PAPER *Special Section of Selected Papers from the 10th Karuizawa Workshop on Circuits and Systems*

Covering Problems in the p-Collection Problems

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SUMMARY The lower-bounded p-collection problem is the problem where to locate p sinks in a flow network with lower bounds such that the value of a maximum flow is maximum. This paper discusses the cover problems corresponding to the lower bounded p-collection problem. We consider the complexity of the cover problem, and we show polynomial time algorithms for its subproblems in a network with tree structure.

key words: *location problem, network flows, NP-complete, optimization problem*

1. Introduction

Location problems are an important in a lot of fields, applied mathematics, computer science, operations research, management science and industrial engineering (see the papers [1], [2], or the book [3]). The p-collection problem is the problem where to locate p sinks in a flow network such that the value of a maximum flow is maximum [6]. This problem is an important location problem in a flow network because one can apply to locating p resources (e.g. data bases, file-servers, etc.) in a computer network such that as many terminals (clients) can utilize these resources as possible. The paper [4] deal with cover problems in flow networks. This paper discusses new cover problems, which correspond to the p-collection problems.

Let $D = (V, A)$ be the digraph with a vertex set V and an arc set A such that $(v, u) \notin A$ for any arc (u, v) of A. Let b^- and b^+ be functions: $V \to Z$ (the set of integers) such that $b^-(v) \leqq b^+(v)$ for any v of V, let c^- and c^+ be functions: $A \to Z$ such that $c^-(a) \leqq c^+(a)$ for any a of A, and let d^- and d^+ be functions: $V \to Z \cup \{\infty\}$ such that $d^-(v) \leqq d^+(v)$ for any v of V. The functions $b^+(v)$ and $b^-(v)$ imply respectively an upper and an lower bounds of flows from the source to v. The functions $c^+(a)$ and $c^-(a)$ correspond to respectively the capacity and the lower bound of a. The functions $d^+(v)$ and $d^-(v)$ mean respectively an upper and an lower bounds of flows from v to the sink. We call a 7-tuple:

$$N = (D, b^-, b^+, c^-, c^+, d^-, d^+)$$

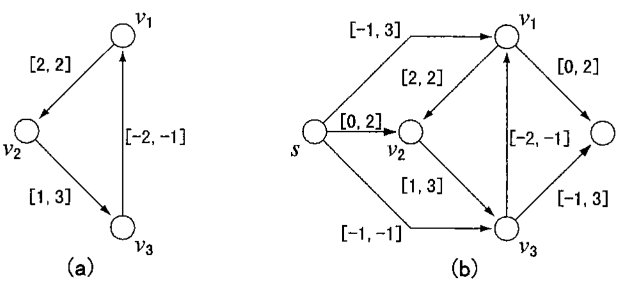
a *network with lower bounds* (usually, described as $N = (D, b^\pm, c^\pm, d^\pm)$, and called a network, for convenience). Figure 1 (a) illustrates an example of a network with lower bounds. Let s and t be new specified vertices called the source and the sink, respectively. We represent V^*, A^s and A^t as the vertex set $V \cup \{s, t\}$, the arc sets $\{(s, v) : v \in V\}$ and $\{(v, t) : v \in V\}$ respectively. Let X be an arbitrary subset of V, let $A^t_X = \{(x, t) : x \in X\}$, and let $A^*_X = A \cup A^s \cup A^t_X$. We define functions e^- and e^+: $A^*_V \to Z \cup \{\infty\}$ as

$$e^\pm(u, v) = \begin{cases} b^\pm(v) & \text{if } u = s \\ d^\pm(u) & \text{if } v = t \\ c^\pm(u, v) & \text{otherwise.} \end{cases}$$

(In this paper, double signs in an equation are in same order.) And we define the adjoint network N_X of N with respect to a subset X of V as the s-t flow network with lower bounds:

$$N_X = (D_X, s, t, e^+|A^*_X, e^-|A^*_X),$$

where $D_X = (V^*, A^*_X)$, and where $e^-|A^*_X$ (respectively, $e^+|A^*_X$) denotes the restrictions of e^- (e^+ respectively) on A^*_X. The adjoint network N_X is *feasible* if there exists a flow f in A_X meeting the following conditions.
Capacity Constrain: For any $a \in A^*_X$,



$b^-(v_1) = -1,\quad b^+(v_1) = 3,\quad d^-(v_1) = 0,\quad d^+(v_1) = 2$
$b^-(v_2) = 0,\quad b^+(v_2) = 2,\quad d^-(v_2) = 1,\quad d^+(v_2) = 3$
$b^-(v_3) = -1,\quad b^+(v_3) = -1,\quad d^-(v_3) = -1,\quad d^+(v_3) = 3.$

Fig. 1 (a) A network N, (b) the adjoint network of N with respect to $X = \{v_1, v_3\}$.

Manuscript received June 30, 1997.
Manuscript revised September 12, 1997.
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$$e^-(a) \leqq f(a) \leqq e^+(a).$$

Flow Conservation: The flow f is conserved in any vertex v of V in N_X. That is,

$$\sum_{u \in \mathrm{ad}(v)} f(v, u) = 0,$$

where $\mathrm{ad}(v)$ is the set of vertices adjacent to-or-from v in N_X, and where $f(v, u) = -f(u, v)$ for any (u, v) of A_X^*.

A flow has skew symmetry, which is already used in the condition of Flow Conservation. That is, we can assume that, if (u, v) is an arc of A_X^*, then $f(v, u) = -f(u, v)$. For any (u, v) of A_X^*, we can also suppose that $c^\pm(v, u) = -c^\mp(u, v)$, and $e^\pm(v, u) = -e^\mp(u, v)$, since the capacity constrain holds for the pair (v, u).

If N_X is feasible, the value of f is $\mathrm{val}(f) = \sum_{a \in A_X^t} f(a)$ for any flow f in N_X. Given a network N, we define $h_N(X)$ as

$$h_N(X) = \begin{cases} \max_f \mathrm{val}(f) & \text{if } N_X \text{ is feasible} \\ -\infty & \text{otherwise.} \end{cases}$$

(Notice that $\max_f \mathrm{val}(f)$ is the value of a maximum flow in N_X.) If p is a positive integer with $p \leqq |V|$, let $H_N(p) = \max\{h_N(X) : |X| = p\}$. (We usually omit the subscripts N of h_N and H_N when N is clear from context.) A subset X^* of V with $|X^*| = p$ is a *maximum p-collection set* of N if $h(X^*) = H(p)$. We call the optimization problem of searching a maximum p-collection set of a network the *lower-bounded p-collection problem*, and we write LBC as this problem. For example let's solve LBC for the network illustrated in Fig. 1 (a). We have

$$h(v_1, v_2) = -\infty,\ h(v_2, v_3) = 4,\ h(v_3, v_1) = 3.$$

Thus $H(2) = 4$, and $\{v_2, v_3\}$ is the maximum 2-collection set.

The p-collection problem discussed in [6] is equivalent to the subproblem of LBC such that

$$\begin{array}{ll} b^-(v) = 0 & \text{for any } v \text{ of } V, \\ c^-(a) \leqq 0 \leqq c^+(a) & \text{for any } a \text{ of } A, \\ d^-(v) = 0 \text{ and } d^+(v) = \infty & \text{for any } v \text{ of } V. \end{array}$$

In this paper we describe this problem to PRC, the primary p-collection problem.

The sink-capacitated p-collection problem discussed in [7] is equivalent to the subproblem of LBC such that

$$\begin{array}{ll} b^-(v) = 0 & \text{for any } v \text{ of } V, \\ c^-(a) \leqq 0 \leqq c^+(a) & \text{for any } a \text{ of } A, \\ d^-(v) = 0 & \text{for any } v \text{ of } V. \end{array}$$

In this paper we describe this problem to SCC.

We introduce description $\alpha/\beta/\gamma$ for subproblems of the p-collection problem. The first item α denotes is the value of p. If $\alpha = p$, suppose that p is arbitrarily fixed. The second item β indicates a network topology. This paper deals with two topology types: general network ($\beta = N$) and network with tree structure ($\beta = T$) where a network with tree structure is a network whose underlying graph is a tree. The last item γ means a subproblem (PRC, SCC or LBC) on which $b^\pm$, $c^\pm$ and $d^\pm$ are restricted. For example, the problem $p/T/$LBC denotes the lower-bounded p-collection problem in a network with tree structure.

A subset X of V is a *cover* of N if $h(X) = \sum_{v \in V} b^+(v)$. (Notice that there is not always a cover.) The cover problem corresponding to LBC is the problem of finding a minimum cover of N. We refer C$/\beta/\gamma$ as the cover problem corresponding to $p/\beta/\gamma$. In the network illustrated in Fig. 1 (a), the subset $\{v_2, v_3\}$ is a minimum cover, since each subset with cardinality 1 is not a cover.

In the paper [6] the authors presented an $O(n)$ algorithm for $1/T/$PRC, and an $O(p^2n^3C^2)$ algorithm for $p/T/$PRC, where $n = |V|$ and C denotes the maximum of weight and capacity. Tsukiyama [5] proposed an $O(p^2n^2C^2)$ algorithm for $p/T/$PRC. The paper [8] contains some complexity results that $p/N/$PRC is strongly NP-hard, and that $p/T/$PRC is weakly NP-hard. The paper [9] proves the weak NP-hardness of $p/T/$LBC, and presents an $O(n)$ algorithm for $1/T/$LBC and an $O(p^2n^2C^2)$ algorithm for $p/T/$LBC.

This paper discusses the cover problems. In Sect. 2 we obtain complexity results for cover problems. Section 3 presents an $O(n)$ algorithm for C$/T/$PRC, and Sect. 4 presents an $O(n^3)$ algorithm for C$/T/$SCC with dynamic programming type. Hence we know that, although $p/T/$PRC and $p/T/$SCC are NP-hard, the corresponding cover problems are solvable in polynomial time.

2. Complexity Results

In this section we shall obtain complexity results for cover problems. The paper [8] shows strong NP-hardness for $p/N/$PRC by polynomial reduction from the vertex cover problem to this problem. The reduced instance in there is also the instance of C$/N/$PRC. Hence C$/N/$PRC is strong NP-hard. Hence we obtain the following theorem.

Theorem 1: The problem C$/N/$PRC, C$/N/$SCC and C$/N/$LBC are strongly NP-hard.

We can know weak NP-hardness for C$/T/$LBC in much the same way as the proof for $p/T/$LBC in the paper [9]. Thus we obtain the following theorem.

Theorem 2: The problem C$/T/$LBC is weakly NP-hard.

In Sects. 3 and 4, we shall design polynomial time algorithms for C$/T/$PRC and C$/T/$SCC, respectively. Thus, although $p/T/$PRC and $p/T/$SCC are NP-hard, the following theorem holds for the corresponding cover problems.

Table 1 Complexity results for cover problems.

Topology type	PRC	SCC	LBC
Network	Strongly NPH	Strongly NPH	Strongly NPH
Tree	P	P	Weakly NPH

Theorem 3: The covering problems C/T/PRC and C/T/SCC belong to the class P.

The obtained results so far are brought together in Table 1.

3. The Problem C/T/PRC

Let $N = (T, b^{\pm}, c^{\pm}, d^{\pm})$ be a network with tree structure, where $T = (V, A)$. For a cover problem, the function $b^{-}(\cdot)$ has few meaning. For convenience, we assign $b^{-}(v)$ to $-\infty$ for any v of V. We present Algorithm 1 with complexity $O(n)$ for C/T/PRC, and we prove Theorem 4.

Theorem 4: Algorithm 1 solves C/T/PRC exactly.

Proof: Suppose that the algorithm returns a vertex set X for a network $N = (T, b^{\pm}, c^{\pm}, d^{\pm})$ where $T = (V, A)$. Let v_0 denote the vertex indicated by u in Step 1.20, let $B(u)$ denote the value of $b^{+}(u)$ in Step 1.6, and let $B(v_0)$ denote the value of $b^{+}(u)$ in Step 1.21. If v is a vertex of $V - \{v_0\}$, then $n(v)$ denotes the vertex adjacent to v and nearest to v_0 in N, and then $U(v)$ denotes the set of vertices adjacent to-or-from v excluding $n(v)$. And let $U(v_0)$ denote the set of vertices adjacent to-or-from v_0. To begin with, we show that X is a cover of N. We define the function f on A^{*}_{X} as follows. For any (s, v) of A^{s}, let $f(s, v) = b^{+}(v)$. For any u of $V - \{v_0\}$, let $f(u, v)$ be the value of D in Step 1.17. Then f satisfies the capacity constrain in $A^{s} \cup A$. We correct f to be flow-conserved in each vertex of $V - X$.

Consider the case where $v_0 \notin X$. Then $B(v_0) = \sum_{u \in U(v_0)} f(u, v_0) + b^{+}(v_0) \leqq 0$. Since $b^{+}(v_0) \geqq 0$, we can increase some $f(u, v_0)$s with negative value without exceeding 0 for f to be flow-conserved on v_0. If $v_0 \in X$, then we do not need to have above the consideration. However, if x is a vertex of X, then let $f(u, x) = 0$ for any u of $U(x)$ such that $f(u, x) < 0$. We call $f(u, v)$ corrected if $v \in X \cup \{v_0\}$, and if $u \in U(v)$. Let v be a vertex of $V - \{v_0\}$. If $f(v, n(v))$ is corrected, and if $f(u, v)$ is not corrected for any u of $U(v)$, then we perform the following operation, hoping for f to be flow-conserved on v.

Step 1. If $B(v) < 0$, then increase some $f(u, v)$s with negative value without exceeding 0 to meet the flow conservation on v.

Step 2. Let $f(u, v)$ be corrected for any u of $U(v)$. □

If $B(v) \geqq 0$, then f is flow-conserved on v. Hence Step 1 corrects f if $B(v) < 0$. Thus

$$B(v) = \sum_{u \in U(v)} f(u, v) + b^{+}(v) < 0.$$

Algorithm 1

```
1.1  begin
1.2    X ← ∅;
1.3    while |V(N)| > 1 do
1.4      u ← an arbitrary leaf of N;
1.5      v ← the vertex of N adjacent to-or-from u;
1.6      if b+(u) ≧ 0 then
1.7        if b+(u) ≦ c+(u,v) then
1.8          D ← b+(u);
1.9        else
1.10         b+(u) ← −∞;
1.11         X ← X ∪ {u};
1.12       fi;
1.13     fi;
1.14     if b+(u) < 0 then
1.15       D ← max{b+(u), c−(u,v)};
1.16     fi;
1.17     b+(v) ← b+(v) + D;
1.18     remove u from N;
1.19   od;
1.20   u ← the vertex of N;
1.21   if b+(u) > 0 then X ← X ∪ {u} fi;
1.22   return(X);
1.23 end.
```

Since $b^{+}(v) \geqq 0$, we can always perform Step 1. Hence this operation changes f to be flow-conserved on v. We repeat the preceding operation, until $f(u, v)$ is corrected for any (u, v) of A. After the iteration completes, the function f meets the capacity constrain in $A^{s} \cup A$ and the flow conservation in $V - X$. We finally define

$$f(x, t) = \sum_{v \in \mathrm{ad}(x)} f(v, x) + b^{+}(x),$$

and the function f has changed to a flow in N_X such that $f(s, v) = b^{+}(v)$ for any (s, v) of A^{s}. Hence X covers N.

There remains to verify whether X is minimum. If we remove the arc $(v, n(v))$ from T for any v of $X - \{v_0\}$, then we obtain the disconnected digraph. This disconnected digraph consists of the connected components each with either v_0 or a vertex of X. Let x be a vertex of $X \cup \{v_0\}$, let T_x be the component with x, and let $V_x = V(T_x)$. Now let W be a subset of $V - V_x$, and assume that W is a cover of N. Then there exists a maximum flow g in N_W such that $g(s, v) = b^{+}(v)$ for any v of V. For any v of V_x, the following inequality must hold.

$$B(v) \leqq \sum_{u \in U(v)} g(u, v) + g(s, v).$$

Consider the case where $x = v_0$, and where $x \in X$. Then

$$\sum_{u \in U(x)} g(u,x) + g(s,x) > 0.$$

Since $x \notin W$, the flow g is not conserved in x. This result contradicts that g is a flow. Hence W is not cover. Now consider the case where $x \in X - \{v_0\}$. Since $c^+(x, n(x)) < B(x)$, we obtain

$$c^+(x, n(x)) < g(x, n(x)).$$

This inequality contradicts the capacity constrain. Hence W is not a cover of N. By the above two considerations, a cover of N must contain a vertex of V_x for any x of X. Hence $|X| \leqq |X^*|$ where X^* denotes a minimum cover of N. We know X is a cover of N, and so we have $|X^*| = |X|$. Consequently X is a minimum cover. □

4. The Problem $C/T/$SCC

In this section we present an algorithm with dynamic programming type for C/T/SCC. Given a network, let v_1 be an arbitrary vertex of it. For convenience, we add to the given network a new vertex v_0 and a new arc (v_1, v_0) such that $c^\pm(v_1, v_0) = \pm\infty$. Let $N = (T, b^\pm, c^\pm, d^\pm)$ denote the new obtained network. We suppose that v_0 is the root, and regard N as rooted. For any vertex v of N, we define the level lev(v) of v as the number of all the arcs in a v-v_0 path between v and v_0. Let $L(l) = \{v \in V : \mathrm{lev}(v) = l\}$ for any nonnegative l, and let l^* denote the maximum integer such that $L(l^*) \neq \emptyset$. If v is a vertex of $V - \{v_0\}$, then $p(v)$ denotes the parent of v, the vertex adjacent to v and nearest to v_0 in N, and then Ch(v) denotes the set of all the children, all the vertices adjacent to-or-from v excluding $p(v)$. Let v be a vertex of T, and let U' be a subset of Ch(v). If we remove the arcs of $\{(v, p(v))\} \cup \{(u, v) : u \in \mathrm{Ch}(v) - U'\}$ from T, then we obtain the subtree $T^v_{U'}$ involving v. Fixing U', let $W = V(T^v_{U'}) - \{v\}$, and let $N^v_{U'}$ denotes the restriction of N on $T^v_{U'}$. Let k be an integer such that $0 \leqq k \leqq |W|$, let X be a subset of W with $|X| = k$, and let $(N^v_{U'})_X$ be the restriction of the adjoint network N_X on the digraph obtained by adding to $T^v_{U'}$ the source s, the sink t, the arc set $\{(s, w) : w \in W\}$, and $\{(x, t) : x \in X\}$. The subnetwork $N^v_{U'}$ is k-covered if there exists a function f_X on the arc set of $(N^v_{U'})_X$ for some X, satisfying the capacity constrain for any arc of $(N^v_{U'})_X$, and the flow conservation for any vertex of W, such that $f_X(s, w) = b^+(s, w)$ for any w of W. (Notice that flows does not always conserved on v.) We define

$$F^v_{U'}(k) = \begin{cases} \min\limits_{|X|=k} \min\limits_{f_X} \sum\limits_{u \in U'} f_X(u, v) & \text{if } N' \text{ is } k\text{-covered} \\ \infty & \text{otherwise.} \end{cases}$$

The value of $F^v_{U'}(k)$ means the minimum value of a flow on the cut from U' to v that covers all the vertices of W.

Let's examine the meaning of $F^{v_0}_{\{v_1\}}(k)$. There exist a subset X of W with $|X| = k$ and f_X on $(N^{v_0}_{\{v_1\}})_X$ such that $f_X(v_0, v_1) = 0$ if and only if there exists a cover with size k of the given network. From the definition of $F^v_{U'}(k)$, if $F^{v_0}_{\{v_1\}}(k) > 0$, then there is not such X and f_X; otherwise however there exist. Hence $F^{v_0}_{\{v_1\}}(k) \leqq 0$ if and only if there exists a cover with size k. Thus we shall find the minimum of k such that $F^{v_0}_{\{v_1\}}(k) \leqq 0$. We shall obtain an algorithm C/T/SCC by simplifying the algorithm for p/T/SCC in the paper [7]. That algorithm stores all feasible values of flows on each cut. In this case the considered algorithm stores only the maximum value of feasible flows. Thus C/T/SCC is solvable in polynomial time. Now study the properties of $F^v_{U'}(k)$ separating three cases.

Case 1: $|U'| = 1$, and $u \in U'$ is a leaf.
It is easy to see the following equation for $k = 0$.

$$F^v_{U'}(0) = \begin{cases} b^+(u) & \text{if } b^+(u) \leqq c^+(u, v) \\ \infty & \text{otherwise.} \end{cases}$$

And for $k = 1$,

$$F^v_{U'}(1) = \begin{cases} b^+(u) - d^+(u) & \text{if } b^+(u) - d^+(u) \leqq c^+(u, v) \\ \infty & \text{otherwise.} \end{cases}$$

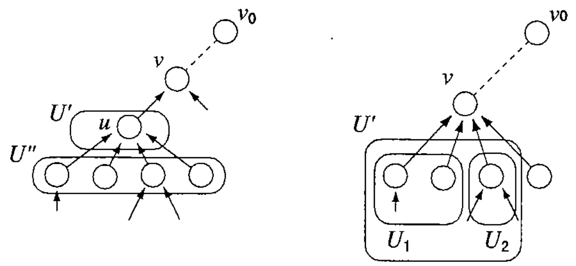
Case 2: $|U'| = 1$, and $u \in U'$ is not a leaf.
Let $U'' = \mathrm{Ch}(u)$, and furthermore consider two cases. Fig. 2 (a) shows state in this case.
Case 2.1: u is not adjacent to t, and $k < |W|$.
Let $F'_{U'}(k)$ denote the value of $F^v_{U'}(k)$ in this case. If $F^u_{U''}(k) + b^+(u) \leqq c^+(u, v)$, then we obtain

$$F'_{U'}(k) = \min\{F^u_{U''}(k) + b^+(u), c^-(u, v)\};$$

otherwise $F'_{U'}(k) = \infty$.
Case 2.2: u is adjacent to t, and $k \geqq 1$.
Let $F''_{U'}(k)$ denote the value of $F^v_{U'}(k)$ in this case. If $F^u_{U''}(k-1) + b^+(u) - d^+(u) \leqq c^+(u, v)$, then we obtain

$$F''_{U'}(k) = \min\{F^u_{U''}(k-1) + b^+(u) - d^+(u), c^-(u, v)\};$$

otherwise $F''_{U'}(k) = \infty$.



(a) (b)

Fig. 2 (a) Case 2, (b) Case 3.

Algorithm 2

2.1 begin
2.2 $\quad i \leftarrow l^* - 1$;
2.3 $\quad$ while $i \geqq 0$ do
2.4 $\quad\quad$ for all $v \in L(i)$ do
2.5 $\quad\quad\quad$ for all $u \in \mathrm{Ch}(v)$ do
2.6 $\quad\quad\quad\quad U' \leftarrow \{u\}$;
2.7 $\quad\quad\quad\quad$ if u is a leaf then
2.8 $\quad\quad\quad\quad\quad$ calculate $F^v_{U'}(0)$ and $F^v_{U'}(1)$; {Case 1}
2.9 $\quad\quad\quad\quad\quad X_{U'}(0) \leftarrow \emptyset$, and $X_{U'}(1) \leftarrow \{u\}$;
2.10 $\quad\quad\quad\quad$ else
2.11 $\quad\quad\quad\quad\quad m \leftarrow |V(T^v_{U'})| - 1$;
2.12 $\quad\quad\quad\quad\quad$ for all $0 \leqq k < m$ do calculate $F'_{U'}(k)$ od; {Case 2.1}
2.13 $\quad\quad\quad\quad\quad$ for all $1 \leqq k \leqq m$ do calculate $F''_{U'}(k)$ od; {Case 2.2}
2.14 $\quad\quad\quad\quad\quad F^v_{U'}(0) \leftarrow F'_{U'}(0)$, $X_{U'}(0) \leftarrow X_{\mathrm{Ch}(u)}(0)$, $F^v_{U'}(m) \leftarrow F''_{U'}(m)$, and $X_{U'}(m) \leftarrow X^u_{\mathrm{Ch}(u)}(k)$;
2.15 $\quad\quad\quad\quad\quad$ for all $1 \leqq k < m$ do
2.16 $\quad\quad\quad\quad\quad\quad$ if $F'_{U'}(k) \leqq F''_{U'}(k)$ then $F^v_{U'}(k) \leftarrow F'_{U'}(k)$, and $X_{U'}(k) \leftarrow X_{\mathrm{Ch}(u)}(k)$;
2.17 $\quad\quad\quad\quad\quad\quad$ else $F^v_{U'}(k) \leftarrow F''_{U'}(k)$, and $X_{U'}(k) \leftarrow X_{\mathrm{Ch}(u)}(k-1) \cup \{u\}$
2.18 $\quad\quad\quad\quad\quad\quad$ fi;
2.19 $\quad\quad\quad\quad\quad$ od;
2.20 $\quad\quad\quad\quad$ fi;
2.21 $\quad\quad\quad$ od;
2.22 $\quad\quad\quad U_1 \leftarrow$ the set of a vertex of $\mathrm{Ch}(v)$;
2.23 $\quad\quad\quad$ while $U_1 \neq \mathrm{Ch}(v)$ do
2.24 $\quad\quad\quad\quad U_2 \leftarrow$ the set of a vertex of $\mathrm{Ch}(v) - U_1$, and $U' \leftarrow U_1 \cup U_2$;
2.25 $\quad\quad\quad\quad$ for all $0 \leqq k < |V(T^v_{U'})|$ do
2.26 $\quad\quad\quad\quad\quad$ calculate $F^v_{U'}(k)$; {Case 3}
2.27 $\quad\quad\quad\quad\quad X_{U'}(k) \leftarrow X_{U_1}(k_1) + X_{U_2}(k_2)$; {where k_1, k_2 are integers such that $F^v_{U'}(k) = F^v_{U_1}(k_1) + F^v_{U_2}(k_2)$}
2.28 $\quad\quad\quad\quad$ od;
2.29 $\quad\quad\quad\quad U_1 \leftarrow U'$;
2.30 $\quad\quad\quad$ od;
2.31 $\quad\quad$ od;
2.32 $\quad\quad i \leftarrow i - 1$;
2.33 $\quad$ od;
2.34 $\quad k^* \leftarrow$ the minimum integer such that $F^{v_0}_{\{v_1\}}(k) \leqq 0$;
2.35 $\quad$ return$(X_{v_1}(k^*))$;
2.36 end.

Using the results in Case 2.1 and 2.2, we can obtain

$$F^v_{U'}(k) = \begin{cases} F'_{U'}(k) & \text{if } k = 0 \\ \min\{F'_{U'}(k), F''_{U'}(k)\} & \text{if } 0 < k < |W| \\ F''_{U'}(k) & \text{if } k = |W|. \end{cases}$$

We can compute the value of $F^v_{U'}(k)$ if we know the values of $F^u_{U''}(k)$ for all children u of v.

Case 3: $|U'| \geqq 2$.

Let U_1 and U_2 be non-empty subsets of U' such that $U_1 \cup U_2 = U'$, and such that $U_1 \cap U_2 = \emptyset$. Figure 2 (b) shows state in this case. Then

$$F^v_{U'}(k) = \min_{k_1,k_2}\{F^v_{U_1}(k_1) + F^v_{U_2}(k_2)\},$$

where k_1 and k_2 are integers such that $k_1 + k_2 = k$, such that $0 \leqq k_1 \leqq \min\{k, |V(T_{U_1})| - 1\}$, and such that $0 \leqq k_2 \leqq \min\{k, |V(T_{U_2})| - 1\}$. We can estimate the value of $F^v_{U'}(k)$ if we know the values of $F^v_{U_1}(k_1)$ and $F^v_{U_2}(k_2)$ for all k_1 and k_2. (In this algorithm we assume that $|U_2| = 1$.)

From the above results we obtain the recurring formulas (Case 2 and 3) and the boundary condition (Case 1) for dynamic programming. Suppose that we know the value of $F^u_{U''}(k)$ for any k and for any u of $\mathrm{Ch}(v)$. Then we can estimate the value of $F^v_{\{u\}}(k)$ for any k and for any u (Fig. 2 (a)). Next we can obtain the value of $F^v_{\mathrm{Ch}(v)}(k)$ for any k (Fig. 2 (b)). We can design Algorithm 2. First the algorithm calculate $F^v_{\mathrm{Ch}(v)}(k)$ for any $v \in L(l^*-1)$, next for any $v \in L(l^*-2)$, $\cdots$, and last for the root v_0.

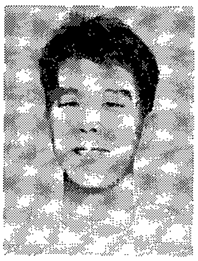
In algorithm 2, the step 2.5 is executed $O(n)$ times. It costs $O(n)$ time to calculate each $F'_{U'}(k)$, $F''_{U'}(k)$ and $F^v_{U'}(k)$. Since $0 \leqq k \leqq n$, the complexity of the algorithm is $O(n^3)$.

5. Conclusion

This paper deals with the cover problems corresponding to PRC, SCC and LBC. In Sect. 2, we obtain the complexity results for the covering problems. Section 3 presents an $O(n)$ algorithm for $C/T/$PRC, and Sect. 4 presents an $O(n^3)$ algorithm with dynamic programming type for $C/T/$SCC. Hence, although $p/T/$PRC and $p/T/$SCC are NP-hard, the corresponding cover problems, $C/T/PRC$ and $C/T/SCC$, are solvable in polynomial time. We hope to improve Algorithm 2.

References

[1] O. Kariv and S.L. Hakimi, "An algorithmic approach to network location problems. I: The p-centers," SIAM J., APPL., MATH., vol.37, no.3, pp.515–538, Dec. 1979.
[2] O. Kariv and S.L. Hakimi, "An algorithmic approach to network location problems. II: The p-medians," SIAM J., APPL., MATH., vol.37, no.3, pp.539–560, Dec. 1979.
[3] P.B. Mirchandani and R.L. Francis, ed., "Discrete location theory," John Wiley & Sons, Inc., 1990.
[4] H. Tamura, M. Sengoku, S. Shinoda, and T. Abe, "Some covering problems in location theory on flow networks," IEICE Trans., Fundamentals, vol.E75-A, no.6, pp.678–684, June 1992.
[5] S. Tsukiyama, "A new algorithm for p-collection problem on a tree-type flow network," Proc. of ITC-CSCC, vol.2, pp.721–724, 1996.
[6] K. Watanabe, H. Tamura, and M. Sengoku, "Problems of where to locate p-sinks in a flow network," IEICE Trans., vol.J78-A, no.8, pp.938–946, 1995.
[7] K. Watanabe, H. Tamura, and M. Sengoku, "Problems of where to locate capacitated p sinks in a flow network," Proc. of JTC-CSCC, pp.758–761, 1995.
[8] K. Watanabe, H. Tamura, and M. Sengoku, "The problems of where to locate p-sinks in a flow network: Complexity approach," IEICE Trans. Fundamentals, vol.E79-A, no.9, pp.1495–1503, Sept. 1996.
[9] K. Watanabe, H. Tamura, and M. Sengoku, "The p-collection problem in a flow network with lower bounds," IEICE Trans., Fundamentals, vol.E80-A, no.4, pp.651–657, April 1997.



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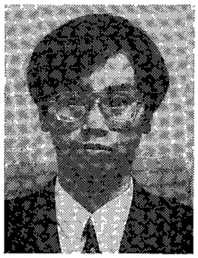

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