

## Mosaic and trace formulae of log-hyponormal operators

Dedicated to Professor Michiaki Watanabe in celebration of his having been  
honoured as an Emeritus Professor of Niigata University

By Muneo CHŌ and Tadasu HURUYA

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**Abstract.** The purpose of this paper is to introduce mosaics of log-hyponormal operators and give a Helton-Howe type trace formula.

### 1. Introduction.

J. D. Pincus and D. Xia, in [14], studied mosaics and principal functions of semi-hyponormal operators and gave the trace formula. In [19], Xia announced trace formulae for semi-hyponormal operators. In [4], we gave trace formulae of  $p$ -hyponormal operators for  $0 < p \leq 1/2$ . In particular we proved a Helton-Howe type trace formula (cf. [13], p. 240, Theorem 2.4). In this paper, we introduce mosaics and principal functions of log-hyponormal operators and prove a Helton-Howe type trace formula of it.

Let  $\mathcal{H}$  be a complex separable Hilbert space and  $B(\mathcal{H})$  be the algebra of all bounded linear operators on  $\mathcal{H}$ . An operator  $T \in B(\mathcal{H})$  is said to be  $p$ -hyponormal if  $(T^*T)^p - (TT^*)^p \geq 0$ . If  $p = 1$ ,  $T$  is called hyponormal and if  $p = 1/2$ ,  $T$  is called semi-hyponormal. The set of all semi-hyponormal operators in  $B(\mathcal{H})$  is denoted by SH. Let SHU denote the set of all operators in SH with equal defect and nullity (cf. [19], p. 4). Hence we may assume that the operator  $U$  in the polar decomposition  $T = U|T|$  is unitary if  $T \in \text{SHU}$ . An operator  $T \in B(\mathcal{H})$  is said to be log-hyponormal if  $T$  is invertible and  $\log T^*T \geq \log TT^*$ . Since the function  $\log(\cdot)$  is operator monotone, an operator  $T$  is log-hyponormal if  $T$  is an invertible  $p$ -hyponormal operator. In [15] K. Tanahashi gave a counter example of log-hyponormal operator which is not  $p$ -hyponormal. When  $\log|T| \geq 0$ , he also proved that  $T' = U \log|T|$  is semi-hyponormal if  $T = U|T|$  is log-hyponormal. If  $T = U|T|$  is log-hyponormal, then we can choose a number  $c > 0$  such that  $\log((1/c)|T|) \geq 0$ . Indeed, it is  $c = \inf\{r : r \in \sigma(|T|)\}$ . Hence

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we have  $U \log((1/c)|T|) \in \text{SUH}$ . We often use this property and the following result.

**THEOREM A** (Tanahashi [16], Lemma 6). *Let  $T = U|T|$  be log-hyponormal with  $\log|T| \geq 0$  and  $T' = U \log|T|$ . Then*

$$\sigma(T) = \{e^r \cdot e^{i\theta} : re^{i\theta} \in \sigma(T')\}.$$

Let  $T = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$ ,  $\Sigma$  be the set of all Borel sets in  $T$ ,  $m$  be a measure on the measurable space  $(T, \Sigma)$  such that  $dm(\theta) = (1/2\pi)d\theta$  and  $\mathcal{D}$  be a separable Hilbert space. The Hilbert space of all vector-valued, strongly-measurable and square-integrable functions with values in  $\mathcal{D}$  and with inner product

$$(f, g) = \int_T (f(e^{i\theta}), g(e^{i\theta}))_{\mathcal{D}} dm(\theta)$$

is denoted by  $L^2(\mathcal{D})$ ; Hardy space is denoted by  $H^2(\mathcal{D})$ , and the projection from  $L^2(\mathcal{D})$  to  $H^2(\mathcal{D})$ , by  $\mathcal{P}$ . If  $f \in L^2(\mathcal{D})$ , then

$$(\mathcal{P}(f))(e^{i\theta}) = \lim_{r \rightarrow 1-0} \frac{1}{2\pi i} \int_{|z|=1} f(z)(z - re^{i\theta})^{-1} dz.$$

Let  $\nu$  be a singular measure on  $(T, \Sigma)$ ,  $F \in \Sigma$  be a set such that  $\nu(T \setminus F) = 0$  and  $m(F) = 0$ . Put  $\mu = m + \nu$ . Let  $R(\cdot)$  be a standard operator-valued strongly-measurable function defined on  $\Omega = (T, \Sigma, \mu)$  with values being the projection in  $\mathcal{D}$ ,  $L^2(\Omega, \mathcal{D})$  be a Hilbert space of all  $\mathcal{D}$ -valued strongly measurable and square-integrable functions on  $\Omega$  with inner product  $(f, g) = \int_T (f(e^{i\theta}), g(e^{i\theta}))_{\mathcal{D}} d\mu$ , and

$$\tilde{H} = \{f : f \in L^2(\Omega, \mathcal{D}), R(e^{i\theta})f(e^{i\theta}) = f(e^{i\theta}), e^{i\theta} \in T\}.$$

Then  $\tilde{H}$  is a subspace of  $L^2(\Omega, \mathcal{D})$ . The space  $L^2(\mathcal{D})$  is identified with a subspace of  $L^2(\Omega, \mathcal{D})$ . Hence  $\mathcal{P}$  extends to  $L^2(\Omega, \mathcal{D})$  such that

$$\mathcal{P}f = 0 \quad \text{for } f \in L^2(\Omega, \mathcal{D}) \ominus L^2(\mathcal{D}).$$

We define an operator  $\mathcal{P}_0$  from  $L^2(\Omega, \mathcal{D})$  to  $\mathcal{D}$  as follows:

$$\mathcal{P}_0(f) = \int f(e^{i\theta}) dm(\theta).$$

Then  $\mathcal{P}_0$  is the projection from  $L^2(\Omega, \mathcal{D})$  to  $\mathcal{D}$  (cf. [19], p. 50). Let  $\alpha(\cdot)$  and  $\beta(\cdot)$  be operator valued, uniformly bounded, and strongly measurable functions on  $\Omega$  such that  $\alpha(e^{i\theta})$  and  $\beta(e^{i\theta})$  are linear operators in  $\mathcal{D}$ , satisfying

$$R(e^{i\theta})\alpha(e^{i\theta}) = \alpha(e^{i\theta})R(e^{i\theta}) = \alpha(e^{i\theta}), \quad R(e^{i\theta})\beta(e^{i\theta}) = \beta(e^{i\theta})R(e^{i\theta}) = \beta(e^{i\theta})$$

and  $\beta(e^{i\theta}) \geq 0$ .

Furthermore, suppose that  $\alpha(e^{i\theta}) = 0$  if  $e^{i\theta} \in F$ . And we denote  $(\alpha f)(e^{i\theta}) = \alpha(e^{i\theta})f(e^{i\theta})$ . An operator  $\tilde{U}$  in  $\mathcal{H}$  is defined by

$$(\tilde{U}f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta}).$$

Since  $\beta(e^{i\theta}) \geq 0$  and  $\mathcal{P}$  is a projection on  $L^2(\mathcal{D})$ , we have

$$(\alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta}), f(e^{i\theta}))_{\mathcal{D}} \geq 0.$$

See details [19]. And the following results hold.

**THEOREM B** (Xia [17], Theorem 6). *With the above notations, let  $\tilde{T}$  be an operator in  $\mathcal{H}$  defined by*

$$(\tilde{T}f)(e^{i\theta}) = e^{i\theta}(Af)(e^{i\theta}),$$

*where  $(Af)(e^{i\theta}) = \alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta})$ . Then  $\tilde{T}$  is semi-hyponormal and the corresponding polar differential operator  $|\tilde{T}| - \tilde{U}|\tilde{T}|\tilde{U}^*$  is*

$$((|\tilde{T}| - \tilde{U}|\tilde{T}|\tilde{U}^*)f)(e^{i\theta}) = \alpha(e^{i\theta})^*\mathcal{P}_0(\alpha f).$$

**THEOREM C** (Xia [17], Theorem 7). *Let  $T = U|T|$  be a semi-hyponormal operator in  $\mathcal{H}$  such that  $U$  is unitary. Then there exist a function space  $\mathcal{H}$ , and operators  $\tilde{T}$  and  $\tilde{U}$  in  $\mathcal{H}$  which have the forms in Theorem B such that*

$$WTW^{-1} = \tilde{T} \quad \text{and} \quad WUW^{-1} = \tilde{U},$$

*where  $W$  is a unitary operator from  $\mathcal{H}$  to  $\mathcal{H}$ . Moreover  $\alpha(\cdot) \geq 0$ .*

$\tilde{T}$  is said to be the singular integral model of  $T$ .

## 2. Mosaic of log-hyponormal operators.

By the singular integral model of a semi-hyponormal operator  $T = U|T|$ , it holds the following

**THEOREM D** (Xia [19], Theorem V.2.5). *With the above notations, let  $T = U|T|$  be in SHU and  $\alpha(\cdot)$ ,  $\beta(\cdot)$  be of Theorems B and C of the singular integral model of  $T$ . Then the following statements hold.*

(1) *There exists a unique  $B(\mathcal{D})$ -valued measurable function of two variables,  $B(e^{i\theta}, r)$  ( $e^{i\theta} \in \mathbf{T}, r \in [0, \infty)$ ), satisfying*

$$0 \leq B(e^{i\theta}, r) \leq I$$

*such that*

$$I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - \ell)^{-1}\alpha(e^{i\theta}) = \exp \int_0^\infty \frac{B(e^{i\theta}, r)}{r - \ell} dr.$$

(2) For any bounded Baire function  $\psi$  on  $\sigma(|T|)$ , the function  $B(e^{i\theta}, r)$  has

$$\int \psi(r) B(e^{i\theta}, r) dr = \alpha(e^{i\theta}) \int_0^1 \psi(\beta(e^{i\theta}) + k \cdot \alpha(e^{i\theta})^2) dk \alpha(e^{i\theta}).$$

*Especially, it holds*

$$\int \frac{B(e^{i\theta}, r)}{r - \ell} dr = \alpha(e^{i\theta}) \int_0^1 (\beta(e^{i\theta}) + k \cdot \alpha(e^{i\theta})^2 - \ell)^{-1} dk \alpha(e^{i\theta}).$$

REMARK 1. The function  $B(e^{i\theta}, r)$  is defined on  $[0, \infty]$ . But, following Theorems V 2.4 and 2.5 of [19], we may assume that  $B(e^{i\theta}, t) = 0$  for  $t < \inf\{r : r \in \sigma(|T|)\}$ .

DEFINITION 1. For  $T \in \text{SHU}$ , the function  $B(\cdot, \cdot)$  in Theorem D is said to be the *mosaic* of  $T$ . We denote the mosaic of  $T$  by  $B_T(\cdot, \cdot)$ .

DEFINITION 2. Let  $T = U|T|$  be a log-hyponormal operator and  $T' = U \log|T|$ . Let  $c = \inf\{r : r \in \sigma(|T|)\} > 0$ . Since  $U(\log|T| - \log c) = U \log((1/c)|T|) \in \text{SHU}$ , there exists the mosaic  $B_{U \log((1/c)|T|)}(\cdot, \cdot)$  of  $U \log((1/c)|T|)$  and by Remark 1 we define

$$B_{T'}(e^{i\theta}, r) := B_{U \log((1/c)|T|)}(e^{i\theta}, r - \log c)$$

and

$$\mathcal{B}_T(e^{i\theta}, r) := \begin{cases} B_{T'}(e^{i\theta}, \log r) & \text{if } r \geq c \\ 0 & \text{if } r < c. \end{cases}$$

For a log-hyponormal operator  $T$ , we call  $\mathcal{B}_T(\cdot, \cdot)$  and  $B_{T'}(\cdot, \cdot)$  the *mosaics* of  $T$  and  $T'$ , respectively.

Let  $t$  be  $t \geq 0$ . For an operator  $T = U|T| \in \text{SHU}$ , since  $U(|T| + t) \in \text{SHU}$ , by Theorem D (1) it holds

$$\begin{aligned} \exp \int_0^\infty \frac{B_{U(|T|+t)}(e^{i\theta}, r)}{r - \ell} dr &= I + \alpha(e^{i\theta})(\beta(e^{i\theta}) + t - \ell)^{-1}\alpha(e^{i\theta}) \\ &= \exp \int_0^\infty \frac{B_T(e^{i\theta}, r)}{r - (\ell - t)} dr = \exp \int_t^\infty \frac{B_T(e^{i\theta}, r - t)}{r - \ell} dr. \end{aligned}$$

Hence, by the uniqueness of the mosaic in Theorem D (1) and Remark 1 we have

$$B_{U(|T|+t)}(e^{i\theta}, r) = B_{U|T|}(e^{i\theta}, r - t). \quad (*)$$

**THEOREM 1.** Let  $T = U|T|$  be a log-hyponormal. For  $0 < k \leq c = \inf\{r : r \in \sigma(|T|)\}$ , it holds that

$$B_{U \log((1/c)|T|)}(e^{i\theta}, r - \log c) = B_{U \log((1/k)|T|)}(e^{i\theta}, r - \log k).$$

**PROOF.** Since  $U(\log((1/k)|T|)) = U(\log|T| - \log k)$  is semi-hyponormal, we have

$$\begin{aligned} B_{U(\log(1/k)|T|)}(e^{i\theta}, r - \log k) &= B_{U(\log((1/c)|T|) + \log(c/k))}(e^{i\theta}, r - \log k) \\ &= B_{U(\log((1/c)|T|))}\left(e^{i\theta}, r - \log k - \log \frac{c}{k}\right) \quad \left(\text{by } (*) \text{ and } \log \frac{c}{k} > 0\right) \\ &= B_{U(\log((1/c)|T|))}(e^{i\theta}, r - \log c). \end{aligned}$$

Hence the proof is complete.  $\square$

By Theorem 1, the mosaic  $\mathcal{B}_T(e^{i\theta}, r)$  of a log-hyponormal operator  $T$  is independent from the choice of  $B_{U \log((1/k)|T|)}(e^{i\theta}, r - \log k)$  ( $0 < k \leq c$ ). Therefore, if a log-hyponormal operator  $T = U|T|$  satisfies  $\log|T| \geq 0$ , then we may take  $c = 1$ . From now on, let  $c = \inf\{r : r \in \sigma(|T|)\}$ .

**REMARK 2.** For a log-hyponormal operator  $T = U|T|$  with  $\log|T| \geq 0$ , by (\*)

- (1) if  $r \geq c$ ,  $\mathcal{B}_T(e^{i\theta}, r) = B_{T'}(e^{i\theta}, \log r) = B_{U(\log|T| - \log c)}(e^{i\theta}, \log r - \log c) = B_{U \log|T|}(e^{i\theta}, \log r)$ ,
- (2) if  $r < c$ ,  $\mathcal{B}_T(e^{i\theta}, r) = 0 = B_{U \log|T|}(e^{i\theta}, \log r)$  (because by Remark 1 and  $\log r < \inf\{\rho : \rho \in \sigma(\log|T|)\}$ ).

Hence in this case two mosaics of  $T' = U \log|T|$  in Definitions 1 and 2 are the same.

**DEFINITION 3.**

- (1) If  $T \in \text{SHU}$ , then the *determining set*  $D(T)$  of  $T$  is defined by

$$D(T) = C - \bigcup \{G : G \text{ is open in } C \text{ and } B_T(e^{i\theta}, r) = 0 \text{ for a.e. } re^{i\theta} \in G\}.$$

- (2) If  $T$  is a log-hyponormal operator, then the *determining set*  $D(T)$  of  $T$  is defined by

$$D(T) = C - \bigcup \{G : G \text{ is open in } C \text{ and } \mathcal{B}_T(e^{i\theta}, r) = 0 \text{ for a.e. } re^{i\theta} \in G\}.$$

For a log-hyponormal operator  $T = U|T|$ , since  $S = U \log((1/c)|T|) \in \text{SHU}$ , we have

$$D(S) = \{(\log(r/c)) \cdot e^{i\theta} : re^{i\theta} \in D(T)\}. \quad (**)$$

An operator  $T$  is called completely nonnormal if it has no nontrivial reducing subspace on which it is normal. We show the following

THEOREM 2. *Let  $T = U|T|$  be a log-hyponormal operator. Then*

$$\mathbf{D}(T) \subseteq \sigma(T).$$

Moreover, if  $T$  is completely nonnormal, then  $\mathbf{D}(T) = \sigma(T)$ .

PROOF. Let  $c = \inf\{r : r \in \sigma(|T|)\}$ . (1) Let  $r$  be  $0 \leq r < c$ . Then it is well known  $re^{i\theta} \notin \sigma(T)$ . By the definition we have  $\mathcal{B}_T(e^{i\theta}, r) = 0$ . Hence, we have  $re^{i\theta} \notin \mathbf{D}(T) \cup \sigma(T)$ . (2) Let  $r$  be  $r \geq c$  and  $T' = U \log|T|$ . Since

$$\mathcal{B}_T(e^{i\theta}, r) = \mathcal{B}_{T'}(e^{i\theta}, \log r) = \mathcal{B}_{U \log((1/c)|T|)}\left(e^{i\theta}, \log \frac{r}{c}\right),$$

by (\*\*) we have

$$\mathbf{D}\left(U \log\left(\frac{1}{c}|T|\right)\right) = \left\{\left(\log \frac{r}{c}\right) \cdot e^{i\theta} : re^{i\theta} \in \mathbf{D}(T)\right\}.$$

Since  $U \log((1/c)|T|) \in \text{SHU}$ , by Theorem V.3.2 of [19] we have

$$\mathbf{D}\left(U \log\left(\frac{1}{c}|T|\right)\right) \subseteq \sigma\left(U \log\left(\frac{1}{c}|T|\right)\right).$$

By Theorem A,

$$\sigma\left(U \log\left(\frac{1}{c}|T|\right)\right) = \sigma\left(U \exp\left(\log\left(\frac{1}{c}|T|\right)\right)\right) = \left\{e^r e^{i\theta} : re^{i\theta} \in \sigma\left(U \log\left(\frac{1}{c}|T|\right)\right)\right\}.$$

Hence if  $re^{i\theta} \in \mathbf{D}(T)$ , then

$$\frac{r}{c} \cdot e^{i\theta} = e^{\log(r/c)} e^{i\theta} \in \sigma\left(U \exp\left(\log\left(\frac{1}{c}|T|\right)\right)\right) = \frac{1}{c} \cdot \sigma(U|T|),$$

so that

$$\mathbf{D}(T) \subseteq \sigma(T).$$

If  $T$  is completely nonnormal, then by Theorem 3 of [7] it holds that  $U \log((1/c)|T|)$  is completely nonnormal. Since  $U \log((1/c)|T|)$  is semi-hyponormal, it holds that  $\mathbf{D}(U \log((1/c)|T|)) = \sigma(U \log((1/c)|T|))$  by Theorem V.3.2 of [19]. By the above it holds that

$$re^{i\theta} \in \mathbf{D}(T) \Leftrightarrow \left(\log \frac{r}{c}\right) \cdot e^{i\theta} \in \mathbf{D}\left(U \log\left(\frac{1}{c}|T|\right)\right)$$

and

$$re^{i\theta} \in \sigma(T) \Leftrightarrow \left(\log \frac{r}{c}\right) \cdot e^{i\theta} \in \sigma\left(U \log\left(\frac{1}{c}|T|\right)\right).$$

Hence we have  $\mathbf{D}(T) = \sigma(T)$ . So the proof is complete.  $\square$

**THEOREM 3.** *Let  $T = U|T|$  be a log-hyponormal operator. Then*

$$\|\log|T| - \log|T^*|\| \leq \frac{1}{2\pi} \iint_{\mathbf{D}(T)} r^{-1} dr d\theta.$$

**PROOF.** Let  $c = \inf\{r : r \in \sigma(|T|)\}$ . Since  $U \log((1/c)|T|)$  is semi-hyponormal, by Theorem V.3.5 of [19] it holds that

$$\left\| \log\left(\frac{1}{c}|T|\right) - \log\left(\frac{1}{c}|T^*|\right) \right\| \leq \frac{1}{2\pi} \iint_{\mathbf{D}(U \log((1/c)|T|))} dp d\theta.$$

Since

$$\mathbf{D}\left(U \log\left(\frac{1}{c}|T|\right)\right) = \left\{ e^{i\theta} \cdot \left(\log \frac{r}{c}\right) : re^{i\theta} \in \mathbf{D}(T) \right\}$$

and  $\|\log((1/c)|T|) - \log((1/c)|T^*|)\| = \|\log|T| - \log|T^*|\|$ , by the transformation  $\rho = \log(r/c)$ , we have

$$\|\log|T| - \log|T^*|\| \leq \frac{1}{2\pi} \iint_{\mathbf{D}(T)} r^{-1} dr d\theta.$$

So the proof is complete.  $\square$

Hence we have the following corollary.

**COROLLARY 4.** *Let  $T$  be a log-hyponormal operator with  $m_2(\mathbf{D}(T)) = 0$ . Then  $T$  is normal, where  $m_2(\cdot)$  is the planar Lebesgue measure.*

### 3. Trace formulae of log-hyponormal operators.

For the trace formula of a log-hyponormal operator  $T$ , we define the principal function of  $T$ .

**DEFINITION 4.** Let  $\text{Tr}_{\mathcal{D}}(\cdot)$  be the trace on  $\mathcal{D}$ .

(1) For  $T \in \text{SHU}$ , the *principal function*  $g_T(e^{i\theta}, r)$  of  $T$  is defined by

$$g_T(e^{i\theta}, r) = \text{Tr}_{\mathcal{D}}(\mathbf{B}_T(e^{i\theta}, r)).$$

(2) For a log-hyponormal operator  $T = U|T|$ , put  $T' = U \log|T|$ . The *principal functions*  $g_T(e^{i\theta}, r)$  and  $g_{T'}(e^{i\theta}, r)$  of  $T$  and  $T'$  are defined by

$$g_T(e^{i\theta}, r) = \text{Tr}_{\mathcal{D}}(\mathcal{B}_T(e^{i\theta}, r)) \quad \text{and} \quad g_{T'}(e^{i\theta}, r) = \text{Tr}_{\mathcal{D}}(\mathcal{B}_{T'}(e^{i\theta}, r))$$

where  $\mathcal{B}_T(\cdot, \cdot)$  and  $\mathcal{B}_{T'}(\cdot, \cdot)$  are the mosaics of  $T$  and  $T'$ , respectively.

Subscripts will usually be suppressed when clear from the context.

REMARK 3. For a log-hyponormal operator  $T = U|T|$ , let  $c = \inf\{r : r \in \sigma(|T|)\}$ ,  $T' = U \log|T|$  and  $S = U \log((1/c)|T|)$ . Let  $g_T(e^{i\theta}, r)$ ,  $g_{T'}(e^{i\theta}, r)$  and  $g_S(e^{i\theta}, r)$  be the principal functions of  $T$ ,  $T'$  and  $S$ , respectively. Then by Definition 3 we have

$$g_T(e^{i\theta}, r) = g_{T'}(e^{i\theta}, \log r) = g_S(e^{i\theta}, \log r - \log c).$$

THEOREM 5. Let  $T = U|T|$  and  $S = V|S|$  be log-hyponormal operators. If  $T$  and  $S$  are unitarily equivalent, then

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r).$$

PROOF. Let  $k$  be  $0 < k \leq \inf\{r : r \in \sigma(|T|) \cup \sigma(|S|)\}$ . By Theorem 1, we may consider the principal functions corresponding to the operators  $T' = U \log((1/k)|T|)$  and  $S' = V \log((1/k)|S|)$ . Since theorem holds for semi-hyponormal operators by Theorem VII.2.4 of [19], we may only prove that  $T'$  and  $S'$  are unitarily equivalent. We assume that  $W^*TW = S$  for a unitary operator  $W$ . Since  $W^*|T|W = |S|$ , we have  $W^*(\log((1/k)|T|))W = \log((1/k)|S|)$  and

$$W^*UW|S| = W^*UWW^*|T|W = W^*TW = S = V|S|.$$

Hence  $W^*UWx = Vx$  for  $x \in \text{ran}(|S|)$ . Since  $|S|$  is invertible, we have  $W^*UW = V$ . Therefore, we have

$$\begin{aligned} W^*T'W &= W^*U \left( \log \left( \frac{1}{k} |T| \right) \right) W = W^*UWW^* \left( \log \left( \frac{1}{k} |T| \right) \right) W \\ &= W^*UW \left( \log \left( \frac{1}{k} |S| \right) \right) = V \left( \log \left( \frac{1}{k} |S| \right) \right) = S'. \end{aligned}$$

So the proof is complete.  $\square$

Hence, the principal function  $g_T(\cdot, \cdot)$  of  $T$  is independent of the concrete model of  $T$ .

Here we denote the trace class of operators by  $\mathcal{C}_1$ . For operators  $A$  and  $B$ , the commutator  $AB - BA$  is denoted by  $[A, B]$ . By  $\mathcal{A}_2$ , we denote the linear space of all Laurent polynomials  $p(x, y)$  of two variables such that  $p(x, y) = \sum_{j=0}^N \sum_{k=-N}^N a_{jk} x^j y^k$ , where  $N$  is an arbitrary positive integer. For an operator  $X$  and an invertible operator  $Y$ , we define  $p(X, Y)$  by



$$p(X, Y) = \sum_{j,k} a_{jk} X^j Y^k.$$

For  $p(x, y), q(x, y) \in \mathcal{A}_2$ , we denote the Jacobian  $\frac{\partial p}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial p}{\partial y} \frac{\partial q}{\partial x}$  by  $J(p, q)$  and  $(J(p, q))(r, e^{i\theta}) = \left(\frac{\partial p}{\partial x}\right)(r, e^{i\theta}) \cdot \left(\frac{\partial q}{\partial y}\right)(r, e^{i\theta}) - \left(\frac{\partial p}{\partial y}\right)(r, e^{i\theta}) \cdot \left(\frac{\partial q}{\partial x}\right)(r, e^{i\theta})$ .

Then in [4] we proved the following

**THEOREM E** (Chō and Huruya [4], Theorem 9). *Let  $T = U|T| \in \text{SHU}$  and  $g_T(\cdot, \cdot)$  be the principal function of  $T$  and  $[|T|, U] \in \mathcal{C}_1$ . Then, for  $p, q \in \mathcal{A}_2$ ,*

$$\text{Tr}([p(|T|, U), q(|T|, U)]) = \iint (J(p, q))(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

We show two trace formulae associated with a log-hyponormal operator. First one is

**THEOREM 6.** *Let  $T = U|T|$  be a log-hyponormal operator such that  $[\log|T|, U] \in \mathcal{C}_1$ . Let  $T' = U \log|T|$  and  $g_{T'}$  be the principal function of  $T'$ . Then, for  $p, q \in \mathcal{A}_2$ ,*

$$\text{Tr}([p(\log|T|, U), q(\log|T|, U)]) = \int_{\log c}^{\infty} \left( \int_T J(p, q)(r, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, r) dm(\theta) \right) dr,$$

where  $c = \inf\{r : r \in \sigma(|T|)\}$ .

**PROOF.** Put  $S = U \log((1/c)|T|)$ . Then  $S \in \text{SHU}$  and  $[|S|, U] = [\log|T|, U] \in \mathcal{C}_1$ . Put  $\tilde{p}(x, y) = p(x + \log c, y)$  and  $\tilde{q}(x, y) = q(x + \log c, y)$ . Then it holds

$$\text{Tr}([p(\log|T|, U), q(\log|T|, U)]) = \text{Tr}([\tilde{p}(|S|, U), \tilde{q}(|S|, U)]).$$

By Theorem E, we have

$$\begin{aligned} \text{Tr}([\tilde{p}(|S|, U), \tilde{q}(|S|, U)]) &= \iint_{D(S)} J(\tilde{p}, \tilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r) dr dm(\theta) \\ &= \int_0^{\infty} \left( \int_T J(\tilde{p}, \tilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r) dm(\theta) \right) dr \\ &= \int_0^{\infty} \left( \int_T J(p, q)(r + \log c, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r) dm(\theta) \right) dr. \quad (\dagger) \end{aligned}$$

By the transformation  $t = r + \log c$ , from Remark 3 we have

$$\begin{aligned}
(\dagger) &= \int_{\log c}^{\infty} \left( \int_T J(p, q)(t, e^{i\theta}) e^{i\theta} \dot{g}_S(e^{i\theta}, t - \log c) dm(\theta) \right) dt \\
&= \int_{\log c}^{\infty} \left( \int_T J(p, q)(t, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, t) dm(\theta) \right) dr.
\end{aligned}$$

So the proof is complete.  $\square$

For the second one, we prepare the following

**THEOREM 7.** *Let  $T = U|T| \in \text{SHU}$  and  $g_T(\cdot)$  be the principal function of  $T$ . Let  $[|T|, U] \in \mathcal{C}_1$ . Then, for  $p, q \in \mathcal{A}_2$ ,*

$$\text{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r) dr dm(\theta).$$

**PROOF.** For  $m = 1, 2, \dots$  and  $n = \pm 1, \pm 2, \dots$ , by Theorem 8 of [4] we have

$$\begin{aligned}
(1) \quad \text{Tr}(|T|^m, U^n) &= \iint m n e^{in\theta} r^{m-1} g_T(e^{i\theta}, r) dr dm(\theta) \\
&= \iint n e^{in\theta} \frac{d}{dr} (r^m) g_T(e^{i\theta}, r) dr dm(\theta)
\end{aligned}$$

and by the proof of Theorem 9 of [4]

$$\begin{aligned}
(2) \quad \text{Tr}(|T|^m - U|T|^m U^{-1}) &= \iint m r^{m-1} g_T(e^{i\theta}, r) dr dm(\theta) \\
&= \iint \frac{d}{dr} (r^m) g_T(e^{i\theta}, r) dr dm(\theta).
\end{aligned}$$

For an operator  $S$ , we denote the trace norm of  $S$  by  $\|S\|_1$ . Since

$$[|T|^m, U^n] = |T|^{m-1} [|T|, U^n] + |T|^{m-2} [|T|, U^n] |T| + \dots + [|T|, U^n] |T|^{m-1}$$

and

$$|T|^m - U|T|^m U^{-1} = [|T|^m, U] U^{-1},$$

we have

$$\| [|T|^m, U^n] \|_1 \leq m \| |T| \|^{m-1} \| [|T|, U^n] \|_1$$

and

$$\| |T|^m - U|T|^m U^{-1} \|_1 \leq m \| |T| \|^{m-1} \| [|T|, U] \|_1.$$

Since  $\mathcal{C}_1$  is complete, in  $\mathcal{C}_1$  we have

$$\lim_{\ell \rightarrow \infty} \left[ \left( \sum_{h=0}^{\ell} \frac{1}{h!} |T|^h \right)^m, U^n \right] = [(\exp(|T|))^m, U^n]$$

and

$$\lim_{\ell \rightarrow \infty} \left\{ \left( \sum_{h=0}^{\ell} \frac{1}{h!} |T|^h \right)^m - U \left( \sum_{h=0}^{\ell} \frac{1}{h!} |T|^h \right)^m U^{-1} \right\} = (\exp(|T|))^m - U(\exp(|T|))^m U^{-1}.$$

Since  $|\text{Tr}(D)| \leq \|D\|_1$  for  $D \in \mathcal{C}_1$ , by (1) we obtain

$$\begin{aligned} \text{Tr}([(\exp(|T|))^m, U^n]) &= \lim_{\ell \rightarrow \infty} \iint n e^{in\theta} \cdot \frac{d}{dr} \left( \sum_{h=0}^{\ell} \frac{1}{h!} r^h \right)^m g_T(e^{i\theta}, r) dr dm(\theta) \\ &= \iint n e^{in\theta} \cdot m e^{mr} g_T(e^{i\theta}, r) dr dm(\theta) \end{aligned}$$

and similarly by (2)

$$\text{Tr}([( \exp(|T|))^m - U(\exp(|T|))^m U^{-1}]) = \iint m e^{mr} g_T(e^{i\theta}, r) dr dm(\theta).$$

Putting  $e^r = s$ , we have

$$(3) \quad \text{Tr}([( \exp(|T|))^m, U^n]) = \iint n e^{in\theta} \cdot m s^{m-1} g_T(e^{i\theta}, \log s) ds dm(\theta)$$

and

$$(4) \quad \text{Tr}((\exp(|T|))^m - U(\exp(|T|))^m U^{-1}) = \iint m s^{m-1} g_T(e^{i\theta}, \log s) ds dm(\theta).$$

Define a bilinear form  $(\cdot, \cdot)$  on  $\mathcal{A}_2$  by

$$(p, q) = \text{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)])$$

for  $p, q \in \mathcal{A}_2$ . Let  $p_2(x, y) = y$ . Then by the proof of Theorem 9 of [4] we can define a linear functional  $\ell$  on  $\mathcal{A}_2$  by, for  $q \in \mathcal{A}_2$ ,

$$\ell\left(\frac{\partial q}{\partial y}\right) = (p_2, q).$$

Then by the similar way of the proof of Theorem E we have

$$(5) \quad (p, q) = -\ell(J(p, q)).$$

Next we define a linear functional  $\ell_0$  on  $\mathcal{A}_2$  by, for  $p \in \mathcal{A}_2$ ,

$$\ell_0(p) = \iint p(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r) dr dm(\theta).$$

Since (3) and (4) hold, by the similar way of the proof of Theorem E we have

$$(6) \quad \ell_0 = -\ell.$$

Therefore, by (5) and (6) we have

$$\begin{aligned} \text{Tr}([p(\exp(|T|), U), q(\exp(|T|), U)]) &= (p, q) = \ell_0(J(p, q)) \\ &= \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \log r) dr dm(\theta) \end{aligned}$$

for  $p, q \in \mathcal{A}_2$ . So the proof is complete.  $\square$

Hence, we have the following

**THEOREM 8.** *Let  $T = U|T|$  be a log-hyponormal operator with  $\log|T| \geq 0$  and  $g_T(\cdot, \cdot)$  be the principal function of  $T$ . Assume that  $[\log|T|, U] \in \mathcal{C}_1$ . Then, for any  $p, q \in \mathcal{A}_2$ , it holds that*

$$\text{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

**PROOF.** Let  $T' = U \log|T|$  and  $g_{T'}$  be the principal function of  $T'$ . Since  $T' \in \text{SHU}$ , by Theorem 7 we have

$$\text{Tr}([p(\exp(|T'|), U), q(\exp(|T'|), U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_{T'}(e^{i\theta}, \log r) dr dm(\theta).$$

Since  $g_{T'}(e^{i\theta}, \log r) = g_T(e^{i\theta}, r)$  and  $\exp(|T'|) = |T|$ , we have

$$\text{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

So the proof is complete.  $\square$

Finally, we show the following main result.

**THEOREM 9.** *Let  $T = U|T|$  be a log-hyponormal operator and  $g_T(\cdot, \cdot)$  be the principal function of  $T$ . Assume that  $[\log|T|, U] \in \mathcal{C}_1$ . Then, for any  $p, q \in \mathcal{A}_2$ , it holds that*

$$\text{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

**PROOF.** For  $c = \inf\{r : r \in \sigma(|T|)\}$ , let  $R = U((1/c)|T|)$ . Put  $\tilde{p}(r, z) = p(c \cdot r, z)$  and  $\tilde{q}(r, z) = q(c \cdot r, z)$ . Then we have

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \mathrm{Tr}([\tilde{p}(|R|, U), \tilde{q}(|R|, U)]).$$

Since  $R$  is log-hyponormal with  $\log|R| = \log((1/c)|T|) \geq 0$ , by Theorem 8 we have

$$\begin{aligned} \mathrm{Tr}([\tilde{p}(|R|, U), \tilde{q}(|R|, U)]) &= \iint J(\tilde{p}, \tilde{q})(t, e^{i\theta}) e^{i\theta} g_R(e^{i\theta}, t) dt dm(\theta) \\ &= \iint c \cdot J(p, q)(c \cdot t, e^{i\theta}) e^{i\theta} g_R(e^{i\theta}, t) dt dm(\theta). \quad (\dagger\dagger) \end{aligned}$$

By the transformation  $r = c \cdot t$ , we have

$$\begin{aligned} (\dagger\dagger) &= \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_R\left(e^{i\theta}, \frac{r}{c}\right) dr dm(\theta) \\ &= \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta), \end{aligned}$$

because  $g_T(e^{i\theta}, r) = \mathrm{Tr}(\mathbf{B}_{U \log((1/c)|T|)}(e^{i\theta}, \log(r/c))) = g_R(e^{i\theta}, (r/c))$  by Definition 2 and Remark 3. So the proof is complete.  $\square$

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Muneo CHŌ

Department of Mathematics  
Kanagawa University  
Yokohama 221-8686  
Japan  
E-mail: chiyom01@kanagawa-u.ac.jp

Tadasi HURUYA

Faculty of Education and Human Sciences  
Niigata University  
Niigata 950-2181  
Japan  
E-mail: huruya@ed.niigata-u.ac.jp