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Singular inner functions of L^1-type II

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Abstract. In the first paper of the same title, we introduced the concept of singular inner functions of L^1-type and obtained results for singular inner functions which are reminiscent of the results for weak infinite powers of Blaschke products. In this paper, we investigate singular inner functions for discrete measures. We give equivalent conditions on a measure for which it is a Blaschke type. And we prove that two discrete measures are mutually singular if and only if the associated common zero sets of singular inner functions of l_+^∞-type do not meet.

1. Introduction.

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disc Δ. We denote by $\mathscr{M} = M(H^\infty)$ the maximal ideal space of H^∞, the space of nonzero multiplicative linear functionals of H^∞ with the weak*-topology. We view Δ as $\Delta \subset \mathscr{M}$, and Δ is an open subset of $\mathscr{M}$. By Carleson's corona theorem [2], Δ is dense in $\mathscr{M}$. Identifying a function in H^∞ with its Gelfand transform, we view H^∞ as the closed subalgebra of $C(\mathscr{M})$, the space of continuous functions on $\mathscr{M}$. We also identify a function in H^∞ with its boundary function and view H^∞ as an (essentially) supremum norm closed subalgebra of L^∞, the usual Lebesgue space on the unit circle $\partial\Delta$. Then we view the maximal ideal space $M(L^\infty)$ of L^∞ as a subset of $\mathscr{M}$ and $M(L^\infty)$ is the Shilov boundary of H^∞. A function f in H^∞ is called inner if $|f| = 1$ on $M(L^\infty)$. For a function f in H^∞, we put

$$\{|f| < 1\} = \{x \in \mathscr{M}\setminus\Delta; |f(x)| < 1\} \quad \text{and} \quad Z(f) = \{x \in \mathscr{M}\setminus\Delta; f(x) = 0\}.$$

We note that these sets are considered in $\mathscr{M}\setminus\Delta$. For $e^{i\theta} \in \partial\Delta$, let $\mathscr{M}_{e^{i\theta}} = \{x \in \mathscr{M}; z(x) = e^{i\theta}\}$, where z is the identity function on Δ. Then $\mathscr{M}\setminus\Delta = \bigcup\{\mathscr{M}_{e^{i\theta}}; e^{i\theta} \in \partial\Delta\}$. For a subset E of $\mathscr{M}$, we denote by $\bar{E}$ the weak*-closure of E in $\mathscr{M}$. See [4], [9], [10] for studies of the structure of H^∞ and $\mathscr{M}$.

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For a sequence $\{z_n\}_n$ in Δ satisfying $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$, we have a Blaschke product

$$b(z)=\prod_{n=1}^{\infty}\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}, \quad z\in\Delta.$$

We denote by $\mathscr{P}(b)$ the set of sequences of positive integers $p=(p_1,p_2,\dots)$ such that $\sum_{n=1}^{\infty}p_n(1-|z_n|)<\infty$ and $p_n\to\infty$ as $n\to\infty$. For $p=(p_1,p_2,\dots)\in\mathscr{P}(b)$ we have an associated Blaschke product defined by

$$b^p(z)=\prod_{n=1}^{\infty}\left(\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right)^{p_n}, \quad z\in\Delta.$$

In [**13**], the author called Blaschke products $b^p, p\in\mathscr{P}(b)$, weak infinite powers of b and studied them.

In [**14**], we obtained results for singular inner functions which are reminiscent of the results for Blaschke products in [**13**]. This paper is a continuation of these papers and we use the same notations as in them. We denote by $M(\partial\Delta)$ the Banach space of bounded Borel measures on $\partial\Delta$ with the total variation norm. Since $M(\partial\Delta)$ is the dual space of $C=C(\partial\Delta)$, the space of continuous functions on $\partial\Delta$, we can consider the weak*-topology on $M(\partial\Delta)$. Let M_s^+ be the set of positive (nonzero) singular measures in $M(\partial\Delta)$ with respect to the Lebesgue measure on $\partial\Delta$.

For each $\mu\in M_s^+$, let

$$\psi_\mu(z)=\exp\left(-\int_{\partial\Delta}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(e^{i\theta})\right), \quad z\in\Delta.$$

Then ψ_μ is inner and called a singular inner function. We note that

$$|\psi_\mu(z)|=\exp\left(-\int_{\partial\Delta}P_z(e^{i\theta})\,d\mu(e^{i\theta})\right), \quad z\in\Delta,$$

where P_z is the Poisson kernel. Hence if $0\le\nu\le\mu, \nu,\mu\in M_s^+$, then $|\psi_\mu|\le|\psi_\nu|$ on $\mathscr{M}$. Let

$$L_+^1(\mu)=\{\nu\in M_s^+; 0\le\nu\ll\mu, \nu\neq 0\}.$$

Then we have a family of singular inner functions $\{\psi_\nu;\nu\in L_+^1(\mu)\}$. In [**14**], we call these functions singular inner functions of L^1-type for the measure μ. Let $\mathscr{R}(\mu)=\bigcup\{\{|\psi_\nu|<1\};\nu\in L_+^1(\mu)\}$ and $\mathscr{R}_0(\mu)=\bigcup\{Z(\psi_\nu);\nu\in L_+^1(\mu)\}$. In [**14**], we study $\mathscr{R}(\mu)$ and $\mathscr{R}_0(\mu)$, and obtain similar theorems as in [**13**].

In [**13**], the author proved that

$$(\alpha)\qquad \overline{\{|b|<1\}}=\bigcap\{\{|b^p|<1\};p\in\mathscr{P}(b)\}=\bigcap\{Z(b^p);p\in\mathscr{P}(b)\}$$

for every Blaschke product b. In this paper, we investigate similar type of theorems for singular inner functions. It is not difficult to show that $\bigcap\{\{|\psi_\nu| < 1\}; \nu \in L^1_+(\mu)\} = \emptyset$ for $\mu \in M_s^+$ with $\mu \neq c\delta_{e^{i\theta}}$, where $\delta_{e^{i\theta}}$ is the unit point mass at $e^{i\theta}$ and c is a positive number. So to have similar theorems like (α), we need to consider subclasses of $L^1_+(\mu)$.

In this paper, we concentrate on discrete measures. We denote by $M_{s,d}^+$ the set of positive discrete measures in M_s^+. We call $\psi_\mu, \mu \in M_{s,d}^+$, discrete singular functions. When $\mu \in M_{s,d}^+$ is a sum of finitely many point measures, it is easy to study properties of ψ_μ. So in this paper, we assume that μ is a sum of infinitely many point measures, and we can write

$$\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}}, \quad \sum_{n=1}^{\infty} a_n < \infty, \quad a_n > 0 \text{ for every } n, \text{ and } e^{i\theta_n} \neq e^{i\theta_k} \text{ for } n \neq k.$$

Then

$$|\psi_\mu(z)| = \prod_{n=1}^{\infty} |\psi_{e^{i\theta_n}}(z)|^{a_n}, \quad z \in \varDelta.$$

As an analogy of $\mathscr{P}(b)$, where b is a Blaschke product, we define $\mathscr{P}(\mu)$ as the set of sequences of positive numbers $p = (p_1, p_2, \dots)$ such that $\sum_{n=1}^{\infty} p_n a_n < \infty$, $p_n \geq 1$, and $p_n \to \infty$ as $n \to \infty$. For $p \in \mathscr{P}(\mu)$, we put

$$\mu^p = \sum_{n=1}^{\infty} p_n a_n \delta_{e^{i\theta_n}}.$$

Measures $\psi_{\mu^p}, p \in \mathscr{P}(\mu)$, are called weak infinite powers of ψ_μ. Then $\mu^p \in L^1_+(\mu)$ and it is expected that the set $\{\psi_{\mu^p}; p \in \mathscr{P}(\mu)\}$ acts as $\{b^p; p \in \mathscr{P}(b)\}$.

In Section 2, we prove

$$\begin{aligned}\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} &= \left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right) \cup \left(\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}\right) \\ &\subset \overline{\{|\psi_\mu| < 1\}}.\end{aligned}$$

In Section 3, we give equivalent conditions on $\mu \in M_{s,d}^+$ for which

$$\overline{\{|\psi_\mu| < 1\}} = \bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}.$$

For every Blaschke product b, (α) holds. Hence, when the above condition is satisfied, we say that ψ_μ is a discrete singular inner function of Blaschke type.

Since $\mu \in M_{s,d}^+$, we can consider another subclass of $L^1_+(\mu)$. Let l^∞_+ be the set of sequences of bounded and positive numbers. For $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$ and $p = (p_1, p_2, \dots) \in l^\infty_+$, we have $\sum_{n=1}^{\infty} p_n a_n < \infty$, so that we can define μ^p in the same way as before. And we have

$$\{|\psi_{\mu^p}| < 1\} \subset \{|\psi_{\mu^q}| < 1\} \quad \text{and} \quad Z(\psi_{\mu^p}) \subset Z(\psi_{\mu^q})$$

for every $p \in l_+^\infty$ and $q \in \mathscr{P}(\mu)$. We call ψ_{μ^p}, $p \in l_+^\infty$, discrete singular functions of l_+^∞-type.

In Section 4, we study the sets $\bigcap\{Z(\psi_{\mu^p}); p \in l_+^\infty\}$ and $\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\}$. We prove that the both sets can be described only using $\{e^{i\theta_n}\}_n$ and are strictly smaller than $\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}$ and $\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}$, respectively. And we prove that if $\mu \perp \lambda, \mu, \lambda \in M_{s,d}^+$ then

$$\left(\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\}\right) \cap \left(\bigcap\{\{|\psi_{\lambda^q}| < 1\}; q \in l_+^\infty\}\right) = \varnothing.$$

2. Weak infinite powers of discrete singular functions.

Let $\mu = \sum_{n=1}^\infty a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Put

$$\mu_n = \sum_{k=n}^\infty a_k \delta_{e^{i\theta_k}} \quad \text{for every } n.$$

Then $\mu_1 = \mu, \mu_n \geq \mu_{n+1}$, and $\|\mu_n\| \to 0$ as $n \to \infty$. Recall that $\mathscr{P}(\mu)$ is the set of sequences of positive numbers $p = (p_1, p_2, \dots)$ such that $\sum_{n=1}^\infty p_n a_n < \infty$, $p_n \geq 1$, and $p_n \to \infty$ as $n \to \infty$. Then $\mu \leq \mu^p$ for every $p \in \mathscr{P}(\mu)$. We note that $\{|\psi_{\delta_{e^{i\theta}}}| < 1\} \subset \mathscr{M}_{e^{i\theta}}$, and use this fact without mentioning it. In this section, we prove the following two theorems.

THEOREM 2.1. *Let $\mu = \sum_{n=1}^\infty a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Put $\mu_n = \sum_{k=n}^\infty a_k \delta_{e^{i\theta_k}}$ for every n. Then*

$$\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\} = \left(\bigcup_{n=1}^\infty Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^\infty \{|\psi_{\mu_n}| < 1\}} \subset \overline{\{|\psi_\mu| < 1\}}.$$

THEOREM 2.2. *Let $\mu = \sum_{n=1}^\infty a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Put $\mu_n = \sum_{k=n}^\infty a_k \delta_{e^{i\theta_k}}$ for every n. Then*

(i) $$\bigcap\{\{|\psi_{\mu^p}|<1\}; p \in \mathscr{P}(\mu)\} = \left(\bigcup_{n=1}^\infty \{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right) \cup \left(\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}\right)$$

$$= \left(\bigcup_{n=1}^\infty \{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right) \cup \overline{\bigcap_{n=1}^\infty \{|\psi_{\mu_n}|<1\}} \subset \overline{\{|\psi_\mu|<1\}}.$$

(ii) $$\overline{\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}} = \bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in \mathscr{P}(\mu)\} = \overline{\{|\psi_\mu| < 1\}}.$$

To prove our theorem, we need some lemmas.

LEMMA 2.1. *Let* $\mu=\sum_{n=1}^{\infty} a_n\delta_{e^{i\theta_n}} \in M_{s,d}^{+}$ *such that* $a_n>0$ *for every* n *and* $e^{i\theta_n} \neq e^{i\theta_k}$ *for* $n \neq k$. *Put* $\mu_n=\sum_{k=n}^{\infty} a_k\delta_{e^{i\theta_k}}$. *Then*

(i)
$$Z(\psi_\mu)=\left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \left(\bigcap_{n=1}^{\infty} Z(\psi_{\mu_n})\right).$$

(ii)
$$\{|\psi_\mu|<1\}=\left(\bigcup_{n=1}^{\infty} \{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right) \cup \left(\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}|<1\}\right).$$

PROOF. We have

$$\mu=\sum_{k=1}^{n-1} a_k\delta_{e^{i\theta_k}}+\mu_n.$$

Then

$$\psi_\mu=\psi_{\mu_n}\prod_{k=1}^{n-1} \psi_{\delta_{e^{i\theta_k}}}^{a_k}.$$

Hence $\psi_\mu(x)=0$ if and only if either $x \in Z(\psi_{\delta_{e^{i\theta_n}}})$ for some n or $x \in Z(\psi_{\mu_n})$ for every n. Also $|\psi_\mu(x)|<1$ if and only if either $|\psi_{\delta_{e^{i\theta_n}}}(x)|<1$ for some n or $|\psi_{\mu_n}(x)|<1$ for every n.

LEMMA 2.2. *Let* $\mu=\sum_{n=1}^{\infty} a_n\delta_{e^{i\theta_n}} \in M_{s,d}^{+}$ *such that* $a_n>0$ *for every* n *and* $e^{i\theta_n} \neq e^{i\theta_k}$ *for* $n \neq k$. *Put* $\mu_n=\sum_{k=n}^{\infty} a_k\delta_{e^{i\theta_k}}$. *Then*

(i) $|\psi_{\mu_n}| \leq |\psi_{\mu_{n+1}}|$ *on* $\mathscr{M}$ *for every* n.

(ii) $|\psi_{\mu^p}| \leq |\psi_\mu| \leq |\psi_{\mu_n}| \leq |\psi_{\delta_{e^{i\theta_n}}}|$ *on* $\mathscr{M}$ *for every* $p \in \mathscr{P}(\mu)$ *and* n.

(iii) $\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\} \subset \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}$.

PROOF. (i) and (ii) follow from $\delta_{e^{i\theta_n}} \leq \mu_n \leq \mu_{n-1} \leq \mu \leq \mu^p$ for every $p \in \mathscr{P}(\mu)$ and n.

(iii) Let $x \in \mathscr{M} \backslash \varDelta$ satisfying

$$|\psi_{\mu_n}(x)|<1 \quad \text{for every } n, \tag{2.1}$$

and $p=(p_1,p_2,\ldots) \in \mathscr{P}(\mu)$. Put $p_n'=\inf\{p_k; k \geq n\}$. Then $p'=(p_1',p_2',\ldots) \in \mathscr{P}(\mu)$ and

$$p_n'\mu_n \leq \sum_{k=n}^{\infty} p_n' a_k\delta_{e^{i\theta_k}} \leq \sum_{k=n}^{\infty} p_k a_k\delta_{e^{i\theta_k}} \leq \mu^p.$$

Hence

$$|\psi_{\mu^p}| \leq |\psi_{\mu_n}|^{p_n'} \quad \text{on } \mathscr{M} \text{ for every } n. \tag{2.2}$$

Let n_0 be the smallest positive integer such that $x \notin \bigcup_{n=n_0}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}$. Then we have $|\psi_{\mu_n}(x)| = |\psi_{\mu_{n_0}}(x)|$ for $n \geq n_0$. Since $p_n' \to \infty$ as $n \to \infty$, by (2.1) and (2.2) we have $\psi_{\mu^p}(x) = 0$.

For $f \in H^\infty$, we put $\{f \neq 0\} = \{x \in \mathscr{M}\backslash\Delta; f(x) \neq 0\}$.

Lemma 2.3. *Let $\mu = \sum_{n=1}^{\infty} a_n\delta_{e^{i\theta_n}} \in M_{s,d}^+$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Then we have the following.*

(i) *For each $q \in \mathscr{P}(\mu)$, there exists $p \in \mathscr{P}(\mu)$ such that $|\psi_{\mu^p}| = 1$ on $\{\psi_{\mu^q} \neq 0\}\backslash\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}$.*

(ii) *Let $x \in \mathscr{M}\backslash\Delta$. If $x \notin \overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}$ and $\mu_{\mu^q}(x) \neq 0$ for some $q \in \mathscr{P}(\mu)$, then there exists $p \in \mathscr{P}(\mu)$ such that $x \notin \overline{\{|\psi_{\mu^p}|<1\}}$.*

Proof. (i) Let $x \in \mathscr{M}\backslash\Delta$ such that $\psi_{\mu^q}(x) \neq 0$ and $x \notin \bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}$. Since $q \in \mathscr{P}(\mu)$, there exists a sequence of increasing positive numbers $\{t_n\}_n$ such that $p \in \mathscr{P}(\mu)$ and $t_n \to \infty$, where $p = (p_1, p_2, \ldots) = (q_1/t_1, q_2/t_2, \ldots)$. Then

$$\begin{aligned}
t_n\mu^p &= t_n\sum_{k=1}^{n-1} p_k a_k \delta_{e^{i\theta_k}} + \sum_{k=n}^{\infty} t_n p_k a_k \delta_{e^{i\theta_k}} \\
&\leq t_n\sum_{k=1}^{n-1} p_k a_k \delta_{e^{i\theta_k}} + \sum_{k=n}^{\infty} q_k a_k \delta_{e^{i\theta_k}} \\
&\leq t_n\sum_{k=1}^{n-1} p_k a_k \delta_{e^{i\theta_k}} + \mu^q.
\end{aligned}$$

Since $|\psi_{\delta_{e^{i\theta_n}}}(x)| = 1$ for every n, by the above we have $|\psi_{\mu^q}(x)| \leq |\psi_{\mu^p}(x)|^{t_n}$. Since $\psi_{\mu^q}(x) \neq 0$ and $t_n \to \infty$, we have $|\psi_{\mu^p}(x)| = 1$.

(ii) By our assumption, there exists an open subset V of $\mathscr{M}\backslash\Delta$ such that

$$x \in V \quad \text{and} \quad V \cap \overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}} = \varnothing.$$

Let $U = \{y \in V; |\psi_{\mu^q}(y)| > |\psi_{\mu^q}(x)|/2\}$. Then U is an open subset of $\mathscr{M}\backslash\Delta$ and $x \in U$. By (i), there exists $p \in \mathscr{P}(\mu)$ such that $|\psi_{\mu^p}| = 1$ on U. This implies our assertion.

Now we give the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. By Lemma 2.2(ii) and (iii), we have

$$\left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}} \subset \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}. \tag{2.3}$$

To prove

$$(2.4)\qquad \bigcap\{Z(\psi_{\mu^p});\, p \in \mathscr{P}(\mu)\} \subset \left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}},$$

let $x \in \mathscr{M}\backslash\varDelta$ such that

$$(2.5)\qquad x \notin \left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

We shall prove the existence of

$$(2.6)\qquad \nu = \mu^p, p \in \mathscr{P}(\mu), \quad \text{such that } \psi_\nu(x) \neq 0.$$

By (2.5) and Lemma 2.1(i),

$$(2.7)\qquad \psi_\mu(x) \neq 0$$

and by the corona theorem [**2**] there exists an open subset U of $\varDelta$ such that

$$(2.8)\qquad x \in \bar{U}$$

and

$$(2.9)\qquad \bar{U} \cap \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}} = \varnothing.$$

Here we show that

$$(2.10)\qquad |\psi_{\mu_n}| \to 1 \quad \text{uniformly on } U \quad \text{as} \quad n \to \infty.$$

To prove this, suppose not. Then there exist $0 < \delta < 1$ and a sequence $\{z_n\}_n$ in U such that

$$(2.11)\qquad |\psi_{\mu_n}(z_n)| < \delta \quad \text{for every } n.$$

Since $\|\mu_n\| \to 0$, $|\psi_{\mu_n}| \to 1$ uniformly on compact subsets of $\varDelta$ as $n \to \infty$. Hence by (2.11), $|z_n| \to 1$. By Lemma 2.2(i), $|\psi_{\mu_n}(z_k)| < \delta$ for every positive integer k and n with $k \geq n$. Let $y \in \overline{\{z_k\}_k}\backslash\{z_k\}_k$. Then $|\psi_{\mu_n}(y)| \leq \delta$ for every n. Since $y \in \bar{U}$, this contradicts (2.9). Thus we get (2.10).

Let $\{\varepsilon_k\}_k$ be a sequence of positive numbers such that

$$(2.12)\qquad \prod_{k=1}^{\infty} \varepsilon_k > 0 \quad \text{and} \quad 0 < \varepsilon_k < 1.$$

For each k, by (2.10) there exists a positive integer n_k such that

$$(2.13)\qquad |\psi_{\mu_{n_k}}| \geq \varepsilon_k \quad \text{on } U.$$

Since $\|\mu_n\| \to 0$ as $n \to \infty$, taking a sufficiently large n_k we may assume moreover that

$$\sum_{k=1}^{\infty} \|\mu_{n_k}\| < \infty.$$

Put

$$\sigma = \sum_{k=1}^{\infty} \mu_{n_k} \quad \text{and} \quad \nu = \mu + \sigma. \tag{2.14}$$

Then $\nu = \mu^p$ for some $p \in \mathscr{P}(\mu)$. We have $\psi_\nu(x) = \psi_\mu(x)\psi_\sigma(x)$. Hence to prove (2.6), by (2.7) it is sufficient to prove

$$\psi_\sigma(x) \neq 0. \tag{2.15}$$

By (2.13) and (2.14), we have

$$|\psi_\sigma| = \prod_{k=1}^{\infty} |\psi_{\mu_{n_k}}| \geq \prod_{k=1}^{\infty} \varepsilon_k \quad \text{on } U.$$

Hence by (2.12),

$$|\psi_\sigma(z)| \geq \prod_{k=1}^{\infty} \varepsilon_k > 0 \quad \text{for every } z \in U.$$

Thus by (2.8) we have (2.15), so that (2.6) holds. Therefore (2.4) holds.

By (2.3) and (2.4),

$$\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\} = \left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

By Lemma 2.1(ii), we have

$$\left(\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}} \subset \overline{\{|\psi_\mu| < 1\}}.$$

This completes the proof.

PROOF OF THEOREM 2.2. (i) By Lemma 2.2(ii),

$$\{|\psi_\mu| < 1\} \subset \bigcap\{\{|\mu_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}. \tag{2.16}$$

We shall prove that

$$\left(\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}\right)\backslash\{|\psi_\mu| < 1\} \subset \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}. \tag{2.17}$$

Let

$$x \in \left(\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}\right) \setminus \{|\psi_\mu| < 1\}.$$

Then $|\psi_{\delta_{e^{i\theta_n}}}(x)| = 1$ for every n and $|\psi_{\mu^p}(x)| < 1$ for every $p \in \mathscr{P}(\mu)$. By Lemma 2.3(i), we have $x \in \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}$. Thus we get (2.17).

Now we have

$$\begin{aligned}\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} &\subset \{|\psi_\mu| < 1\} \cup \left(\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}\right) \quad \text{by (2.17)} \\ &\subset \bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} \quad \text{by (2.16)}.\end{aligned}$$

Hence by Lemma 2.1(ii) and Theorem 2.1, we have

$$\begin{aligned}\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} &= \{|\psi_\mu| < 1\} \cup \left(\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}\right) \\ &= \left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right) \cup \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.\end{aligned}$$

(ii) We have

$$\begin{aligned}\overline{\{|\psi_\mu| < 1\}} &= \overline{\left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right)} \cup \overline{\left(\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}\right)} \quad \text{by Lemma 2.1(ii)} \\ &= \overline{\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}} \quad \text{by (i)} \\ &\subset \bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in \mathscr{P}(\mu)\}.\end{aligned}$$

To prove

$$\bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in \mathscr{P}(\mu)\} \subset \overline{\{|\psi_\mu| < 1\}}, \tag{2.18}$$

let $x \notin \overline{\{|\psi_\mu| < 1\}}$. Then by Theorem 2.1, $x \notin \cap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}$. Hence there exists $q \in \mathscr{P}(\mu)$ such that $\psi_{\mu^q}(x) \neq 0$. Since $x \notin \overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}}$, by Lemma 2.3(ii), there exists $p \in \mathscr{P}(\mu)$ such that $x \notin \overline{\{|\psi_{\mu^p}| < 1\}}$. Therefore $x \notin \bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in \mathscr{P}(\mu)\}$, so that (2.18) holds. Thus (ii) holds. This completes the proof.

3. Discrete singular functions of Blaschke type.

Recall that for a Blaschke product b,

$$\overline{\{|b| < 1\}} = \bigcap\{\overline{\{|b^p| < 1\}}; p \in \mathscr{P}(b)\} = \bigcap\{Z(b^p); p \in \mathscr{P}(b)\}.$$

A measure $\mu \in M_{s,d}^{+}$ satisfying $\overline{\{|\psi_\mu| < 1\}} = \bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}$ is called a Blaschke type. In this section, we study discrete singular measures of Blaschke type. The following is our theorem.

THEOREM 3.1. *Let* $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^{+}$ *such that* $a_n > 0$ *for every* n *and* $e^{i\theta_n} \neq e^{i\theta_k}$ *for* $n \neq k$. *Put* $\mu_n = \sum_{k=n}^{\infty} a_k \delta_{e^{i\theta_k}}$ *for every* n. *Then the following conditions are equivalent.*

(a) $$\overline{\{|\psi_\mu| < 1\}} = \bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}.$$

(b) $$\overline{\{|\psi_\mu| < 1\}} = \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}.$$

(c) $$\overline{\{|\psi_\mu| < 1\}} = \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(d) $$\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} = \bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}.$$

(e) $$\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} = \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(f) $$\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\} = \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(g) $$\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(h) $$\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\} \quad \textit{is closed}.$$

Hence if $\mu \in M_{s,d}^{+}$ *is of Blaschke type then each subset appeared in conditions* (a)–(f) *coincides with the others.*

To prove Theorem 3.1, we need some lemmas.

LEMMA 3.1. *Let* $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^{+}$ *such that* $a_n > 0$ *and* $e^{i\theta_n} \neq e^{i\theta_k}$ *for* $n \neq k$. *Put* $\mu_n = \sum_{k=n}^{\infty} a_k \delta_{e^{i\theta_k}}$. *Then we have the following.*

(i) (a) *holds if and only if*

$$\overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}} \Big\backslash \left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right) \subset \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(ii) (c) *holds if and only if*

$$\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset \overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}| < 1\}}.$$

(iii) (d) *holds if and only if*

$$\bigcup_{n=1}^{\infty}\left(\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\backslash Z(\psi_{\delta_{e^{i\theta_n}}})\right)\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}.$$

(iv) (e) *holds if and only if*

$$\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}.$$

(v) (f) *holds if and only if*

$$\bigcup_{n=1}^{\infty}Z(\delta_{e^{i\theta_n}})\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}.$$

PROOF. (i) follows Lemma 2.1(ii) and Theorem 2.2(i). (ii) follows Lemma 2.1(ii). (iii) follows Theorems 2.1 and 2.2. (iv) follows Theorem 2.2(i). (v) follows Theorem 2.1.

Let $QC=(H^\infty+C)\cap\overline{(H^\infty+C)}$, where C is the space of continuous functions on $\partial\Delta$ and $\overline{H^\infty+C}=\{\bar{f};f\in H^\infty+C\}$. For $\zeta\in M(L^\infty)$, let $Q=\{\xi\in M(L^\infty);f(\xi)=f(\zeta)$ for every $f\in QC\}$. This set Q is called a QC-level set. For $x\in\mathcal{M}$, there is a unique probability measure μ_x on $M(L^\infty)$ such that $\int_{M(L^\infty)}f\,d\mu_x=f(x)$ for every $f\in H^\infty$. We denote by $\operatorname{supp}\mu_x$ the closed support set of μ_x. It is known that for $x\in\mathcal{M}\backslash\Delta$ and a QC-level set Q, $\operatorname{supp}\mu_x\subset Q$ or $\operatorname{supp}\mu_x\cap Q=\emptyset$. And there exists a unique QC-level set Q_x such that $\operatorname{supp}\mu_x\subset Q_x$. See **[11]**, **[12]**, **[15]** for the study of QC-level sets.

LEMMA 3.2. *Let φ be an inner function and x a point in $\mathcal{M}\backslash\Delta$ such that $|\varphi(x)|<1$. Then we have the following.*

(i) *There exists $y\in\mathcal{M}$ such that $\operatorname{supp}\mu_y\subset\operatorname{supp}\mu_x$ and $0<|\varphi(y)|<1$.*

(ii) *There exists $\zeta\in\mathcal{M}$ such that $\operatorname{supp}\mu_\zeta\subset\operatorname{supp}\mu_x$ and $\zeta\in\overline{\{|\varphi|<1\}}\backslash\{|\varphi|<1\}$.*

SKETCH OF PROOF. See [3] in detail. Let $H^\infty_{|\operatorname{supp}\mu_x}$ be the restriction algebra on $\operatorname{supp}\mu_x$. Then $H^\infty_{|\operatorname{supp}\mu_x}$ is a closed subalgebra of $C(\operatorname{supp}\mu_x)$. Let $M(H^\infty_{|\operatorname{supp}\mu_x})$ be the maximal ideal space of $H^\infty_{|\operatorname{supp}\mu_x}$. Then we view $M(H^\infty_{|\operatorname{supp}\mu_x})$ as

$$M(H^\infty_{|\operatorname{supp}\mu_x})=\{\eta\in\mathcal{M};\operatorname{supp}\mu_\eta\subset\operatorname{supp}\mu_x\}.$$

Since $|\varphi(x)|<1$ and $x\in M(H^\infty_{|\operatorname{supp}\mu_x})$, φ is not invertible in $H^\infty_{|\operatorname{supp}\mu_x}$. Since $|\varphi|=1$ on $\operatorname{supp}\mu_x$, we have

$$\varphi(M(H^\infty_{|\operatorname{supp}\mu_x}))=\{|z|\le 1\}.$$

By this fact, (i) is clear. By Shilov's idempotent theorem, $\{\eta \in M(H^\infty_{|\operatorname{supp}\mu_x}); |\varphi(\eta)| < 1\}$ is not closed in $M(H^\infty_{|\operatorname{supp}\mu_x})$. Hence we get (ii).

For an inner funcition φ, let

$$N(\bar{\varphi}) = \overline{\bigcup\{\operatorname{supp}\mu_x; |\varphi(x)| < 1\}}. \tag{3.1}$$

The properties of $N(\bar{\varphi})$ are studied in [**11**], [**12**]. The following two lemmas are keys to prove Theorem 3.1.

Lemma 3.3. *Let φ be an inner function. Then*

(i) $N(\bar{\varphi}) = \bigcup\{Q_x; x \in Z(\varphi)\} = \bigcup\{Q_x; x \in \{|\varphi| < 1\}\}$.

(ii) $N(\bar{\varphi}) = \bigcup\{Q_x; x \in \overline{\{|\varphi| < 1\}}\}$.

(iii) $N(\bar{\varphi}) = \bigcup\{Q_x; x \in \{|\varphi| < 1\}\backslash Z(\varphi)\}$.

(iv) $N(\bar{\varphi}) = \bigcup\{Q_x; x \in \overline{\{|\varphi| < 1\}}\backslash\{|\varphi| < 1\}\}$.

Proof. (i) is proved in [**11**].

(ii) Let $x \in \overline{\{|\varphi| < 1\}}$. Then there is a net $\{x_\alpha\}_\alpha$ in $\{|\varphi| < 1\}$ such that $x_\alpha \to x$. Then $\mu_{x_\alpha} \to \mu_x$ in the weak*-topology of the space of bounded Borel measures on $M(L^\infty)$, see [**4**, p. 375]. By (3.1), $\operatorname{supp}\mu_{x_\alpha} \subset N(\bar{\varphi})$ and $N(\bar{\varphi})$ is a closed subset of $M(L^\infty)$, so that we have $\operatorname{supp}\mu_x \subset N(\bar{\varphi})$. By (i), $N(\bar{\varphi})$ is a union set of QC-level sets, so that we have $Q_x \subset N(\bar{\varphi})$. Thus we get (ii).

(iii) Let $x \in Z(\varphi)$. Then by Lemma 3.2(i), there is $y \in \mathcal{M}$ such that $\operatorname{supp}\mu_y \subset \operatorname{supp}\mu_x$ and $0 < |\varphi(y)| < 1$. Then $Q_y = Q_x$, so that by (i) we get (iii).

(iv) Let $x \in \{|\varphi| < 1\}$. Then by Lemma 3.2(ii), there is $y \in \mathcal{M}$ such that $\operatorname{supp}\mu_y \subset \operatorname{supp}\mu_x$ and $y \in \overline{\{|\varphi| < 1\}}\backslash\{|\varphi| < 1\}$. Then $Q_y = Q_x$, so that by (ii) we get (iv).

The following lemma follows from [**11**, Corollary 4].

Lemma 3.4. *Let φ and ψ be inner functions. Then $\{|\psi| < 1\} \subset \{|\varphi| < 1\}$ if and only if $N(\bar{\psi}) \subset N(\bar{\varphi})$.*

Applying Lemmas 3.3 and 3.4, we have the following.

Lemma 3.5. *Let φ and ψ be inner functions. Then the following conditions are equivalent.*

(i) $\{|\psi| < 1\} \subset \{|\varphi| < 1\}$.

(ii) $\{|\psi| < 1\} \subset \overline{\{|\varphi| < 1\}}$.

(iii) $Z(\psi) \subset \overline{\{|\varphi| < 1\}}$.

(iv) $\{|\psi| < 1\} \backslash Z(\psi) \subset \overline{\{|\varphi| < 1\}}$.

(v) $\overline{\{|\psi| < 1\}} \backslash \{|\psi| < 1\} \subset \overline{\{|\varphi| < 1\}}$.

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv), and (ii) $\Rightarrow$ (v) are trivial.

(iii) $\Rightarrow$ (i) By condition (iii) and Lemma 3.3(i), we have $N(\bar{\psi}) \subset N(\bar{\varphi})$. Hence by Lemma 3.4, we get (i).

(iv) $\Rightarrow$ (i) By condition (iv) and Lemma 3.3(ii) and (iii), we have $N(\bar{\psi}) \subset N(\bar{\varphi})$. Hence by Lemma 3.4, we get (i).

(v) $\Rightarrow$ (i) By condition (v) and Lemma 3.3(ii) and (iv), $N(\bar{\psi}) \subset N(\bar{\varphi})$. Also by Lemma 3.4, we get (i).

LEMMA 3.6. *Let $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}}$ be a measure on $\partial \Delta$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Put $\mu_n = \sum_{k=n}^{\infty} a_k \delta_{e^{i\theta_k}}$. Then for each positive integer k, the following conditions are equivalent.*

(i) $\{|\psi_{\delta_{e^{i\theta_k}}}| < 1\} \subset \bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\}$.

(ii) $\{|\psi_{\delta_{e^{i\theta_k}}}| < 1\} \subset \overline{\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\}}$.

(iii) $Z(\psi_{\delta_{e^{i\theta_k}}}) \subset \overline{\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\}}$.

(iv) $\{|\psi_{\delta_{e^{i\theta_k}}}| < 1\} \backslash Z(\psi_{\delta_{e^{i\theta_k}}}) \subset \overline{\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\}}$.

(v) $\overline{\{|\psi_{\delta_{e^{i\theta_k}}}| < 1\}} \backslash \{|\psi_{\delta_{e^{i\theta_k}}}| < 1\} \subset \overline{\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\}}$.

PROOF. Put $\zeta = e^{i\theta_k}$. Then we have $|\psi_{\mu_n}| = |\psi_{\mu_{k+1}}|$ on $\mathscr{M}_\zeta$ for $n \geq k+1$. Hence

$$\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\} \cap \mathscr{M}_\zeta = \{|\psi_{\mu_{k+1}}| < 1\} \cap \mathscr{M}_\zeta. \tag{3.2}$$

By Frostman's theorem, see [**9**], there exists a Blaschke product b such that

$$\{|\psi_{\delta_\zeta}| < 1\} = \{|b| < 1\}. \tag{3.3}$$

Then for $p \in \mathscr{P}(b)$ we have

$$\overline{\{|b| < 1\}} \subset Z(b^p), \tag{3.4}$$

see [**9**], [**13**]. Since $|b^p| = 1$ on $\cup\{M_\xi; \xi \neq \zeta, \xi \in \partial\Delta\}$, by (3.3) and (3.4),

$$\overline{\{|\psi_{\delta_\zeta}| < 1\}} \cap \overline{\bigcup\{M_\xi; \xi \neq \zeta\}} = \varnothing. \tag{3.5}$$

We have

$$\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\} = \left(\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\} \cap M_\zeta \right) \cup \left(\bigcap_{n=1}^{\infty} \{|\psi_{\mu_n}| < 1\} \cap \left(\bigcup \{M_\xi; \xi \neq \zeta\} \right) \right)$$

and

$$\{|\psi_{\mu_{k+1}}|<1\}=\left(\{|\psi_{\mu_{k+1}}|<1\}\cap\mathscr{M}_\zeta\right)\cup\left(\{|\psi_{\mu_{k+1}}|<1\}\cap\left(\bigcup\{\mathscr{M}_\xi;\xi\neq\zeta\}\right)\right).$$

Hence by (3.2) and (3.5),

$$\{|\psi_{\delta_\zeta}|<1\}\subset\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}\quad\text{if and only if}\quad\{|\psi_{\delta_\zeta}|<1\}\subset\{|\psi_{\mu_{k+1}}|<1\}.$$

We also have

$$\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}=\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}\cap M_\zeta}\cup\overline{\left(\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}\cap\left(\bigcup\{M_\xi;\xi\neq\zeta\}\right)\right)}$$

and

$$\overline{\{|\psi_{\mu_{k+1}}|<1\}}=\overline{\{|\psi_{\mu_{k+1}}|<1\}\cap\mathscr{M}_\zeta}\cup\overline{\left(\{|\psi_{\mu_{k+1}}|<1\}\cap\left(\bigcup\{\mathscr{M}_\xi;\xi\neq\zeta\}\right)\right)}.$$

Let E be a subset of $\overline{\{|\psi_{\delta_\zeta}|<1\}}$. Then by (3.2) and (3.5), we have

$$E\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}\quad\text{if and only if}\quad E\subset\overline{\{|\psi_{\mu_{k+1}}|<1\}}.$$

Thus we may replace $\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}$ with $\{|\psi_{\mu_{n+1}}|<1\}$ in conditions (i) through (v) above and apply Lemma 3.5 to obtain the result.

Proof of Theorem 3.1. By Theorems 2.1 and 2.2,

$$\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}\subset\bigcap\{Z(\psi_{\mu^p});p\in\mathscr{P}(\mu)\}\subset\bigcap\{\{|\psi_{\mu^p}|<1\};p\in\mathscr{P}(\mu)\}\subset\overline{\{|\psi_\mu|<1\}}.$$

Hence we have (c) $\Rightarrow$ (b) $\Rightarrow$ (a). By Lemmas 3.1 and 3.6, we have that conditions (c), (d), (e), (f), and (g) are equivalent.

(a) $\Leftrightarrow$ (h) By Theorem 2.2(i), (h) holds if and only if

$$\overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}\Big\backslash\left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right)\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}.$$

Hence by Lemma 3.1(i), we have (a) $\Leftrightarrow$ (h).

(a) $\Rightarrow$ (c) Suppose that (a) holds. By Lemma 3.1(i), we have

$$\overline{\{|\psi_{\delta_{e^{i\theta_k}}}|<1\}}\backslash\{|\psi_{\delta_{e^{i\theta_k}}}|<1\}\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}\quad\text{for every } k.$$

Then by Lemma 3.6,

$$\bigcup_{k=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_k}}}|<1\}\subset\overline{\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}}.$$

Hence by Lemma 3.1(ii), (c) holds. Thus we get our theorem.

In [**14**, Theorem 5.1], we gave equivalent conditions on $\mu\in M_{s,d}^{+}$ for which the following condition is satisfied.

($\sharp_1$) There exists $\nu\in L_{+}^{1}(\mu)$ such that $\overline{\{|\psi_\mu|<1\}}\subset Z(\psi_\nu)$.

We note that if $\mu\in M_{s,d}^{+}$ satisfies ($\sharp_1$), then λ also satisfies ($\sharp_1$) for every $\lambda\in M_{s,d}^{+}$ with $\lambda\ll\mu$ and $\mu\ll\lambda$. In Theorem 3.1, we give equivalent conditions on $\mu\in M_{s,d}^{+}$ for which the following condition is satisfied.

($\sharp_2$) $$\overline{\{|\psi_\mu|<1\}}=\bigcap\{Z(\psi_{\mu^p});p\in\mathscr{P}(\mu)\}.$$

Since $\mu^p\in L_{+}^{1}(\mu)$, we have ($\sharp_2$) $\Rightarrow$ ($\sharp_1$). In [**14**, Example 5.2], we show the existence of a measure μ satisfying condition (g) in Theorem 3.1, so that this μ satisfies ($\sharp_2$).

Here we have the following.

Proposition 3.1. *Let $\lambda\in M_{s,d}^{+}$ satisfying condition ($\sharp_1$). Then there exists $\mu\in L_{+}^{1}(\lambda)$ such that μ satisfies ($\sharp_1$) but does not satisfy ($\sharp_2$).*

Proof. Since λ satisfies ($\sharp_1$), by [**14**, Corollary 4.1 and Theorem 5.1], the closed support set of λ, denoted $S(\lambda)$, does not have an isolated point. Hence λ is not a finite sum of point measures, so that we can write λ as $\lambda=\sum_{n=1}^{\infty}a_n\delta_{e^{i\theta_n}}$, where $a_n>0$ for every n and $e^{i\theta_n}\neq e^{i\theta_k}$ for $n\neq k$. It is not difficult to find a measure $\tau=\sum_{n=2}^{\infty}c_n\delta_{e^{i\theta_n}}\in L_{+}^{1}(\lambda)$ such that $c_n>0$ for every $n\geq 2$ and

(3.6) $$|\psi_{\delta_{e^{i\theta_1}}}|=1\quad\text{on } Z(\psi_\tau).$$

Let $c_1=1$ and $\mu=\sum_{n=1}^{\infty}(c_n/n)\delta_{e^{i\theta_n}}$. Since $\lambda\ll\mu$, $\mu\ll\lambda$, and λ satisfies ($\sharp_1$), μ satisfies ($\sharp_1$). We let $p_0=(1,2,\dots)$. Then $p_0\in\mathscr{P}(\mu)$ and $\mu^{p_0}=\delta_{e^{i\theta_1}}+\tau$. Then $Z(\mu^{p_0})=Z(\psi_{\delta_{e^{i\theta_1}}})\cup Z(\psi_\tau)$. Hence by (3.6), $\overline{\{|\psi_{\delta_{e^{i\theta_1}}}|<1\}}\not\subset Z(\mu^{p_0})$. But we have $\overline{\{|\psi_{\delta_{e^{i\theta_1}}}|<1\}}\subset\overline{\{|\psi_\mu|<1\}}$. Therefore μ does not satisfy ($\sharp_2$).

4. Discrete singular functions of l_{+}^{∞}-type.

Let $\mu=\sum_{n=1}^{\infty}a_n\delta_{e^{i\theta_n}}\in M_{s,d}^{+}$ such that $a_n>0$ for every n and $e^{i\theta_n}\neq e^{i\theta_k}$ for $n\neq k$. Let l_{+}^{∞} be the set of bounded sequences $p=(p_1,p_2,\dots)$ such that $p_n>0$ for every n. Then $\{|\psi_{\mu^p}|<1\}\subset\{|\psi_\mu|<1\}$ for every $p\in l_{+}^{\infty}$ and

$$\bigcap\{\{|\psi_{\mu^p}|<1\};p\in l_{+}^{\infty}\}\subset\bigcap\{\{|\psi_{\mu^p}|<1\};p\in\mathscr{P}(\mu)\}.$$

In this section, we study the sets $\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}$ and $\bigcap\{Z(\psi_{\mu^p}); p\in l_+^\infty\}$. We prove the following.

THEOREM 4.1. *Let $\mu=\sum_{n=1}^\infty a_n\delta_{e^{i\theta_n}}\in M_{s,d}^+$ such that $a_n>0$ for every n and $e^{i\theta_n}\neq e^{i\theta_k}$ for $n\neq k$. Then we have the following.*

(i) $$\bigcap\{Z(\psi_{\mu^p}); p\in l_+^\infty\}=\overline{\bigcup_{n=1}^\infty Z(\psi_{\delta_{e^{i\theta_n}}})}.$$

(ii) $$\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}=\left(\bigcup_{n=1}^\infty\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right)\cup\overline{\bigcup_{n=1}^\infty Z(\psi_{\delta_{e^{i\theta_n}}})}.$$

(iii) $$\overline{\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}}=\bigcap\{\overline{\{|\psi_{\mu^p}|<1\}}; p\in l_+^\infty\}=\overline{\bigcup_{n=1}^\infty\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}.$$

To prove this theorem, we need a lemma. In the same way as the proof of Lemma 2.3, we have the following.

LEMMA 4.1. *Let $\mu=\sum_{n=1}^\infty a_n\delta_{e^{i\theta_n}}\in M_{s,d}^+$ such that $a_n>0$ for every n and $e^{i\theta_n}\neq e^{i\theta_k}$ for $n\neq k$. Then we have the following.*

(i) *For each $q\in l_+^\infty$, there exists $p\in l_+^\infty$ such that $|\psi_{\mu^p}|=1$ on $\{\psi_{\mu^q}\neq 0\}\setminus\bigcup_{n=1}^\infty\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}$.*

(ii) *Let $x\in\mathscr{M}\setminus\Delta$. If $x\notin\overline{\bigcup_{n=1}^\infty\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}$ and $\mu_{\mu^q}(x)\neq 0$ for some $q\in l_+^\infty$, then there exists $p\in l_+^\infty$ such that $x\notin\overline{\{|\psi_{\mu^p}|<1\}}$.*

PROOF OF THEOREM 4.1. (i) Let E be a closed subset of Δ such that

$$\bar{E}\cap\overline{\bigcup_{n=1}^\infty Z(\psi_{\delta_{e^{i\theta_n}}})}=\varnothing. \tag{4.1}$$

Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers such that

$$\prod_{n=1}^\infty\varepsilon_n>0\quad\text{and}\quad 0<\varepsilon_n<1\quad\text{for every } n. \tag{4.2}$$

By (4.1), for each n we have

$$\inf\{|\psi_{\delta_{e^{i\theta_n}}}(z)|; z\in E\}>0.$$

Then we can take a small positive number p_n such that

$$\inf\{|\psi_{\delta_{e^{i\theta_n}}}(z)|^{a_np_n}; z\in E\}\geq\varepsilon_n.$$

We may assume that $p=(p_1,p_2,\dots)\in l_+^\infty$. Hence by (4.2) we have

$$\inf\{|\psi_{\mu^p}(z)|; z\in E\}\geq\prod_{n=1}^\infty\varepsilon_n>0.$$

This implies that $\bar{E} \cap Z(\psi_{\mu^p}) = \emptyset$. Therefore we have

$$\bigcap\{Z(\mu^p); p \in l_+^\infty\} \subset \overline{\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})}.$$

The reverse inclusion is obvious. Thus we get Theorem 4.1(i).

(ii) It is clear that

$$\left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right) \cup \overline{\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})} \subset \bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\}.$$

To prove the reverse inclusion, let $x \in \mathscr{M}\backslash\Delta$ such that

$$x \notin \left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}\right) \cup \overline{\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})}.$$

Then by Theorem 4.1(i) there exists $q \in l_+^\infty$ such that $\psi_{\mu^q}(x) \neq 0$. By Lemma 4.1(i), there exists $p \in l_+^\infty$ such that $|\psi_{\mu^p}(x)| = 1$. Thus we get Theorem 4.1(ii).

(iii) By Theorem 4.1(ii),

$$\overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}} = \overline{\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\}} \subset \bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in l_+^\infty\}.$$

To prove the reverse inclusion, let

$$x \notin \overline{\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}}.$$

Then

$$x \notin \overline{\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{i\theta_n}}})},$$

so that by (i) there exists $q \in l_+^\infty$ such that $\psi_{\mu^q}(x) \neq 0$. Hence by Lemma 4.1(ii), $x \notin \overline{\{|\psi_{\mu^p}| < 1\}}$ for some $p \in l_+^\infty$. Therefore $x \notin \bigcap\{\overline{\{|\psi_{\mu^p}| < 1\}}; p \in l_+^\infty\}$. Thus we get Theorem 4.1(iii).

In Section 3, we show that for some $\mu \in M_{s,d}^+$, $\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in \mathscr{P}(\mu)\}$ is a closed subset of $\mathscr{M}$. But we have the following.

Proposition 4.1. *Let $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$ such that $a_n > 0$ for every n and $e^{i\theta_n} \neq e^{i\theta_k}$ for $n \neq k$. Then we have the following.*

(i) *$\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\}$ is not a closed subset of $\mathscr{M}$.*

(ii) *$\bigcap\{\{|\psi_{\mu^p}| < 1\}; p \in l_+^\infty\} \neq \bigcap\{Z(\psi_{\mu^p}); p \in l_+^\infty\}$.*

Proof. By Theorem 4.1(ii), we have

$$\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\} = \left(\bigcup_{n=1}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}\right)\cup\overline{\bigcup_{n=1}^{\infty}Z(\psi_{\delta_{e^{i\theta_n}}})}.$$

By (3.5), we have

$$\overline{\{|\psi_{\delta_{e^{i\theta_k}}}|<1\}}\cap\overline{\bigcup_{n=1}^{\infty}Z(\psi_{\delta_{e^{i\theta_n}}})} = Z(\psi_{\delta_{e^{i\theta_k}}}).$$

Hence

$$\overline{\{|\psi_{\delta_{e^{i\theta_k}}}|<1\}}\not\subset\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\},$$

so that $\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}$ is not closed.

Theorem 4.2. *Let $\mu=\sum_{n=1}^{\infty}a_n\delta_{e^{i\theta_n}}\in M_{s,d}^+$ such that $a_n>0$ for every n and $e^{i\theta_n}\neq e^{i\theta_k}$ for $n\neq k$. Then*

(i) $$\bigcap\{Z(\psi_{\mu^p}); p\in l_+^\infty\}\subsetneqq\bigcap\{Z(\psi_{\mu^p}); p\in\mathscr{P}(\mu)\}.$$

(ii) $$\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}\subsetneqq\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in\mathscr{P}(\mu)\}.$$

(iii) $$\overline{\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in l_+^\infty\}}\subsetneqq\overline{\bigcap\{\{|\psi_{\mu^p}|<1\}; p\in\mathscr{P}(\mu)\}}.$$

Let $\{z_n\}_n$ be a sequence in Δ such that

$$\lim_{k\to\infty}\prod_{n\neq k}\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right|=1.$$

Such a sequence and an associated Blaschke product are called sparse (or thin). By [5], for every sequence $\{w_n\}_n$ in Δ with $|w_n|\to 1$ there exists a sparse subsequence of $\{w_n\}_n$. See, [7], [8] for the study of sparse Blaschke products.

Proof of Theorem 4.2. We note that by our assumption, μ is an infinite sum of point measures. Let $e^{i\theta_0}$ be a cluster point of $\{e^{i\theta_n}\}_n$ in $\partial\Delta$. Put

$$\tau=\mu-\mu(\{e^{i\theta_0}\})\delta_{e^{i\theta_0}}.$$

Then we have $|\psi_{\mu_n}|=|\psi_\tau|$ on $\mathscr{M}_{e^{i\theta_0}}$ for sufficiently large n. Hence

$$\left(\bigcap_{n=1}^{\infty}\{|\psi_{\mu_n}|<1\}\right)\cap\mathscr{M}_{e^{i\theta_0}}=\{|\psi_\tau|<1\}\cap\mathscr{M}_{e^{i\theta_0}}$$

and

$$\{|\psi_\tau|<1\}\cap\mathscr{M}_{e^{i\theta_0}}\subset\{|\psi_\mu|<1\}\cap\mathscr{M}_{e^{i\theta_0}}\subset(\{|\psi_\tau|<1\}\cap\mathscr{M}_{e^{i\theta_0}})\cup\{|\psi_{\delta_{e^{i\theta_0}}}|<1\}.$$

Therefore by Theorems 2.1, 2.2, and 4.1, to show our assertions it is sufficient to prove that

$$(4.3)\qquad (\{|\psi_\tau|<1\}\cap \mathscr{M}_{e^{i\theta_0}})\setminus\overline{\bigcup_{n=0}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}\neq\emptyset.$$

Since $e^{i\theta_0}$ is a cluster point of $\{e^{i\theta_n}\}_n$, there exists a point $x\in\mathscr{M}_{e^{i\theta_0}}$ such that

$$(4.4)\qquad |\psi_{\delta_{e^{i\theta_0}}}(x)|=1 \quad\text{and}\quad \psi_\tau(x)=0.$$

For, take $x_n\in\mathscr{M}_{e^{i\theta_n}}$ such that $\psi_{\delta_{e^{i\theta_n}}}(x_n)=0$, then take $x\in\overline{\{x_n\}_n}\cap\mathscr{M}_{e^{i\theta_0}}$. This x satisfies (4.4). Then by the corona theorem, there exists a sequence $\{z_n\}_n$ in Δ such that $|\psi_{\delta_{e^{i\theta_0}}}(z_n)|\to 1, \psi_\tau(z_n)\to 0$, and $z_n\to e^{i\theta_0}$ as $n\to\infty$. Moreover we may assume that $\{z_n\}_n$ is sparse. Let b be the associated Blaschke product. Then by [**9**, p. 205], $Z(b)=\overline{\{z_n\}_n}\setminus\{z_n\}_n$, and $|\psi_{\delta_{e^{i\theta_0}}}|=1$ and $\psi_\tau=0$ on $Z(b)$. Since b is sparse, we have

$$(4.5)\qquad |\psi_{\delta_{e^{i\theta_0}}}|=1 \quad\text{and}\quad \psi_\tau=0 \quad\text{on } \{|b|<1\},$$

see [**1**], [**6**], [**8**]. Then $\{|b|<1\}\cap\{|\psi_{\delta_{e^{i\theta_0}}}|<1\}=\emptyset$. Hence by [**11**], $\overline{\{|b|<1\}}\cap\overline{\{|\psi_{\delta_{e^{i\theta_0}}}|<1\}}=\emptyset$. Since $z_n\to e^{i\theta_0}$ as $n\to\infty$, also by (4.5) we have

$$\{|b|<1\}\cap\overline{\bigcup_{n=0}^{\infty}\{|\psi_{\delta_{e^{i\theta_n}}}|<1\}}=\emptyset \quad\text{and}\quad \{|b|<1\}\subset\{|\psi_\tau|<1\}\cap\mathscr{M}_{e^{i\theta_0}}.$$

Hence we obtain (4.3), completing the proof.

The following is the last theorem in this paper.

THEOREM 4.3. *Let $\mu,\lambda\in M_{s,d}^+$ be sums of infinitely many point measures. Then $\mu\perp\lambda$ if and only if*

$$\left(\bigcap\{\overline{\{|\psi_{\mu^p}|<1\}};p\in l_+^\infty\}\right)\cap\left(\bigcap\{\overline{\{|\psi_{\lambda^q}|<1\}};q\in l_+^\infty\}\right)=\emptyset.$$

This theorem follows from Theorem 4.1 and the following proposition.

PROPOSITION 4.2. *Let $\{e^{is_k}\}_k$ and $\{e^{it_n}\}_n$ be sequences in $\partial\Delta$ such that $\overline{\{e^{is_k}\}_k}\cap\overline{\{e^{it_n}\}_n}=\emptyset$. Then*

$$\overline{\bigcup_{k=1}^{\infty}Z(\psi_{\delta_{e^{is_k}}})}\cap\overline{\bigcup_{n=1}^{\infty}Z(\psi_{\delta_{e^{it_n}}})}=\emptyset.$$

PROOF. We may assume that $e^{is_n}\neq e^{is_k}$ and $e^{it_n}\neq e^{it_k}$ for $n\neq k$. For each positive integer n, let

$$(4.6)\qquad \Delta_n=\left\{z;|z-\frac{1}{2}e^{it_n}|<\frac{1}{2}\right\}.$$

Then $\partial\Delta_n \cap \partial\Delta = \{e^{it_n}\}$. Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers such that

$$\prod_{n=1}^{\infty} \varepsilon_n > 0 \quad \text{and} \quad 0 < \varepsilon_n < 1 \quad \text{for every } n. \tag{4.7}$$

Since $\{e^{is_k}\}_k \cap \{e^{it_n}\}_n = \varnothing$, for positive integers k and j we have

$$|\psi_{\delta_{e^{is_k}}}(z)| \to 1 \quad \text{as } |z| \to 1 \text{ and } z \in \Delta_j. \tag{4.8}$$

Hence there exists $a_k > 0$ such that

$$|\psi_{\delta_{e^{is_k}}}|^{a_k} \geq \varepsilon_k \quad \text{on } \bigcup_{j=1}^{k} \Delta_j. \tag{4.9}$$

Taking sufficiently small a_k, we may assume that $\sum_{k=1}^{\infty} a_k < \infty$. Put

$$\mu = \sum_{k=1}^{\infty} a_k \delta_{e^{is_k}}. \tag{4.10}$$

Then for each positive integer n, by (4.9) we have

$$|\psi_\mu| = \left(\prod_{k=1}^{n-1} |\psi_{\delta_{e^{is_k}}}|^{a_k}\right)\left(\prod_{k=n}^{\infty} |\psi_{\delta_{e^{is_k}}}|^{a_k}\right) \geq \left(\prod_{k=1}^{n-1} |\psi_{\delta_{e^{is_k}}}|^{a_k}\right) \prod_{k=n}^{\infty} \varepsilon_k \quad \text{on } \Delta_n. \tag{4.11}$$

By (4.6), $Z(\delta_{e^{it_n}}) \subset \overline{\Delta_n}$, so that by (4.8) and (4.11) we have

$$|\psi_\mu| \geq \prod_{k=n}^{\infty} \varepsilon_k \quad \text{on } Z(\delta_{e^{it_n}}).$$

Hence by (4.7),

$$|\psi_\mu| \geq \prod_{k=1}^{\infty} \varepsilon_k > 0 \quad \text{on } \overline{\bigcup_{n=1}^{\infty} Z(\delta_{e^{it_n}})}.$$

By (4.10),

$$|\psi_\mu| = 0 \quad \text{on } \overline{\bigcup_{k=1}^{\infty} Z(\delta_{e^{is_k}})}.$$

Therefore we obtain

$$\overline{\bigcup_{k=1}^{\infty} Z(\psi_{\delta_{e^{is_k}}})} \cap \overline{\bigcup_{n=1}^{\infty} Z(\psi_{\delta_{e^{it_n}}})} = \varnothing.$$

This completes the proof.

We remark that there exist $\mu, \lambda \in M_{s,d}^+$ such that $\mu \perp \lambda$ and

$$\left(\bigcap\{Z(\psi_{\mu^p}); p \in \mathscr{P}(\mu)\}\right) \cap \left(\bigcap\{Z(\psi_{\lambda^q}); q \in \mathscr{P}(\lambda)\}\right) \neq \varnothing. \tag{4.12}$$

For, let $\nu=\sum_{n=1}^{\infty}a_n\delta_{e^{i\theta_n}}\in M_{s,d}^{+}$ satisfying conditions of Theorem 3.1 and put $\mu=\sum_{n=2}^{\infty}a_n\delta_{e^{i\theta_n}}$. Then it is not difficult to see that μ also satisfies conditions of Theorem 3.1 and we have

$$(4.13)\qquad Z(\psi_{\delta_{e^{i\theta_1}}})\subsetneqq\overline{\{|\psi_\nu|<1\}}=\overline{\{|\psi_\mu|<1\}}=\bigcap\{Z(\psi_{\mu^p});p\in\mathscr{P}(\mu)\}.$$

Let $\lambda=\delta_{e^{i\theta_1}}+\sum_{n=1}^{\infty}b_n\delta_{e^{it_n}}\in M_{s,d}^{+}$ such that $\{e^{it_n}\}_n\cap\{e^{i\theta_n}\}_n=\varnothing$. Since $\delta_{e^{i\theta_1}}\perp\mu$, $\mu\perp\lambda$. Since $\delta_{e^{i\theta_1}}\ll\lambda$, $Z(\psi_{\delta_{e^{i\theta_1}}})\subset\bigcap\{Z(\psi_{\lambda^q});q\in\mathscr{P}(\lambda)\}$. Hence by (4.13), we have (4.12).

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