_2 + 4\psi_3)\,\sigma.$$

Taking account of (5.13) and Green's theorem, we have

$$(5.26)\qquad \int_{U(M)} \psi_2\,\sigma = -2\int_{U(M)} \psi_3\,\sigma - \int_{U(M)} H(\varphi_2 + \varphi_3 + \varphi_4)\,\sigma.$$

By (5.22), (5.23), (5.25) and (5.26), we finally obtain

$$\begin{aligned}(5.27)\int_{U(M)} &\left\{2\psi_1 - 8\psi_3 + H(12\varphi_1 - 4\varphi_3 - 4\varphi_4) + K(x,\mathrm{grad}^v H)\|\mathrm{grad}^v H\|^2\right\}\sigma\\ &= 0.\end{aligned}$$

The integral formula (5.27) together with (5.21) and (5.24) will play an important role in our arguments.

In order to estimate the integration on U_p, we need the following

Proposition 5.2 ([6]) *Let f be a homogeneous polynomial of degree $r (r \geq 1)$ defined on $\mathbb{R}^m$. Then we have*

$$\int_{S^{m-1}} (\Delta^{\mathbb{R}^m} f)\ \sigma_2 = r(m+r-2) \int_{S^{m-1}} (f_{|S^{m-1}})\ \sigma_2,$$

where $\Delta^{\mathbb{R}^m}$ denotes the Laplace operator of $\mathbb{R}^m$.

Let $M = (M, J, g)$ be an m-dimensional connected Einstein non-Kähler nearly Kähler manifold with vanishing first Chern form. For a fixed $p \in M$, $f(x) = -R(x, Jx, x, Jx)$ $(x \in T_pM)$ can be considered as a homogeneous polynomial of degree 4 defined on $T_pM (\cong \mathbb{R}^m)$ and $f_{|U_p} = H$. As an application of Proposition 5.2, we have the followings:

$$\int_{U_p} H\ \sigma_2 = \frac{8\tau}{5m(m+2)} \mathrm{Vol}(S^{m-1}), \tag{5.28}$$

$$\int_{U_p} H^2\ \sigma_2 = \frac{1}{4(m+2)} \int_{U_p} \|\mathrm{grad}^v H\|^2\ \sigma_2 + \frac{64\tau^2}{25m^2(m+2)^2} \mathrm{Vol}(S^{m-1}). \tag{5.29}$$

By (5.16) and (5.29), we have

$$\begin{aligned} &\int_{U_p} \sum_{k=1}^{m} R(x, Jx, x, e_k)^2\ \sigma_2 \\ &= \frac{1}{16} \int_{U_p} \|\mathrm{grad}^v H\|^2\ \sigma_2 + \int_{U_p} H^2\ \sigma_2 \\ &= \frac{m+6}{16(m+2)} \int_{U_p} \|\mathrm{grad}^v H\|^2\ \sigma_2 + \frac{64\tau^2}{25m^2(m+2)^2} \mathrm{Vol}(S^{m-1}). \end{aligned} \tag{5.30}$$

If M is holomorphically δ-pinched, by (5.28), we have

$$\delta l \leq \frac{8\tau}{5m(m+2)} \leq l. \tag{5.31}$$

We are now in position to prove the following

Theorem 5.3 *Let $M = (M, J, g)$ be a 6-dimensional connected complete non-Kähler nearly Kähler manifold satisfying the condition*

$$K(x, y) > \frac{\tau}{120},$$

for linearly independent $x, y \in T_pM$, for all $p \in M$. Then M is isometric to a 6-dimensional sphere $S^6(c)$ of constant curvature $c = \tau/30$.

Proof. First of all, we shall evaluate the values $\varphi_\alpha(\alpha = 1, 3, 4)$. By (1.19) and (5.21), we have

$$\varphi_1(p, x) = \frac{\tau}{30}\left(\frac{\tau}{30} - H(x)\right), \tag{5.32}$$

$$\varphi_3(p, x) = \frac{\tau}{30}\left(\frac{\tau}{6} - H(x)\right), \tag{5.33}$$

Since M is an Einstein manifold with $\rho = 5\rho^*$, by (5.21), we have

$$\varphi_4(p, x) = -\frac{2\tau}{15}H(x). \tag{5.34}$$

By (5.24), (5.30) and (1.19), we have

$$\int_{U_p} \psi_3 \, \sigma_2 = \frac{\tau}{480}\int_{U_p} \|\mathrm{grad}^v H\|^2 \, \sigma_2. \tag{5.35}$$

Since M is compact as we have already remarked, by (5.28), (5.29), (5.32), (5.33), (5.34) and (5.35), the integral formula (5.27) can be written as

$$\int_{U(M)} \left\{2\psi_1 + \left(K(x, \mathrm{grad}^v H) - \frac{\tau}{120}\right) \|\mathrm{grad}^v H\|^2\right\} \sigma = 0. \tag{5.36}$$

Considering that $\psi_1 \geq 0$, the theorem follows immediately from (5.36), Proposition 5.1 and Theorem 4.6. ∎

Furthermore, we have the following

Theorem 5.4 *Let $M = (M, J, g)$ be a 6-dimensional connected complete non-Kähler nearly Kähler manifold. If M is holomorphically $\delta(> 2/5)$-pinched, then M is isometric to a 6-dimensional sphere $S^6(c)$ of constant curvature $c = \tau/30$.*

Proof. By the hypothesis and (1.17), (1.19), (5.8) and (5.31), we have

$$\begin{aligned} K(x, y) &\geq \frac{1}{4}(3\delta - 2)l + \frac{\tau}{40} \\ &> -\frac{1}{5}l + \frac{\tau}{40} \\ &> -\frac{\tau}{60} + \frac{\tau}{40} = \frac{\tau}{120}, \end{aligned}$$

for $x, y \in T_pM$ with $\|x\| = \|y\| = 1$, $g(x, y) = g(x, Jy) = 0$ for all $p \in M$. Thus the theorem follows immediately from Theorem 5.3. ∎

Next, we shall deal with general cases where the dimension of M is not necessarily equal to 6. In connection with the results obtained by R. L. Bishop and S. I. Goldberg ([7, 8, 9]), we have the following

Theorem 5.5 *Let $M = (M, J, g)$ be an $m(= 2n)$-dimensional connected compact non-Kähler nearly Kähler manifold with constant scalar curvature. If M satisfies the condition*

$$K(x,y) + K(x,Jy) + B(x,y) > 0, \tag{5.37}$$

for $x, y \in T_pM$ with $x \neq 0, y \neq 0, g(x,y) = g(x,Jy) = 0$ for all $p \in M$, then the Ricci tensor ρ of M is parallel and the first Chern form γ of M vanishes.

Proof. Since M is compact and the scalar curvature of M is constant, by the result due to S. Tachibana [48], the first Chern form γ is a harmonic 2-form.

For each point $p \in M$, we may choose an orthonormal basis $\{e_i\} = \{e_\alpha, Je_\alpha\}$ $(i = 1, \cdots, m;\ \alpha = 1, \cdots, n)$ which diagonalizes the symmetric linear endomorphism $\rho^1 - 5\rho^{*1}$ of T_pM. By the choice of $\{e_i\}$, we have

$$\gamma_{ij} := \gamma(e_i, e_j) = 0 \quad \text{for} \quad e_j \neq \pm Je_i. \tag{5.38}$$

For the 2-form γ, we put

$$F(\gamma) = \sum_{i,j,k=1}^{m} \rho_{ij}\gamma_{ik}\gamma_{jk} + \frac{1}{2}\sum_{i,j,k,l=1}^{m} R_{ijkl}\gamma_{ij}\gamma_{kl}. \tag{5.39}$$

By (5.38), (5.39) reduce to

$$F(\gamma) = \sum_{\alpha,\beta=1}^{n} \left\{-(R_{\alpha\beta\alpha\beta} + R_{\alpha\bar\beta\alpha\bar\beta})(\gamma_{\alpha\bar\alpha})^2 + R_{\alpha\bar\alpha\beta\bar\beta}\gamma_{\alpha\bar\alpha}\gamma_{\beta\bar\beta}\right\}, \tag{5.40}$$

where we put $R_{\alpha\bar\beta\alpha\bar\beta} = R(e_\alpha, Je_\beta, e_\alpha, Je_\beta)$, $\gamma_{\alpha\bar\alpha} = \gamma(e_\alpha, Je_\alpha), \cdots$, etc. By (1.18) and (5.40), we have

$$F(\gamma) = 2\sum_{\alpha<\beta} \left\{-R_{\alpha\bar\alpha\beta\bar\beta}(\gamma_{\alpha\bar\alpha} - \gamma_{\beta\bar\beta})^2 + 2\|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha}^2 + \gamma_{\beta\bar\beta}^2)\right\}, \tag{5.41}$$

or

$$(5.42)\quad F(\gamma) = 2\sum_{\alpha<\beta}\left\{-(R_{\alpha\beta\alpha\beta} + R_{\alpha\bar\beta\alpha\bar\beta})(\gamma_{\alpha\bar\alpha} - \gamma_{\beta\bar\beta})^2 + 4\|(\nabla_{e_\alpha}J)e_\beta\|^2\gamma_{\alpha\bar\alpha}\gamma_{\beta\bar\beta}\right\}.$$

By (5.41) and (5.42), we have finally

$$(5.43)\qquad F(\gamma) = \sum_{\alpha<\beta}\left\{-(R_{\alpha\beta\alpha\beta} + R_{\alpha\bar\beta\alpha\bar\beta} + R_{\alpha\bar\alpha\beta\bar\beta})(\gamma_{\alpha\bar\alpha} - \gamma_{\beta\bar\beta})^2 \right.$$
$$\left. +2\|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha} + \gamma_{\beta\bar\beta})^2\right\}.$$

Since γ is harmonic and $F(\gamma) \geq 0$, according to Yano-Bochner[65], it follows that $F(\gamma) = 0$ and γ is parallel. Thus, by (5.43), we get

$$(5.44)\qquad \gamma_{\alpha\bar\alpha} - \gamma_{\beta\bar\beta} = 0, \quad \text{and} \quad \|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha} + \gamma_{\beta\bar\beta})^2 = 0,$$

for $1 \leq \alpha < \beta \leq n$. Since M is non-Kählerian, it follows that

$$(5.45)\qquad (\nabla_{e_\alpha}J)e_\beta \neq 0, \quad \text{for some} \ \ \alpha < \beta.$$

By (5.44) and (5.45), we have

$$(5.46)\qquad \gamma = 0, \quad (\text{i.e.,} \ \ \rho = 5\rho^*).$$

Therefore, by (1.14) and (5.46), we have

$$\nabla(\rho - \rho^*) = 0,$$

and hence

$$\nabla\rho = 0. \qquad ∎$$

Furthermore, we have the following

Theorem 5.6 *Let $M = (M, J, g)$ be an m-dimensional connected compact non-Kähler nearly Kähler manifold with constant scalar curvature. If M satisfies the condition $T(\lambda, \mu)$ $(\lambda > 0)$, and is holomorphically $\delta(> \frac{2}{\lambda+3})$ -pinched, then M is an Einstein manifold and the first Chern form γ of M vanishes.*

Proof. By the hypothesis and (1.17), (5.8), (5.9), we have

$$K(x,y) \geq \frac{1}{4}(3\delta - 2 + 3\lambda\delta)l > \frac{\lambda}{\lambda+3}l > 0, \tag{5.47}$$

for $x, y \in T_pM$ with $\|x\| = \|y\| = 1, g(x,y) = g(x,Jy) = 0$ for all $p \in M$. Thus, by (1.18), (5.9) and (5.47), we have

$$\begin{aligned} & K(x,y) + K(x,Jy) + B(x,y) \\ &= 2\{K(x,y) + K(x,Jy) - \|(\nabla_x J)y\|^2\} \\ &\geq (3\delta - 2)l + \|(\nabla_x J)y\|^2 \\ &\geq \{(3+\lambda)\delta - 2\}l > 0, \end{aligned}$$

and hence M satisfies the condition (5.37) in Theorem 5.5. Thus, it follows that

$$\nabla\rho = 0 \quad \text{and} \quad \rho = 5\rho^*.$$

Since M is irreducible by (5.47), we may easily show that M is an Einstein manifold. ∎

In the rest of this section, we shall prove a result (Theorem 5.10) related to Theorem 4.6. We assume that $M = (M, J, g)$ is an m-dimensional connected, Einstein, non-Kähler nearly Kähler manifold with vanishing first Chern form, and furthermore satisfies the condition $T(\lambda,\mu)$ with $5\lambda \geq 4\mu$ and is holomorphically $\delta(> \frac{2}{\lambda+3})$-pinched.

First, we shall estimate the values of the functions $\varphi_\alpha(\alpha = 1, 3, 4)$ on $U(M)$.

Lemma 5.7 *For each point $(p,x) \in U(M)$, we have*

$$\varphi_1(p,x) \geq \frac{l}{8}\{(m+2)(5\lambda - 4\mu)\delta - 8\lambda\}H(x).$$

Proof. Let $\{e_i\}(e_1 = x, e_2 = Jx)$ be an orthonormal basis of T_pM which diagonalizes the matrix $\left(g((\nabla_{e_i}J)x, (\nabla_{e_j}J)x)\right)_{1\leq i,j\leq m}$. Then, by the hypothesis for M and (1.6), (5.8), (5.9), (5.21), (5.31) and (5.47), we get

$$\begin{aligned}
\varphi_1(p,x) &= -\sum_{i,j=1}^{m} R(e_i,x,Je_j,Jx)g((\nabla_{e_i})x,(\nabla_{e_j})x) \\
&= -\sum_{i=1}^{m} R(e_i,x,e_i,x)\|(\nabla_{e_i})x\|^2 - \sum_{i=1}^{m}\|(\nabla_{e_i})x\|^4 \\
&\geq \sum_{i=3}^{m} K(e_i,x)\|(\nabla_{e_i})x\|^2 - \mu H(x)\sum_{i=1}^{m}\|(\nabla_{e_i})x\|^2 \\
&\geq \lambda H(x)\{\rho(x,x)-H(x)\} - \mu H(x)\{\rho(x,x)-\rho^*(x,x)\} \\
&= \lambda H(x)\left(\frac{\tau}{m}-H(x)\right) - \frac{4\tau}{5m}\mu H(x) \\
&= \frac{\tau}{5m}(5\lambda-4\mu)H(x) - \lambda H(x)^2 \\
&\geq \frac{m+2}{8}(5\lambda-4\mu)\delta l H(x) - \lambda l H(x) \\
&= \frac{l}{8}\{(m+2)(5\lambda-4\mu)\delta - 8\lambda\}H(x).
\end{aligned}$$

∎

Lemma 5.8 *For each point $(p,x)\in U(M)$, we have*

$$\varphi_3(p,x) \leq \mu l\left(\frac{\tau}{m}-H(x)\right).$$

Proof. Let $\{e_i\}(e_1=x, e_2=Jx)$ be an orthonormal basis of T_pM as in the proof of Lemma 5.7. Then, by (5.8), (5.9), (5.21) and (5.47), we get

$$\begin{aligned}
\varphi_3(p,x) &= -\sum_{i=1}^{m} R(x,(\nabla_{e_i}J)x,x,(\nabla_{e_i}J)x) \\
&= -\sum_{i=1}^{m} R(x,e_i,x,e_i)\|(\nabla_{e_i}J)x\|^2 \\
&\leq \mu H(x)\left(\frac{\tau}{m}-H(x)\right) \\
&\leq \mu l\left(\frac{\tau}{m}-H(x)\right).
\end{aligned}$$

∎

Lemma 5.9 *For each point $(p,x)\in U(M)$, we have*

$$\varphi_4(p,x) \leq -\frac{m+2}{2}\delta l H(x).$$

Proof. By (1.15), (5.8), (5.21) and (5.31), we get

$$\begin{aligned} \varphi_4(p,x) &= -\frac{4\tau}{5m}H(x) \\ &\leq -\frac{m+2}{2}\delta l H(x). \end{aligned}$$

∎

Next, we estimate the value $\int_{U_p} \psi_3\sigma_2$. By (5.8), (5.9), (5.24), (5.28), (5.29) and (5.30), we have

$$\begin{aligned} \int_{U_p} \psi_3\sigma_2 &\leq \mu l \left\{ \int_{U_p} \sum_{k=1}^{m} R(x,Jx,x,e_k)^2\sigma_2 - \int_{U_p} H^2\sigma_2 \right\} \\ &= \frac{\mu l}{16}\int_{U_p} \|\mathrm{grad}^v H\|^2\sigma_2. \end{aligned} \tag{5.48}$$

Now, we prove the following

Theorem 5.10 *Let $M = (M,J,g)$ be an m-dimensional connected compact non-Kähler nearly Kähler manifold with constant scalar curvature. If M satisfies the condition $T(\lambda,\mu)$ with $5\lambda \geq 4\mu$, $3\lambda \geq 4\mu - 1$, and is holomorphically δ-pinched where $\delta > \frac{2}{\lambda+3}$ and $\delta \geq \frac{3\lambda+4\mu}{15\lambda-12\mu+4}$, then M is isometric to a 6-dimensional sphere $S^6(c)$ of constant curvature $c = \tau/30$.*

Proof. First, we note that

$$1 - \frac{3\lambda+4\mu}{15\lambda-12\mu+4} = \frac{4(3\lambda-4\mu+1)}{15\lambda-12\mu+4} \geq 0.$$

By the hypothesis, we also note that

$$\begin{aligned} &(m+2)(3\lambda+4\mu) - \{5(m+2)\mu + 24\lambda - 8\mu\} \\ &= (m-6)(3\lambda-\mu) \\ &\geq (m-6)(3\mu-2\lambda) \geq 0. \end{aligned} \tag{5.49}$$

Next, from the hypothesis for M and Theorem 5.6, it follows that M is an Einstein manifold with vanishing first Chern form. Furthermore, by (5.27), (5.28), (5.29), (5.47), (5.48), (5.49), and Lemma 5.7 $\sim$ Lemma 5.9, we obtain

$$0 \geq 2\int_{U(M)} \psi_1\sigma \tag{5.50}$$

$$
\begin{aligned}
&+\frac{l}{2}\left\{\frac{(m+2)(15\lambda-12\mu+4)\delta-24\lambda+8\mu-4(m+2)\mu}{4(m+2)}+\frac{2\lambda}{\lambda+3}\right\}\int_{U(M)}\|\mathrm{grad}^v H\|^2\sigma \\
&\quad+\frac{32\tau^2 l\{(m+2)(15\lambda-12\mu+4)\delta-24\lambda+8\mu-5(m+2)\mu\}}{25m^2(m+2)^2}\mathrm{Vol}(S^{m-1})\mathrm{Vol}(M) \\
&\quad\geq 2\int_{U(M)}\psi_1\sigma+\frac{l}{2}\left(\frac{\mu}{4}+\frac{2\lambda}{\lambda+3}\right)\int_{U(M)}\|\mathrm{grad}^v H\|^2\sigma \\
&\quad\geq 0.
\end{aligned}
$$

Thus, from (5.50) and Proposition 5.1, it follows that M is a space of constant holomorphic sectional curvature. Therefore, the theorem follows immediately from Theorem 4.6. ∎

Chapter III

Almost Kähler manifolds

1 Fundamental facts

Let $M = (M, J, g)$ be an $m(= 2n)$-dimensional almost Kähler manifold. Then, by (I.2.5) and (I.1.7), we have

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)), \tag{1.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$, from which it follows that

$$4\|\nabla J\|^2 = \|N\|^2. \tag{1.2}$$

This shows that the almost complex structure J of an almost Kähler manifold M is integrable if and only if (M, J, g) is a Kähler manifold.

It is well known that the tangent bundle $T(M)$ over a non-flat Riemannian manifold M possesses a non-Kähler almost Kähler structure ([49]). The almost complex structure J on $T(M)$ is given by

$$JX^h = -X^v, \quad JX^v = X^h,$$

where X^h (resp. X^v) denotes the horizontal (resp. the vertical) lift of $X \in \mathfrak{X}(M)$. With this almost complex structure J and the Sasaki metric g^s (cf. §5 of Chapter II), $(T(M), J, g^s)$ becomes an almost Kähler manifold. The integrability condition of this almost complex structure J is nothing but the flatness of M.

W. P. Thurston [53] gave an example of a compact almost Kähler manifold which does not admit a Kähler structure, this manifold W^4 was defined as a fiber bundle over the 2-torus. E. Abbena [1] showed that the Thurston manifold W^4 can be expressed in the form G/Γ, where G is a closed connected Lie subgroup of $GL(4, \mathbb{C})$ defined by

$$G = \left\{ A = \begin{bmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{bmatrix} \;\middle|\; a_{12}, a_{13}, a_{23}, a \in \mathbb{R} \right\},$$

and Γ is a discrete subgroup of G defined by

$$\Gamma = \{ A \in G \mid a_{12}, a_{13}, a_{23}, a \in \mathbb{Z} \}.$$

Using this expression, Abbena wrote down the metric and the almost complex structure explicitly. Furthermore, she showed that the Ricci $*$-tensor ρ^* is symmetric in W^4.

After Thurston, several examples of compact non-Kähler almost Kähler manifolds are known ([61, 12]).

It is fundamental to discuss the integrability of the almost complex structure in an almost Kähler manifold. Z. Olszak [28] proved the following

Theorem 1.1 *In dimension ≥ 8, there are no almost Kähler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kählerian.*

Furthermore, D. E. Blair [10] recently proved the following

Theorem 1.2 *In dimension 4, there are no almost Kähler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kählerian.*

In the case of $\dim M = 6$, the existence problem of non-Kähler almost Kähler manifolds of (negative) constant curvature is still open.

Concerning the integrability of an almost complex structure, the following conjecture by S. I. Goldberg is well-known ([15, 61]).

The almost complex structure of a compact Einstein almost Kähler manifold is integrable.

K. Sekigawa [43, 44] gave the following positive answer to this conjecture.

Theorem 1.3 *Let $M = (M, J, g)$ be a compact Einstein almost Kähler manifold whose scalar curvature is non-negative. Then M is a Kähler manifold.*

Let $\{e_i\}_{i=1,\cdots,m}$ be an orthonormal basis of T_pM at any point $p \in M$. In this chapter, we shall adopt the following notational convention:

$$
\begin{aligned}
&R_{ijkl} = R(e_i, e_j, e_k, e_l), \quad G_{ijkl} = G(e_i, e_j, e_k, e_l),\\
&R_{\bar{i}jkl} = R(Je_i, e_j, e_k, e_l), \quad G_{\bar{i}jkl} = G(Je_i, e_j, e_k, e_l),\\
&\quad \cdots\\
&R_{\bar{i}\bar{j}\bar{k}\bar{l}} = R(Je_i, Je_j, Je_k, Je_l), \quad G_{\bar{i}\bar{j}\bar{k}\bar{l}} = G(Je_i, Je_j, Je_k, Je_l),\\
&\rho_{ij} = \rho(e_i, e_j), \quad \rho^*_{ij} = \rho^*(e_i, e_j),\\
&J_{ij} = g(Je_i, e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k),
\end{aligned}
$$

and so on, where the Latin indices run over the range $1, \cdots, m$. It is easy to see that

$$\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0, \tag{1.3}$$

$$\nabla_{\bar{i}} J_{jk} = \nabla_i J_{\bar{j}k} = \nabla_i J_{j\bar{k}}. \tag{1.4}$$

In the course of the proof of Theorem 1.3, Sekigawa has established the following integral formula on a compact almost Kähler manifold.

Proposition 1.4 ([44]) *Let $M = (M, J, g)$ be an m-dimensional compact almost Kähler manifold. Then we have*

$$\int_M \left(f_1 - \frac{1}{2} f_2 + f_3 - 2f_4 \right) dM = 0, \tag{1.5}$$

where f_1, f_2, f_3 and f_4 are smooth functions on M defined respectively by

$$f_1(p) = \sum R_{abij}\{R_{\bar{a}\bar{b}ij} - R_{\bar{a}\bar{b}\bar{i}\bar{j}}\}, \tag{1.6}$$

$$
\begin{aligned}
f_2(p) &= \sum R_{a\bar{a}ij}\{R_{b\bar{b}ij} - R_{b\bar{b}\bar{i}\bar{j}}\}, \\
f_3(p) &= \sum R_{a\bar{a}ij}(\nabla_{\bar{b}} J_{ik})\nabla_b J_{jk}, \\
f_4(p) &= \sum R_{abij}(\nabla_{\bar{b}} J_{ik})\nabla_{\bar{a}} J_{jk},
\end{aligned}
$$

at any point $p \in M$.

The integral formula (1.5) will play an important role in the next section.

2 Almost Kähler manifolds of constant holomorphic sectional curvature

Let $M = (M, J, g)$ be an m-dimensional almost Kähler manifold satisfying the condition (b) in §2 of Chapter I. Then it follows that

$$G_{ijkl} = G_{klij}, \tag{2.1}$$

$$G_{\bar{i}\bar{j}\bar{k}\bar{l}} = G_{ijkl}. \tag{2.2}$$

Hence, by Lemma 2.3 of Chapter I, we have

$$
\begin{aligned}
G_{ijkl} &= \frac{1}{2}\sum_a (\nabla_i J_{ja} - \nabla_j J_{ia})(\nabla_k J_{la} - \nabla_l J_{ka}) \\
&= \frac{1}{2}\sum_a (\nabla_a J_{ij})\nabla_a J_{kl},
\end{aligned}
\tag{2.3}
$$

from which we have

$$\rho_{ij} - \rho^*_{ij} = -\frac{1}{2}\sum_{a,b}(\nabla_a J_{bi})\nabla_a J_{bj}, \tag{2.4}$$

$$\tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2. \tag{2.5}$$

The main purpose of this section is to prove the following theorem.

Theorem 2.1 ([36]) *Let $M = (M, J, g)$ be a compact almost Kähler manifold of constant holomorphic sectional curvature c. If M satisfies the condition (b) and the constant c is non-negative, then M is a Kähler manifold.*

Proof. By virtue of Theorem 3.9 of Chapter I, the curvature tensor of M is given by

$$R_{ijkl} = \frac{c}{4}H_{ijkl} - \frac{1}{4}\{G_{iljk} - G_{ikjl} - 2G_{ijkl}\}, \tag{2.6}$$

where

$$H_{ijkl} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}.$$

and G_{ijkl} is given by (2.3).

From (2.6), we have

$$\rho_{ij} + 3\rho^{*}_{ij} = (m+2)c\delta_{ij}, \tag{2.7}$$

$$\tau + 3\tau^{*} = m(m+2)c, \tag{2.8}$$

(cf. Corollary 3.7 of Chapter I).

We shall evaluate the values of the functions f_1, f_2, f_3 and f_4 in Proposition 1.4.

Lemma 2.2 *For each point $p \in M$, we have*

$$f_1(p) = -\frac{1}{2}\|G\|^2.$$

Proof of Lemma 2.2. By (1.6), (I.1.19), (2.1) and (2.2), we have

$$\begin{aligned}
f_1(p) &= \sum R_{abij}(R_{\bar{a}\bar{b}ij} - R_{\bar{a}\bar{b}\bar{i}\bar{j}}) \\
&= \sum R_{abij}G_{\bar{a}\bar{b}ij} \\
&= \frac{1}{2}\sum(R_{abij}G_{\bar{a}\bar{b}ij} + R_{ab\bar{i}\bar{j}}G_{\bar{a}\bar{b}\bar{i}\bar{j}}) \\
&= -\frac{1}{2}\sum(R_{abij}G_{abij} - R_{ab\bar{i}\bar{j}}G_{abij}) \\
&= -\frac{1}{2}\sum G_{abij}G_{abij} \\
&= -\frac{1}{2}\|G\|^2.
\end{aligned}$$

∎

Lemma 2.3 *For each point $p \in M$, we have*

$$f_2(p) = 0.$$

Proof of Lemma 2.3. By (1.6), (I.1.21) and (2.1), we have

$$\begin{aligned} f_2(p) &= \sum R_{a\bar{a}ij}(R_{b\bar{b}ij} - R_{b\bar{b}\bar{i}\bar{j}}) \\ &= \sum R_{a\bar{a}ij}G_{b\bar{b}ij} \\ &= \sum R_{a\bar{a}ij}G_{ijb\bar{b}} \\ &= 0. \end{aligned}$$

∎

Lemma 2.4 *For each point $p \in M$, we have*

$$f_3(p) = -\frac{(m+2)}{2}c\|\nabla J\|^2 - \|\rho - \rho^*\|^2.$$

Proof of Lemma 2.4. By the Bianchi identity, we have

$$\sum R_{a\bar{a}i\bar{j}} = -2\rho^*_{ij}. \tag{2.9}$$

By (1.6), (1.4), (2.4), (2.5), (2.7) and (2.9), we have

$$\begin{aligned} f_3(p) &= \sum R_{a\bar{a}ij}(\nabla_{\bar{b}}J_{ik})\nabla_b J_{jk} \\ &= \sum R_{a\bar{a}i\bar{j}}(\nabla_{\bar{b}}J_{ik})\nabla_b J_{\bar{j}k} \\ &= -2\sum \rho^*_{ij}(\nabla_b J_{ik})\nabla_b J_{jk} \\ &= 4\sum \rho^*_{ij}(\rho_{ij} - \rho^*_{ij}) \\ &= \sum\{(\rho_{ij} + 3\rho^*_{ij}) - (\rho_{ij} - \rho^*_{ij})\}(\rho_{ij} - \rho^*_{ij}) \\ &= \sum(m+2)c\delta_{ij}(\rho_{ij} - \rho^*_{ij}) - \|\rho - \rho^*\|^2 \\ &= (m+2)c(\tau - \tau^*) - \|\rho - \rho^*\|^2 \\ &= -\frac{(m+2)}{2}c\|\nabla J\|^2 - \|\rho - \rho^*\|^2. \end{aligned}$$

∎

Lemma 2.5 *For each point $p \in M$, we have*

$$f_4(p) = -\frac{3}{4}c\|\nabla J\|^2 - \frac{1}{8}\|G\|^2.$$

Proof of Lemma 2.5. By (1.6), (1.4) and (2.6), we have

$$
\begin{aligned}
(2.10)\quad f_4(p) &= \sum R_{abij}(\nabla_{\bar b} J_{ik})\nabla_{\bar a} J_{jk} \\
&= \sum R_{abij}(\nabla_b J_{i\bar k})\nabla_a J_{j\bar k} \\
&= \sum R_{abij}(\nabla_b J_{ik})\nabla_a J_{jk} \\
&= \frac{c}{4}\sum(\nabla_b J_{ik})\nabla_a J_{jk}(\delta_{aj}\delta_{bi} - \delta_{ai}\delta_{bj} + J_{aj}J_{bi} - J_{ai}J_{bj} - 2J_{ab}J_{ij}) \\
&\quad -\frac{1}{4}\sum(\nabla_b J_{ik})\nabla_a J_{jk}(G_{ajbi} - G_{aibj} - 2G_{abij}).
\end{aligned}
$$

We shall calculate the right hand side of (2.10) term by term. By (1.3), we have

$$(1)\qquad \sum \nabla_b J_{ik}(\nabla_a J_{jk})\delta_{aj}\delta_{bi} = \sum(\nabla_b J_{bk})\nabla_a J_{ak} = 0.$$

Taking account of

$$
\begin{aligned}
\|\nabla J\|^2 &= \sum(\nabla_i J_{jk})\nabla_i J_{jk} \\
&= -\sum(\nabla_j J_{ki} + \nabla_k J_{ij})\nabla_i J_{jk} \\
&= 2\sum(\nabla_j J_{ik})\nabla_i J_{jk},
\end{aligned}
$$

we have

$$(2)\qquad \sum \nabla_b J_{ik}(\nabla_a J_{jk})\delta_{ai}\delta_{bj} = \sum(\nabla_j J_{ik})\nabla_i J_{jk} = \frac{1}{2}\|\nabla J\|^2.$$

Similarly we have

$$
\begin{aligned}
(3)\qquad \sum \nabla_b J_{ik}(\nabla_a J_{jk})J_{aj}J_{bi} &= \sum(\nabla_b J_{\bar b k})\nabla_a J_{\bar a k} \\
&= \sum(\nabla_b J_{b\bar k})\nabla_a J_{a\bar k} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
(4)\qquad \sum \nabla_b J_{ik}(\nabla_a J_{jk})J_{ai}J_{bj} &= \sum(\nabla_b J_{\bar a k})\nabla_a J_{\bar b k} \\
&= \sum(\nabla_b J_{ak})\nabla_a J_{bk} \\
&= \frac{1}{2}\|\nabla J\|^2,
\end{aligned}
$$

$$\sum \nabla_b J_{ik}(\nabla_a J_{jk}) J_{ab} J_{ij} = \sum (\nabla_{\bar{a}} J_{ik}) \nabla_a J_{\bar{i}k} \tag{5}$$
$$= \sum (\nabla_a J_{ik}) \nabla_a J_{ik}$$
$$= \|\nabla J\|^2.$$

By (I.1.18), (2.1) and (2.2), we have

$$\begin{aligned} &\sum \nabla_b J_{ik}(\nabla_a J_{jk}) G_{ajbi} \\ &= \frac{1}{4} \sum (\nabla_b J_{ik} - \nabla_i J_{bk})(\nabla_a J_{jk} - \nabla_j J_{ak}) G_{ajbi} \\ &= \frac{1}{2} \sum G_{biaj} G_{ajbi} \\ &= \frac{1}{2} \sum G_{ajbi} G_{ajbi} \\ &= \frac{1}{2} \|G\|^2. \end{aligned} \tag{6}$$

By (I.1.19), (1.4),

$$\begin{aligned} \sum \nabla_b J_{ik}(\nabla_a J_{jk}) G_{aibj} &= \sum \nabla_{\bar{b}} J_{ik}(\nabla_a J_{\bar{j}k}) G_{ai\bar{b}\bar{j}} \\ &= -\sum \nabla_b J_{i\bar{k}}(\nabla_a J_{j\bar{k}}) G_{aibj} \\ &= -\sum \nabla_b J_{ik}(\nabla_a J_{jk}) G_{aibj}, \end{aligned}$$

from which we have

$$\sum \nabla_b J_{ik}(\nabla_a J_{jk}) G_{aibj} = 0. \tag{7}$$

By the same reason as (7), we have

$$\sum \nabla_b J_{ik}(\nabla_a J_{jk}) G_{abij} = 0. \tag{8}$$

Substituting (1) $\sim$ (8) into (2.10), we obtain Lemma 2.5. ∎

Now we are in position to complete the proof of Theorem 2.1. By virtue of the above lemmas and Proposition 1.4, we have consequently

$$\int_M \left(\frac{m-1}{2} c \|\nabla J\|^2 + \|\rho - \rho^*\|^2 + \frac{1}{4} \|G\|^2 \right) dM = 0. \tag{2.11}$$

From this formula, we may easily show that if the constant c is non-negative, then ∇J vanishes identically on M. Thus we obtain the theorem. ∎

From (2.5) and (2.8), we have

$$m(m+2)c = 4\tau + \frac{3}{2}\|\nabla J\|^2.$$

Therefore, we have immediately the following

Corollary 2.6 *Let $M = (M, J, g)$ be a compact almost Kähler manifold of constant holomorphic sectional curvature. If M satisfies the condition (b) and the scalar curvature of M is non-negative, then M is a Kähler manifold.*

Chapter IV

Hermitian manifolds

1 Hermitian surfaces

An almost Hermitian manifold $M = (M, J, g)$ with integrable almost complex structure J is called a Hermitian manifold. A real 4-dimensional Hermitian manifold is called a Hermitian surface. In this section, we shall recall some basic properties of Hermitian surfaces.

Let $M = (M, J, g)$ be a Hermitian surface. Then it is useful to consider the 1-form $\omega = (\omega_i)$ called the *Lee form* which is defined by

$$\omega = \delta\Omega \circ J \ ; \quad \omega_i = -(\nabla_k J_j{}^k) J_i{}^j, \tag{1.1}$$

where δ is the codifferential. The Lee form ω satisfies the following equalities ([45, 54]):

$$d\Omega = \omega \wedge \Omega, \tag{1.2}$$

$$J^{ij} \nabla_i \omega_j = 0, \tag{1.3}$$

$$2\nabla_i J_{jk} = \omega_a J_j{}^a g_{ik} - \omega_a J_k{}^a g_{ij} + \omega_j J_{ki} - \omega_k J_{ji}, \tag{1.4}$$

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2. \tag{1.5}$$

For a Hermitian manifold M with $\dim M = m = 2n \geq 4$, the Lee form can be defined by

$$\omega = \frac{1}{n-1} \delta\Omega \circ J,$$

but (1.2) may not hold in general. If (1.2) holds and if ω is closed, we have locally $\omega = df$, and $g' = e^{-f}g$ are local Kähler metrics on M. Such a manifold M is called a *locally conformal Kähler manifold.*

Taking account of the formula by K. Sekigawa [45], we have the following

Proposition 1.1 *Let $M = (M, J, g)$ be a Hermitian surface. Then we have*

$$
\begin{aligned}
(1.6)\qquad & 2G(X,Y,Z,W) \\
&= g(Y,W)\left\{(\nabla_X\omega)(Z) + \frac{1}{2}\omega(X)\omega(Z) - \frac{1}{2}g(X,Z)\|\omega\|^2\right\} \\
&-g(X,W)\left\{(\nabla_Y\omega)(Z) + \frac{1}{2}\omega(Y)\omega(Z) - \frac{1}{2}g(Y,Z)\|\omega\|^2\right\} \\
&+\Omega(Y,Z)\left\{(\nabla_X\omega)(JW) + \frac{1}{2}\omega(X)\omega(JW) - \frac{1}{2}\Omega(X,W)\|\omega\|^2\right\} \\
&-\Omega(X,Z)\left\{(\nabla_Y\omega)(JW) + \frac{1}{2}\omega(Y)\omega(JW) - \frac{1}{2}\Omega(Y,W)\|\omega\|^2\right\} \\
&-\Omega(Y,W)\left\{(\nabla_X\omega)(JZ) + \frac{1}{2}\omega(X)\omega(JZ)\right\} \\
&+\Omega(X,W)\left\{(\nabla_Y\omega)(JZ) + \frac{1}{2}\omega(Y)\omega(JZ)\right\} \\
&-g(Y,Z)\left\{(\nabla_X\omega)(W) + \frac{1}{2}\omega(X)\omega(W)\right\} \\
&+g(X,Z)\left\{(\nabla_Y\omega)(W) + \frac{1}{2}\omega(Y)\omega(W)\right\},
\end{aligned}
$$

for $X, Y, Z, W \in \mathfrak{X}(M)$.

In a Hermitian surface $M = (M, J, g)$, there is another important connection $\widetilde{\nabla}$ called the *Hermitian connection* (known also as the *Chern connection*), which is characterized by

$$\widetilde{\nabla}g = 0, \quad \widetilde{\nabla}J = 0, \quad \widetilde{T}(JX, Y) = \widetilde{T}(X, JY),$$

where $\widetilde{T}$ is the torsion tensor of $\widetilde{\nabla}$. The Hermitian connection $\widetilde{\nabla}$ of a Hermitian surface M is explicitly given by

$$(1.7)\qquad \widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\{\omega(Y)X + \omega(JX)Y - g(X,Y)B\},$$

where B is the Lee field defined by $g(B,X) = \omega(X)$ (cf. [57]). Or in terms of local components,

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} - \frac{1}{2}\{\omega_k\delta^i_j + J_j{}^a\omega_a J_k{}^i - g_{jk}\omega^i\}, \tag{1.8}$$

where $\widetilde{\Gamma} = (\widetilde{\Gamma}^i_{jk})$ and $\Gamma = (\Gamma^i_{jk})$ are the coefficients of the connections $\widetilde{\nabla}$ and ∇, respectively.

By (1.8), the curvature tensor $\widetilde{R} = (\widetilde{R}_{ijk}{}^l)$ of the Hermitian connection $\widetilde{\nabla}$ is given by

$$\begin{aligned}\widetilde{R}_{ijk}{}^l \;=\; & R_{ijk}{}^l \\ & +\frac{1}{2}\Big\{(\nabla_j\omega_k + \frac{1}{2}\omega_j\omega_k)\delta^l_i - (\nabla_i\omega_k + \frac{1}{2}\omega_i\omega_k)\delta^l_j \\ & \quad +(\nabla_i\omega^l + \frac{1}{2}\omega_i\omega^l)g_{jk} - (\nabla_j\omega^l + \frac{1}{2}\omega_j\omega^l)g_{ik}\Big\} \\ & +\frac{\|\omega\|^2}{4}\left(g_{ik}\delta^l_j - g_{jk}\delta^l_i - 2J_{ij}J_k{}^l\right) \\ & +\frac{1}{2}J_k{}^l\left\{J_i{}^a(\nabla_j\omega_a + \omega_j\omega_a) - J_j{}^a(\nabla_i\omega_a + \omega_i\omega_a)\right\}.\end{aligned} \tag{1.9}$$

We denote by $\widetilde{\rho} = (\widetilde{\rho}_{ij})$ and $\widetilde{\rho}^* = (\widetilde{\rho}^*_{ij})$ the tensor fields on M of type (0.2) defined respectively by

$$\widetilde{\rho}_{ij} = \widetilde{R}_{kij}{}^k \quad \text{and} \quad \widetilde{\rho}^*_{ij} = \frac{1}{2}J_j{}^s\widetilde{R}_{isa}{}^b J_b{}^a.$$

Furthermore, we put

$$\widetilde{\tau} = g^{ij}\widetilde{\rho}_{ij} \quad \text{and} \quad \widetilde{\tau}^* = g^{ij}\widetilde{\rho}^*_{ij}.$$

Then, by (1.9), we have easily

$$\widetilde{\rho}_{jk} = \rho_{jk} + \frac{1}{2}\left\{\nabla_j\omega_k - J_j{}^a J_k{}^b(\nabla_b\omega_a + \omega_b\omega_a) - (\delta\omega)g_{jk})\right\}, \tag{1.10}$$

$$\begin{aligned}\widetilde{\rho}^*_{jk} \;=\; & \rho^*_{jk} - \frac{1}{2}\left(\nabla_j\omega_k + J_j{}^a J_k{}^b\nabla_b\omega_a\right) \\ & -\frac{3}{4}\left(\omega_j\omega_k + J_j{}^a J_k{}^b\omega_b\omega_a - \|\omega\|^2 g_{jk}\right).\end{aligned} \tag{1.11}$$

Taking account of (1.5), we have

$$\widetilde{\tau} = \tau^* + \frac{1}{2}\|\omega\|^2, \tag{1.12}$$

$$\widetilde{\tau}^* = \frac{1}{2}(\tau + \tau^*) + \|\omega\|^2. \tag{1.13}$$

By (1.9), the holomorphic sectional curvature $\widetilde{H}(x)$ with respect to the Hermitian connection is given by

$$\widetilde{H}(x) = H(x) + \frac{1}{4}\{\ \|\omega\|^2 - \omega(x)^2 - \omega(Jx)^2\ \}, \tag{1.14}$$

for $x \in T_pM$ with $\|x\| = 1$. It follows that

$$\widetilde{H}(x) \geq H(x), \tag{1.15}$$

for any non-zero vector $x \in T_pM$.

Concerning the holomorphic sectional curvature $\widetilde{H}$ with respect to the Hermitian connection, A. Balas and P. Gauduchon [4] proved the following

Theorem 1.2 *Every Hermitian metric of constant non-positive holomorphic sectional curvature (with respect to the Hermitian connection) on a compact complex surface is Kähler.*

Furthermore, they also studied the structure of compact Hermitian surfaces of constant *positive* holomorphic sectional curvature ([4], Theorem 2). In the next section, we shall study the structure of compact Hermitian surfaces of constant holomorphic sectional curvature (in the Riemannian sense).

We recall here two well-known theorems on complex surfaces which will be used later. We denote by $\chi(M)$, $c_1(M)$, $c_2(M)$ and $p_1(M)$ the Euler class, the first Chern class, the second Chern class and the first Pontrjagin class of M, respectively. We note that $c_2(M)$ is equal to $\chi(M)$ and $\frac{1}{3}p_1(M)$ is equal to the Hirzebruch signature of M.

Theorem 1.3 ([63]) *Let $M = (M, J)$ be a compact connected almost complex surface. Then we have*

$$p_1(M) + 2\chi(M) = c_1(M)^2.$$

Theorem 1.4 ([26]) *Let $M = (M, J)$ be a compact connected complex surface. Then we have*

$$c_1(M)^2 \le \max\{\ 2c_2(M),\ 3c_2(M)\ \}.$$

2 Hermitian surfaces of constant holomorphic sectional curvature

Let $M = (M, J, g)$ be a Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then, taking account of Theorem 3.2 of Chapter I and Proposition 1.1, we have

$$(2.1)\quad \begin{aligned} R_{ijkl} &= \frac{1}{4}\|\omega\|^2\big(g_{il}g_{jk} - g_{ik}g_{jl}\big) + \Big(\frac{c(p)}{4} - \frac{1}{16}\|\omega\|^2\Big)H_{ijkl} \\ &\quad + \frac{1}{96}\big\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} \\ &\qquad + J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\big\}, \end{aligned}$$

where

$$\begin{aligned} H_{ijkl} &= g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}, \\ A_{ij} &= 21(\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j) - 3J_i{}^aJ_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b), \\ B_{ij} &= 7(J_j{}^a\nabla_i\omega_a - J_i{}^a\nabla_j\omega_a) - (J_j{}^a\nabla_a\omega_i - J_i{}^a\nabla_a\omega_j) \\ &\quad + 3(J_j{}^a\omega_i\omega_a - J_i{}^a\omega_j\omega_a). \end{aligned}$$

By (2.1) and (1.5), we have

$$(2.2)\qquad \rho_{ij} = \Big\{\frac{3}{2}c + \frac{3}{16}(\tau - \tau^*)\Big\}g_{ij} - \frac{1}{4}T_{ij},$$

$$(2.3)\qquad \rho^*_{ij} = \Big\{\frac{3}{2}c - \frac{1}{16}(\tau - \tau^*)\Big\}g_{ij} + \frac{1}{4}T^*_{ij},$$

where

$$(2.4)\qquad T_{ij} = \nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j - J_i{}^aJ_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b),$$

and

$$(2.5) \qquad T^*_{ij} = \nabla_i \omega_j - \nabla_j \omega_i - J_i{}^a J_j{}^b (\nabla_a \omega_b - \nabla_b \omega_a).$$

By (2.2), we get

$$(2.6) \qquad \tau + 3\tau^* = 24c.$$

By (2.6), the formula (2.2) and (2.3) may be written as

$$(2.2') \qquad \rho_{ij} = \frac{\tau}{4} g_{ij} - \frac{1}{4} T_{ij},$$

$$(2.3') \qquad \rho^*_{ij} = \frac{\tau^*}{4} g_{ij} + \frac{1}{4} T^*_{ij}.$$

From now on we establish some integral formulas which will play an important role in our arguments. Assume that the manifold M is compact and connected. By (2.4),

$$T_{ij}\omega^i\omega^j = 2\omega^i\omega^j\nabla_i\omega_j + \|\omega\|^4 - 2J_i{}^a J_j{}^b (\nabla_a \omega_b)\omega^i\omega^j,$$

and it follows that

$$(2.7) \qquad \int_M T_{ij}\omega^i\omega^j dM = \int_M \{\|\omega\|^2 \delta\omega + \|\omega\|^4 - 2F\} dM,$$

where we define the function F on M by

$$(2.8) \qquad F = J_i{}^a J_j{}^b (\nabla_a \omega_b)\omega^i\omega^j.$$

By (2.2'), (2.3'), (2.5) and (2.7), we get

$$(2.9) \qquad \int_M \{\rho_{ij}\omega^i\omega^j + \rho^*_{ij}\omega^i\omega^j - \frac{1}{2}F\} dM = \frac{1}{4}\int_M \{(\tau + \tau^*)\|\omega\|^2 - \|\omega\|^2\delta\omega - \|\omega\|^4\} dM.$$

By making use of Ricci's identity and Green's theorem, we get

$$(2.10) \qquad \int_M (\nabla_i \omega_j)\nabla^j \omega^i dM = \int_M \{(\delta\omega)^2 - \rho_{ij}\omega^i\omega^j\} dM.$$

Since

$$(\nabla_i\omega_j)\nabla^i\omega^j = \frac{1}{2}(\nabla_i\omega_j - \nabla_j\omega_i)(\nabla^i\omega^j - \nabla^j\omega^i) + (\nabla_i\omega_j)\nabla^j\omega^i,$$

we get

$$(2.11) \qquad \int_M (\nabla_i\omega_j)\nabla^i\omega^j dM = \int_M \{\|d\omega\|^2 + (\delta\omega)^2 - \rho_{ij}\omega^i\omega^j\}dM.$$

Taking account of the relation $\nabla_j\tau = 2\nabla^i\rho_{ij}$, from (1.4), (2.10) and (2.11), we have

$$\begin{aligned}
&\int_M \tau\delta\omega\ dM \\
&= \int_M \omega^j\nabla_j\tau\ dM \\
&= \int_M \{\|d\omega\|^2 + 2(\delta\omega)^2 + 2\|\omega\|^2\delta\omega - 2\rho_{ij}\omega^i\omega^j - 2\rho^*_{ij}\omega^i\omega^j + 2F\}dM.
\end{aligned}$$

Combining with (2.9), it follows that

$$(2.12) \quad \int_M F\ dM = \int_M \{\tau\delta\omega + \frac{1}{4}\|\omega\|^4 - \frac{1}{2}(\tau-\tau^*)^2 + 6c\|\omega\|^2 - \|d\omega\|^2\}dM.$$

Next, we define the functions G_1 and G_2 by

$$G_1 = J_i{}^aJ_j{}^b(\nabla_a\omega_b)\nabla^i\omega^j,$$

$$G_2 = J_i{}^aJ_j{}^b(\nabla_a\omega_b)\nabla^j\omega^i.$$

Then, by similar computations to those of (2.12), we get

$$(2.13) \qquad \begin{aligned}\int_M G_1\ dM &= \int_M G_2\ dM \\ &= \int_M \{\rho^*_{ij}\omega^i\omega^j - \frac{3}{2}F - \frac{3}{4}\|\omega\|^2\delta\omega\}dM.\end{aligned}$$

By (1.5), (2.9), (2.10), (2.11) and (2.13), we get

$$(2.14) \qquad \int_M \|T\|^2\ dM = \int_M \{4\|d\omega\|^2 + 2(\tau-\tau^*)^2 - 4\tau^*\|\omega\|^2\}dM,$$

and

$$(2.15) \qquad \int_M \|T^*\|^2\ dM = 4\int_M \|d\omega\|^2\ dM.$$

By (2.14), we have immediately the following

Lemma 2.1 *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then*

$$(2.16) \qquad \int_M \tau^*\|\omega\|^2 dM \le \int_M \{\|d\omega\|^2 + \frac{1}{2}(\tau-\tau^*)^2\}dM.$$

Equality holds if and only if $T = 0$ (i.e., M is Einstein).

Lemma 2.2 *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature c. Then the Euler class of M is given by*

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{12c^2 - \frac{1}{16}(\tau - \tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2\}dM. \tag{2.17}$$

Proof. We begin with a calculation of the norm of the curvature tensor. Since

$$\begin{aligned}
&\left(g_{il}g_{jk} - g_{ik}g_{jl}\right)R^{ijkl} = 2\tau, \\
&H_{ijkl}R^{ijkl} = 2(\tau + 3\tau^*), \\
&\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} \\
&\qquad + J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\}R^{ijkl} \\
&\qquad = -4A_{ij}\rho^{ij} - 12B_{ik}J_j{}^k\rho^{*ij},
\end{aligned}$$

we have, by (2.1) and (2.6)

$$\|R\|^2 = 12c^2 + \frac{3}{8}\|\omega\|^2(\tau - \tau^*) - \frac{1}{24}A_{ij}\rho^{ij} - \frac{1}{8}B_{ik}J_j{}^k\rho^{*ij}. \tag{2.18}$$

By direct computations, we get

$$\begin{aligned}
A_{ij}\rho^{ij} \;=\; & -\frac{9}{2}\tau(2\delta\omega - \|\omega\|^2) - 6\|\omega\|^4 \\
& -12\{(\nabla_i\omega_j)\nabla^i\omega^j + (\nabla_i\omega_j)\nabla^j\omega^i + \omega^i\nabla_i\|\omega\|^2\} \\
& +12(2F + G_1 + G_2),
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
B_{ik}J_j{}^k\rho^{*ij} \;=\; & \frac{3}{2}\tau^*(2\delta\omega - \|\omega\|^2) \\
& -4\{(\nabla_i\omega_j)\nabla^i\omega^j - (\nabla_i\omega_j)\nabla^j\omega^i - G_1 + G_2\}.
\end{aligned} \tag{2.20}$$

By substituting (2.19), (2.20) into (2.18), we get

$$\begin{aligned}
\|R\|^2 \;=\; & 12c^2 + \frac{3}{16}(\tau - \tau^*)^2 + (\nabla_i\omega_j)\nabla^i\omega^j + \frac{1}{2}\omega^i\nabla_i\|\omega\|^2 \\
& + \frac{1}{4}\|\omega\|^4 - F - G_1.
\end{aligned} \tag{2.21}$$

By (1.5), (2.9), (2.11), (2.13) and (2.21), we have

$$\int_M \|R\|^2 dM = \int_M \{12c^2 + \frac{7}{16}(\tau - \tau^*)^2 + \|d\omega\|^2 - \frac{1}{2}\tau^*\|\omega\|^2\}dM. \tag{2.22}$$

By (2.2') and (2.14), we get

$$\int_M \|\rho\|^2 dM = \frac{1}{4}\int_M \{\tau^2 + \|d\omega\|^2 + \frac{1}{2}(\tau-\tau^*)^2 - \tau^*\|\omega\|^2\} dM. \tag{2.23}$$

Now, (2.17) is an immediate consequence of (2.22), (2.23) and the well known Gauss-Bonnet formula

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\} dM.$$

∎

Lemma 2.3 *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the square of the first Chern class of M is given by*

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\|\omega\|^2 + \|d\omega\|^2\} dM. \tag{2.24}$$

Proof. It is easy to see

$$\gamma \wedge \gamma = \frac{1}{4}\{\gamma_{ij}\gamma_{kl}J^{ij}J^{kl} - 2\gamma_{ij}\gamma_{kl}J^{ik}J^{jl}\} dM. \tag{2.25}$$

By (I.1.13), (1.4) and (2.3'), we get

$$8\pi\gamma_{ij} = (\tau^* + \|\omega\|^2)J_{ij} - J_i{}^k(\omega_j\omega_k + J_j{}^a J_k{}^b \omega_a\omega_b) + J_i{}^k T^*_{jk}. \tag{2.26}$$

By (1.3) and (2.6), we get

$$8\pi\gamma_{ij}J^{ij} = 4\tau^* + 2\|\omega\|^2, \tag{2.27}$$

$$\begin{aligned} 64\pi^2\gamma_{ij}\gamma_{kl}J^{ik}J^{jl} &= 4(\tau^* + \|\omega\|^2)^2 - 4(\tau^* + \|\omega\|^2)\|\omega\|^2 \\ &\quad + 2\|\omega\|^4 - \|T^*\|^2. \end{aligned} \tag{2.28}$$

Substituting (2.27), (2.28) into (2.25), we have

$$\gamma \wedge \gamma = \frac{1}{32\pi^2}\{(\tau^*)^2 + \tau^*\|\omega\|^2 + \frac{1}{4}\|T^*\|^2\} dM.$$

Thus we obtain (2.24). ∎

By virtue of (2.6), Theorem 1.3, Lemma 2.2 and Lemma 2.3, the first Pontrjagin class $p_1(M)$ is given by

$$
\begin{aligned}
(2.29) \qquad p_1(M) &= \frac{1}{32\pi^2}\int_M \Big\{(\tau^*)^2 + \frac{1}{8}(\tau-\tau^*)^2 - 24c^2 + \|d\omega\|^2\Big\} dM \\
&= \frac{1}{32\pi^2}\int_M \Big\{\frac{1}{12}(\tau-3\tau^*)^2 + \|d\omega\|^2\Big\} dM.
\end{aligned}
$$

Thus we have the following

Theorem 2.4 *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the first Pontrjagin class $p_1(M)$ is non-negative. If $p_1(M)$ is equal to zero, then $\tau = 3\tau^*$ and $d\omega = 0$ (and hence, M is a locally conformal Kähler surface).*

We now proceed to prove the following result related to Theorem 1.2.

Theorem 2.5 *Let $M = (M, J, g)$ be a compact Hermitian surface of constant non-positive holomorphic sectional curvature. Then M is a Kähler surface.*

Proof. First of all, we show that $\chi(M) \geq 0$. To show this, suppose that $\chi(M) < 0$. Then, by taking account of Theorem 1.3, Theorem 1.4 and Theorem 2.4, we have

$$0 \leq 2\chi(M) - c_1(M)^2 = -p_1(M) \leq 0.$$

Thus we have

$$(2.30) \qquad \tau = 3\tau^*, \qquad d\omega = 0,$$

and

$$(2.31) \qquad c_1(M)^2 = 2\chi(M) < 0.$$

By (2.6) and (2.30), we get

$$\tau^* = 4c \leq 0, \qquad \tau = 12c \leq 0.$$

Hence, we have

$$\int_M \tau^*\|\omega\|^2 dM = 4c\int_M \|\omega\|^2 dM \leq 0.$$

On the other hand, taking account of (1.5), we have

$$\begin{aligned}\int_M \tau^*\|\omega\|^2 dM &= 4c\int_M \|\omega\|^2 dM \\ &= 4c\int_M (\tau-\tau^*)dM \\ &= 32c^2\mathrm{Vol}(M) \geq 0.\end{aligned}$$

Thus, we have

$$\int_M \tau^*\|\omega\|^2 dM = 0. \tag{2.32}$$

By (2.32) and Lemma 2.3, we have

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M (\tau^*)^2 dM \geq 0.$$

This is a contradiction to (2.31), which shows $\chi(M) \geq 0$.

Since $\chi(M) \geq 0$, by virtue of Theorem 1.4, Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}0 &\leq 32\pi^2\{3\chi(M) - c_1(M)^2\} \\ &= \int_M \{36c^2 - \frac{3}{16}(\tau-\tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2 - (\tau^*)^2 - \|d\omega\|^2\}dM \\ &\leq \int_M \{\frac{1}{16}(\tau+3\tau^*)^2 + \frac{1}{16}(\tau-\tau^*)^2 - (\tau^*)^2 - \frac{1}{2}\|d\omega\|^2\}dM \\ &= \int_M \{3c(\tau-\tau^*) - \frac{1}{2}\|d\omega\|^2\}dM \\ &= \int_M \{3c\|\omega\|^2 - \frac{1}{2}\|d\omega\|^2\}dM \\ &\leq 0.\end{aligned} \tag{2.33}$$

This shows that

(i) $$c < 0 \quad \text{and} \quad \omega = 0,$$

or

(ii) $$c = 0 \quad \text{and} \quad d\omega = 0.$$

Since $\omega = 0$ implies that M is Kählerian, it is sufficient to consider the case (ii). In this case, taking account of $\tau + 3\tau^* = 0$, the second line of (2.33) reduces to

$$0 = \int_M \{\frac{1}{4}(\tau - \tau^*)^2 - \frac{1}{2}\tau^*\|\omega\|^2\} dM = \frac{1}{8}\int_M \|T\|^2 dM. \tag{2.34}$$

Hence, we get $T = 0$. Then, by (2.7), (2.12) and (2.34), we have

$$\begin{aligned} 0 &= \int_M \{\|\omega\|^2\delta\omega + \|\omega\|^4 - 2F\} dM \\ &= \int_M \{-2\tau^*\|\omega\|^2 - 2\tau\delta\omega + (\tau - \tau^*)^2\} dM \\ &= -2\int_M \tau\delta\omega \, dM. \end{aligned}$$

Since $\tau + 3\tau^* = 0$, we get

$$\int_M \tau\delta\omega \, dM = 0 \quad \text{and} \quad \int_M \tau^*\delta\omega \, dM = 0.$$

Then, by (1.5) and (2.34),

$$\begin{aligned} \int_M (\tau - \tau^*)^2 dM &= \int_M (\tau - \tau^*)(2\delta\omega + \|\omega\|^2) dM \\ &= \int_M (\tau - \tau^*)\|\omega\|^2 dM \\ &= -4\int_M \tau^*\|\omega\|^2 dM \\ &= -2\int_M (\tau - \tau^*)^2 dM, \end{aligned}$$

from which we have

$$\tau = \tau^* = 0 \quad \text{and} \quad 2\delta\omega + \|\omega\|^2 = 0.$$

Consequently, we have

$$\int_M \|\omega\|^2 dM = 0,$$

which implies $\omega = 0$ on M. This completes the proof of the theorem. ∎

Next, we shall consider the case where the constant c is positive. We obtain the following

Theorem 2.6 *Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise positive constant holomorphic sectional curvature. Then the Euler number $\chi(M)$ and the Chern number $c_1(M)^2$ are positive, and the Pontrjagin number $p_1(M)$ is non-negative (and hence, M is an algebraic surface with positive Euler number and non-negative signature).*

Proof. We first observe that $\chi(M) \geq 0$. In fact, as in the proof of Theorem 2.5, if $\chi(M) < 0$ then we see that $\tau = 3\tau^*$, $d\omega = 0$ and $c_1(M)^2 = 2\chi(M) < 0$. Thus, we have $\tau^* = 4c > 0$ by virtue of (2.6), and hence

$$\int_M \tau^* \|\omega\|^2 dM \geq 0.$$

From this and (2.24), we have $c_1(M)^2 \geq 0$. But this is a contradiction.

Since $\chi(M) \geq 0$, by (2.17), we get

$$\int_M \tau^* \|\omega\|^2 dM \geq \int_M \{\frac{1}{8}(\tau - \tau^*)^2 - 24c^2\} dM. \tag{2.35}$$

From (2.6), (2.24) and (2.35), we have

$$\begin{aligned} 32\pi^2 c_1(M)^2 &\geq \int_M \{(\tau^*)^2 + \frac{1}{8}(\tau - \tau^*)^2 - 24c^2 + \|d\omega\|^2\} dM \\ &= \int_M \{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\} dM \\ &\geq 0. \end{aligned} \tag{2.36}$$

Thus, we have

$$c_1(M)^2 \geq 0.$$

Suppose that $c_1(M)^2 = 0$. Then, by (2.36), we have $\tau = 3\tau^*$, $d\omega = 0$ and hence $\tau^* = 4c > 0$ by (2.6). Thus, by (2.24), we have

$$\begin{aligned} 0 = c_1(M)^2 &= \frac{1}{32\pi^2} \int_M \{(\tau^*)^2 + \tau^* \|\omega\|^2\} dM \\ &= \frac{1}{32\pi^2} \int_M \{16c^2 + 4c\|\omega\|^2\} dM \\ &> 0. \end{aligned}$$

But, this is a contradiction. Thus, we have finally

$$c_1(M)^2 > 0.$$

Therefore, by Theorem 1.4, we have

$$3\chi(M) \geq c_1(M)^2 > 0.$$

The rest of the proof of the theorem follows immediately from Theorem 2.4 and the well-known classification of compact complex surfaces ([24, 11]). ∎

Furthermore, we have the following

Theorem 2.7 *Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise positive constant holomorphic sectional curvature. Then M is biholomorphically equivalent to a complex projective surface $P^2(\mathbf{C})$.*

Proof. Let u and v be the two scalar curvatures of Hermitian geometry introduced in the work of A. Balas [3]. Then we have

$$u = \frac{1}{2}\tilde{\tau}^* \quad \text{and} \quad v = \frac{1}{2}\tilde{\tau}. \tag{2.37}$$

By (1.12), (1.13), (2.6) and (2.37), we have

$$\begin{aligned} u + v &= \frac{1}{2}(\tilde{\tau} + \tilde{\tau}^*) \\ &= \frac{1}{4}(\tau + 3\tau^*) + \frac{3}{4}\|\omega\|^2 \\ &= 6c + \frac{3}{4}\|\omega\|^2 > 0. \end{aligned} \tag{2.38}$$

Thus, from (2.38), taking account of the result of A. Balas ([3], Theorem 1), we see that the plurigenera of M all vanish, that is, the Kodaira dimension of M is equal to -1. Thus, the Noether formula ([24]) is of the form

$$c_1(M)^2 + c_2(M) = 12(1 - q), \quad (q \geq 0), \tag{2.39}$$

where $q = q(M)$ is the irregularity of M. From (2.39), taking account of Theorem 2.6, we have $q = 0$. Thus, (2.39) reduces to

$$c_1(M)^2 + c_2(M) = 12. \tag{2.40}$$

Referring to the well-known classification of compact complex surfaces (see, for example, [11], p.415), we infer that M is rational, equivalently, obtained by successive blowing up's from a complex projective surface $P^2(\mathbb{C})$ or a (geometrically) ruled surface over a complex projective line $P^1(\mathbb{C})$.

Since $c_2(M) = \chi(M) > 0$ by Theorem 2.6, Miyaoka's inequality (Theorem 1.4) is of the form

$$c_1(M)^2 \leq 3c_2(M). \tag{2.41}$$

By (2.40) and (2.41), we have

$$c_2(M) \geq 3. \tag{2.42}$$

Furthermore, by Theorem 1.3 and Theorem 2.6, we have

$$c_1(M)^2 \geq 2c_2(M). \tag{2.43}$$

So, by (2.40), (2.42) and (2.43), we have

$$c_2(M) = 3 \quad \text{or} \quad 4. \tag{2.44}$$

We assume that $c_2(M) = 4$. Then, by (2.40), we get $c_1(M)^2 = 8$. Thus, by Theorem 1.3, it follows immediately that $p_1(M) = 0$, and hence M is locally conformal Kähler surface with $\tau = 3\tau^*$ by Theorem 2.4. Then, by (2.6), we have

$$\tau = 12c > \tau^* = 4c.$$

Since M is simply connected, M is a globally conformal Kähler surface. Thus, there exists a differentiable function f on M such that $\omega = df$. Then the equality (1.5) reduces to

$$\tau - \tau^* = -2\Delta f + \|\mathrm{grad} f\|^2, \tag{2.45}$$

where $\Delta = -\delta d$. Let p_0 be a point of M such that $f(p_0) = \min_{p \in M} f(p)$. Then, by (2.45), we see that $\tau \leq \tau^*$ at p_0. But this is a contradiction, and hence $c_2(M) = \chi(M) = 3$. Consequently, we have

$$c_1(M)^2 = 9 \quad \text{and} \quad p_1(M) = 3. \tag{2.46}$$

The second Betti number $b_2(M)$ of M is written as usual by $b_2(M) = b_+ + b_-$, and the signature of M $(= \frac{1}{3}p_1(M))$ is equal to $b_+ - b_-$. Since $\chi(M) = 2 + b_2(M) = 2 + b_+ + b_-$ and $b_+ - b_- = 1$ by (2.46), we have

$$b_+ = 1, \quad b_- = 0. \tag{2.47}$$

Summing up the above arguments, we can conclude that M is biholomorphically equivalent to a complex projective surface $P^2(\mathbb{C})$. This completes the proof of the theorem. ∎

Bibliography

[1] E. Abbena: An example of an almost Kähler manifold which is not Kählerian, Boll. Un. Mat. Ital. (6) 3A (1984), 383–392.

[2] A. Balas: Compact Hermitian manifolds of constant holomorphic sectional curvature, Math. Z. 189(1985), 193–210.

[3] A. Balas: On the sum of the Hermitian scalar curvatures of a compact Hermitian manifold, Math. Z. 195(1987), 429–432.

[4] A. Balas and P. Gauduchon: Any Hermitian metric of constant non-positive (Hermitian) holomorphic sectional curvature on a compact complex surface is Kähler, Math. Z. 190(1985), 39–43.

[5] M. Berger: Le spectre des variétés riemanniennes, Rev. Roumaine Math. Pures Appl. 13(1968), 915–931.

[6] M. Berger, P. Gauduchon and E. Mazet: *Le spectre d'une variété riemannienne*, Lecture Notes in Math. 194, Springer (1971).

[7] R. L. Bishop and S. I. Goldberg: On the second cohomology group of a Kähler manifold of positive curvature, Proc. Amer. Math. Soc. 16(1965), 119–122.

[8] R. L. Bishop and S. I. Goldberg: Rigidity of positively curved Kähler manifolds, Proc. Nat. Acad. Sci. 54(1965), 1037–1041.

[9] R. L. Bishop and S. I. Goldberg: On the topology of positively curved Kähler manifolds II, Tôhoku Math. J. 17(1965), 310–318.

[10] D. E. Blair: Nonexistence of 4-dimensional almost Kähler manifolds of constant curvature, Proc. Amer. Math. Soc. 110(1990), 1033–1039.

[11] E. Bombieri and D. Husemoller: Classification and embeddings of surfaces, Proc. Symp. Pure Math. 29(1975), 329–420.

[12] L. A. Cordero, M. Fernandez and M. de Leon: Examples of compact non-Kähler almost Kähler manifolds, Proc. Amer. Math. Soc. 95(1985), 280–286.

[13] A. Frölicher: Zur Differentialgeometrie der komplexen Strukturen, Math. Ann. 129(1955), 50–95.

[14] T. Fukami and S. Ishihara: Almost Hermitian structure on S^6, Tôhoku Math. J. 7(1955), 151–156.

[15] S. I. Goldberg: Integrability of almost Kähler manifolds, Proc. Amer. Math. Soc. 21(1969), 96–100.

[16] A. Gray: Some examples of almost Hermitian manifolds, Illinois J. Math. 10(1966), 353–366.

[17] A. Gray: Almost complex submanifolds of the six sphere, Proc. Amer. Math. Soc. 20(1969), 277–279.

[18] A. Gray: Nearly Kähler manifolds, J. Differential Geometry 4(1970), 283–309.

[19] A. Gray: Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geometry 7(1972), 343–369.

[20] A. Gray: Curvature identities for Hermitian and almost Hermitian manifolds, Tôhoku Math. J. 28(1976), 601–612.

[21] A. Gray: Compact Kähler manifolds with nonnegative sectional curvature, Invent. Math. 41(1977), 33–43.

[22] A. Gray and L. Vanhecke: Almost Hermitian manifolds with constant holomorphic sectional curvature, Časopis pro pěstování mat. 104(1979), 170–179.

[23] S. Kobayashi and K. Nomizu: *Foundations of differential geometry*, I, II, Interscience Publishers, New York, 1963, 1969.

[24] K. Kodaira: On the structure of compact analytic surfaces I, Amer. J. Math. 86(1964), 751–798.

[25] M. Matsumoto: On 6-dimensional almost Tachibana spaces, Tensor, N.S. 23(1972), 250–252.

[26] Y. Miyaoka: On the Chern numbers of surfaces of general type, Invent. Math. 42(1977), 225–237.

[27] A. Newlander and L. Nirenberg: Complex analytic coordinates in almost complex manifolds, Ann. of Math. 65(1957), 391–404.

[28] Z. Olszak: A note on almost Kähler manifolds, Bull. Acad. Polon. Sci. 26(1978), 139–141.

[29] R. Osserman: Curvatures in the eighties, Amer. Math. Monthly 97(1990), 731–756.

[30] T. Sakai: On eigen-values of Laplacian and curvature of Riemannian manifold, Tôhoku Math. J. 23(1971), 589–603.

[31] S. Sasaki: On the differential geometry of tangent bundle of Riemannian manifolds, Tôhoku Math. J. 10(1958), 338–354.

[32] T. Sato: Riemannian 3-symmetric spaces and homogeneous K-spaces, Mem. Fac. Tech. Kanazawa Univ. 12(1979), 137–143.

[33] T. Sato: The spectrum of the Laplacian for some 6-dimensional K-spaces, Kodai Math. J. 4(1981), 129–136.

[34] T. Sato: On a certain class of Riemannian k-symmetric spaces, Tensor, N.S. 42(1985), 55–65.

[35] T. Sato: On some almost Hermitian manifolds with constant holomorphic sectional curvature, Kyungpook Math. J. 29(1989), 11–25.

[36] T. Sato: On some compact almost Kähler manifolds with constant holomorphic sectional curvature, *Geometry of Manifolds*, Ed. by Shiohama, Academic Press,1989, 129–139.

[37] T. Sato and K. Sekigawa: Hermitian surfaces of constant holomorphic sectional curvature, Math. Z. 205(1990), 659–668.

[38] T. Sato and K. Sekigawa: Hermitian surfaces of constant holomorphic sectional curvature II, submitted.

[39] S. Sawaki and K. Sekigawa: Almost Hermitian manifolds with constant holomorphic sectional curvature, J. Differential Geometry 9(1974), 123–134.

[40] S. Sawaki, Y. Watanabe and T. Sato: Notes on a K-space of constant holomorphic sectional curvature, Kōdai Math. Sem. Rep. 26(1975), 438–445.

[41] K. Sekigawa: Notes on homogeneous almost Hermitian manifolds, Hokkaido Math. J. 7(1978), 206–213.

[42] K. Sekigawa: Almost Hermitian manifolds satisfying some curvature conditions, Kodai Math. J. 2(1979), 384–405.

[43] K. Sekigawa: On some 4-dimensional compact Einstein almost Kähler manifolds, Math. Ann. 271(1985), 333–337.

[44] K. Sekigawa: On some compact Einstein almost Kähler manifolds, J. Math. Soc. Japan 39(1987), 677–684.

[45] K. Sekigawa: On some 4-dimensional compact almost Hermitian manifolds, J. Ramanujan Math. Soc. 2(1987), 101–116.

[46] K. Sekigawa and T. Sato: Nearly Kähler manifolds with positive holomorphic sectional curvature, Kodai Math. J. 8(1985), 139–156.

[47] K. Sekigawa and J. Watanabe: On some compact Riemannian 3-symmetric spaces, Sci. Rep. Niigata Univ. 19(1983), 1–17.

[48] S. Tachibana: On infinitesimal conformal and projective transformations of compact K-spaces, Tôhoku Math. J. 13(1961), 386–392.

[49] S. Tachibana and M. Okumura: On the almost-complex structure of tangent bundles of Riemannian spaces, Tôhoku Math. J. 14(1962), 156–161.

[50] K. Takamatsu: Some properties of 6-dimensional K-spaces, Kōdai Math. Sem. Rep. 23(1971), 215–232.

[51] K. Takamatsu and T. Sato: A K-space of constant holomorphic sectional curvature, Kōdai Math. Sem. Rep. 27(1976), 116–127.

[52] S. Tanno: Constancy of holomorphic sectional curvature in almost Hermitian manifolds, Kōdai Math. Sem. Rep. 25(1973), 190–201.

[53] W. P. Thurston: Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc. 55(1976), 467–468.

[54] F. Tricerri and I .Vaisman: On some 2-dimensional Hermitian manifolds, Math. Z. 192(1986), 205–216.

[55] F. Tricerri and L .Vanhecke: Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc. 267(1981), 365–398.

[56] F. Tricerri and L .Vanhecke: *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series 83, Cambridge Univ. Press, 1983.

[57] I. Vaisman: Some curvature properties of complex surfaces, Ann. Mat. Pura Appl. 32(1982), 1–18.

[58] L. Vanhecke: Almost Hermitian manifolds with J-invariant Riemannian curvature tensor, Rend. Sem. Mat. Univ. e Politec. Torino 34(1975-76), 487–498.

[59] L. Vanhecke: Some almost Hermitian manifolds with constant holomorphic sectional curvature, J. Differential Geometry 12(1977), 461–471.

[60] Y. Watanabe and K. Takamatsu: On a K-space of constant holomorphic sectional curvature, Kōdai Math. Sem. Rep. 25(1973), 297–306.

[61] B. Watson: New examples of strictly almost Kähler manifolds, Proc. Amer. Math. Soc. 88(1983), 541–544.

[62] J. A. Wolf and A. Gray: Homogeneous spaces defined by Lie group automorphisms, I, II, J. Differential Geometry 2(1968), 77–114.

[63] W. T. Wu: Sur les classes caractéristiques des structures fibrées sphériques, Actual. Sci. Ind. 1183(1952), 1–89.

[64] K. Yano: *Differential geometry on complex and almost complex spaces*, Pergamon Press, 1965.

[65] K. Yano and S. Bochner: *Curvature and Betti numbers*, Annals of Math. Studies 32, Princeton Univ. Press, 1953.