PAPER *Special Section on Digital Signal Processing*

Multidimensional Multirate Filter and Filter Bank without Checkerboard Effect

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SUMMARY The checkerboard effect is caused by the periodic time-variant property of multirate filters which consist of up-samplers and digital filters. Although the conditions for some one-dimensional (1D) multirate systems to avoid the checkerboard effect have been shown, the conditions for Multidimensional (MD) multirate systems have not been considered. In this paper, some theorems about the conditions for MD multirate filters without checkerboard effect are derived. In addition, we also consider MD multirate filter banks without checkerboard effect. Simulation examples show that the checkerboard effect can be avoided by using the proposed conditions.

key words: *multidimensional signal processing, multirate filter, multirate filter bank, checkerboard effect*

1. Introduction

Multirate signal processing is widely used in subband coding, and adaptive signal processing, etc. [11]. Recently Multidimensional (MD) multirate signal processing is expected in sampling format conversion, and in many other applications of digital video processing.

The checkerboard effect is caused by the periodic time-variant property of the multirate filters which consist of up-samplers and digital filters. The conditions for some one-dimensional (1D) multirate systems to avoid the checkerboard effect have been shown [1]–[7]. In [1], the checkerboard effect for 1D multirate filter banks was pointed out. In [2], [4], the checkerboard effect was considered and some theorems about the conditions to avoid the checkerboard effect were derived. In [3], [5]–[7], the conditions for 1D multirate filter banks were considered and some design methods were given. However, the conditions for MD multirate systems without checkerboard effect have not been considered.

In this paper, some theorems about the conditions for MD multirate filters to avoid the checkerboard effect are derived. We also consider MD multirate filter banks without checkerboard effect. Simulation examples show that the checkerboard effect can be avoided by using the proposed conditions.

All through this work, we use the following notations for D-dimensional systems and signals [11].

$\mathbf{z}$: the $\mathbf{z}$ denotes the $D \times 1$ vector

$$\mathbf{z} = [z_0 z_1 \cdots z_{D-1}]^T. \tag{1}$$

$\mathbf{z}^{(\mathbf{M})}$: the $\mathbf{z}^{(\mathbf{M})}$ is the $D \times 1$ vector whose kth component is obtained as

$$(\mathbf{z}^{(\mathbf{M})})_k = z_0^{M_{0,k}} z_1^{M_{1,k}} \cdots z_{D-1}^{M_{D-1,k}} \tag{2}$$

where $\mathbf{M}$ is a $D \times D$ nonsingular integer matrix, and $M_{k,l}$ denotes the k, lth element of $\mathbf{M}$.

$LAT(\mathbf{M})$: $LAT(\mathbf{M})$ is the set of integer vectors $\mathbf{p}$ defined by

$$\mathbf{p} = \mathbf{M}\mathbf{n}, \quad \mathbf{n} \in \mathcal{N}, \tag{3}$$

where $\mathcal{N}$ is the set of $D \times 1$ integer vectors and $\mathbf{n}$ is a $D \times 1$ integer vector $[n_0, n_1, \cdots, n_{D-1}]^T$. The subscript T on the vector denotes the transposition.

$FPD(\mathbf{M})$: $FPD(\mathbf{M})$ is the set of vectors defined by

$$FPD(\mathbf{M}) = \{\mathbf{M}\mathbf{x} | \mathbf{x} \in [0,1)^D\}, \tag{4}$$

where $[0,1)$ denotes the set of $D \times 1$ vectors $\mathbf{x}$ so that all elements of $\mathbf{x}$ satisfy $0 \leqq x_i < 1, i = 0, \cdots, D-1$.

$\mathcal{N}(\mathbf{M})$: $\mathcal{N}(\mathbf{M})$ is the set of integer vectors in $FPD(\mathbf{M})$.

2. MD Multirate Filter without Checkerboard Effect

In this section, some theorems about the conditions for MD multirate filters to avoid the checkerboard effect are shown.

2.1 MD Multirate Filter

Figure 1 shows an MD multirate filter, where $\uparrow \mathbf{M}_U$ and $\downarrow \mathbf{M}_D$ denote an up-sampler with the factor $\mathbf{M}_U$ and a down-sampler with the factor $\mathbf{M}_D$ respectively. $\mathbf{M}_U$ and $\mathbf{M}_D$ are $D \times D$ non singular integer matrices and they are mutually prime matrices which mean that $LAT(\mathbf{M}_U) \subseteqq LAT(\mathbf{M}_D)$ and $LAT(\mathbf{M}_D) \subseteqq LAT(\mathbf{M}_U)$ are not satisfied [12]. $F(\mathbf{z})$ is the transfer function of an MD FIR filter given as

$$F(\mathbf{z}) = \sum_{\mathbf{n} \in \mathcal{N}} f(\mathbf{n}) \mathbf{z}^{(-\mathbf{n})}, \tag{5}$$

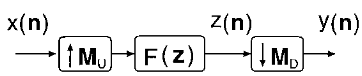

Fig. 1 MD multirate filter.

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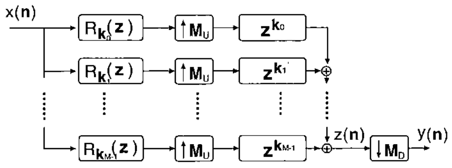


Fig. 2 Polyphase structure.

where $f(\mathbf{n})$ denotes the impulse response of $F(\mathbf{z})$.

The MD multirate filter in Fig. 1 can be equivalently represented as shown in Fig. 2, where $R_{\mathbf{k}_l}(\mathbf{z})$ is a type-II polyphase filter of the filter $F(\mathbf{z})$ and defined as

$$F(\mathbf{z}) = \sum_{l=0}^{M-1} \mathbf{z}^{(\mathbf{k}_l)} R_{\mathbf{k}_l}(\mathbf{z}^{(\mathbf{M}_U)}) \tag{6}$$

where $\mathbf{k}_l$ is a $D \times 1$ integer vector defined as $\mathcal{N}(\mathbf{M}_U)$ [11] and M is the absolute determinant of $\mathbf{M}_U$.

In Fig. 2, the input signal $x(\mathbf{n})$ is expanded by the up-sampler $\uparrow \mathbf{M}_U$ and the digital filter so that the expanded signal $z(\mathbf{n})$ is given as

$$z(\mathbf{n}) = \begin{cases} \sum_{\mathbf{m}\in\mathcal{N}} r_{\mathbf{k}_0}(\mathbf{m}) x(\mathbf{n}+\mathbf{k}_0-\mathbf{m}) \\ \qquad \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_0 \\ \sum_{\mathbf{m}\in\mathcal{N}} r_{\mathbf{k}_1}(\mathbf{m}) x(\mathbf{n}+\mathbf{k}_1-\mathbf{m}) \\ \qquad \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_1 \\ \qquad \vdots \\ \sum_{\mathbf{m}\in\mathcal{N}} r_{\mathbf{k}_{M-1}}(\mathbf{m}) x(\mathbf{n}+\mathbf{k}_{M-1}-\mathbf{m}) \\ \qquad \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_{M-1} \end{cases} \tag{7}$$

where $r_{\mathbf{k}_i}(\mathbf{n})$ denotes the impulse response of $R_{\mathbf{k}_i}(\mathbf{z})$ and $\mathbf{t}$ is a $D \times 1$ arbitrary integer vector. Then, the signal $z(\mathbf{n})$ is decimated by the down-sampler $\downarrow \mathbf{M}_D$ as

$$y(\mathbf{n}) = z(\mathbf{M}_D\mathbf{n}). \tag{8}$$

In 1D multirate systems, it is known that the checkerboard effect is caused by the periodic time-variant property of multirate filters which consist of up-samplers and digital filters [1]–[7]. In the following, we consider the checkerboard effect in the MD multirate filter by using the above expression.

2.2 Periodicity of Step Response

In Fig. 1, when the input signal $x(\mathbf{n})$ is the unit step signal ($u(\mathbf{n}) = 1$, $\mathbf{n} \in [0,\infty)^D$) and the components $n_0, n_1, \cdots, n_{D-1}$ of the vector $\mathbf{n}$ are large enough, the signal $z(\mathbf{n})$ becomes the steady state value $s_{\mathbf{k}}(\mathbf{n})$ as

$$s_{\mathbf{k}}(\mathbf{n}) = \begin{cases} R_{\mathbf{k}_0}(\mathbf{1}), & \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_0 \\ R_{\mathbf{k}_1}(\mathbf{1}), & \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_1 \\ \quad\vdots & \\ R_{\mathbf{k}_{M-1}}(\mathbf{1}), & \mathbf{n} = \mathbf{M}_U\mathbf{t} - \mathbf{k}_{M-1} \end{cases}, \tag{9}$$

where $R_{\mathbf{k}_l}(\mathbf{1})$ is given as

$$R_{\mathbf{k}_l}(\mathbf{1}) = R_{\mathbf{k}_l}(e^{j\omega_0}, e^{j\omega_1}, \cdots, e^{j\omega_{D-1}})|_{\omega_0,\omega_1,\cdots,\omega_{D-1}=0}, \tag{10}$$

and denotes the DC gain of the polyphase filter $R_{\mathbf{k}_l}(\mathbf{z})$. Thus the output signal $y(\mathbf{n})$ is reduced to $ss(\mathbf{n})$ as

$$ss(\mathbf{n}) = s_{\mathbf{k}}(\mathbf{M}_D\mathbf{n}), \tag{11}$$

From Eq. (9), $s_{\mathbf{k}}(\mathbf{n})$ is not constant and has the period $\mathbf{M}_U$. As a result, $ss(\mathbf{n})$ also has the period $\mathbf{M}_U$. Note that $\mathbf{M}_D$ does not cause the checkerboard effect. Thus, we can see that the multirate systems which include up-samplers and digital filters have the periodic step response. The periodic artifact caused by this periodic step response is called the checkerboard effect.

2.3 The Conditions for MD Multirate Filter without Checkerboard Effect

We derive some theorems about the conditions for MD multirate filters without checkerboard effect.

Theorem 1: A necessary and sufficient condition for MD multirate filters to avoid the checkerboard effect is given as

$$R_{\mathbf{k}_0}(\mathbf{1}) = \cdots = R_{\mathbf{k}_{M-1}}(\mathbf{1}). \tag{12}$$

Proof: As shown in the previous subsection, the checkerboard effect is caused by the periodicity of the step response. Therefore, to avoid the checkerboard effect, the steady state values of the step response must be constant. From Eq. (9), it is clear that this condition is equal to Eq. (12). □

Theorem 2: If $F(\mathbf{z})$ is represented as Eq. (13), the checkerboard effect is not caused. Equation (13) is a sufficient condition for MD multirate filters to avoid the checkerboard effect, although it is a necessary and sufficient condition for 1D multirate filters [4].

$$F(\mathbf{z}) = P(\mathbf{z}) \sum_{l=0}^{M-1} \mathbf{z}^{(-\mathbf{k}_l)} \tag{13}$$

where

$$\mathbf{k}_l \in \mathcal{N}(\mathbf{M}_U). \tag{14}$$

Proof: First, let us show that Eq. (13) is a sufficient condition for MD multirate filters. If $F(\mathbf{z})$ can be factorized as Eq. (13), then Fig. 1 can be expressed as Fig. 3. In Fig. 3, when the input signal $x(\mathbf{n})$ is the unit step signal $u(\mathbf{n})$ and the components of the vector $\mathbf{n}$ are enough large, the signal $x'(\mathbf{n})$ becomes the steady state value $s_p(\mathbf{n})$

$$s_p(\mathbf{n}) = \begin{cases} 1, & \mathbf{n} = \mathbf{M}_U\mathbf{t} + \mathbf{k}_0 \\ 1, & \mathbf{n} = \mathbf{M}_U\mathbf{t} + \mathbf{k}_1 \\ \quad\vdots & \\ 1, & \mathbf{n} = \mathbf{M}_U\mathbf{t} + \mathbf{k}_{M-1}. \end{cases} \tag{15}$$

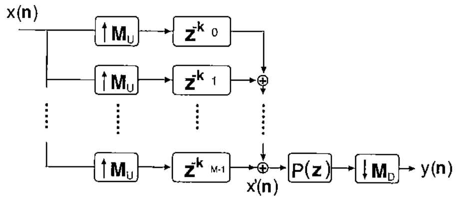


Fig. 3 MD multirate filter satisfying Eq. (14).

From Eq. (15), we can see that $s_p(\mathbf{n})$ is constant. Moreover, since $P(\mathbf{z})$ is a linear time-invariant filter and $\downarrow \mathbf{M}_D$ does not cause the checkerboard effect, the output signal $y(\mathbf{n})$ is also constant and does not have any periodcity. Therefore, under Eq. (13), the checkerboard effect is not caused.

Next, we show that Eq. (13) is not a necessary conditon. As an example, let us consider that dimension $D = 2$ and the following $\mathbf{M}_U$ and $\mathbf{M}_D$

$$\mathbf{M}_U = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{M}_D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{16}$$

From the above result, if $F(\mathbf{z})$ is shown as Eq. (17), the checkerboard effect is not caused.

$$F(z_0, z_1) = (1 + z_0^{-1})P(z_0, z_1) \tag{17}$$

Let us consider another 2D filter $F'(z_0, z_1)$

$$F'(z_0, z_1) = (1 + 2z_0^{-1} + z_1^{-1}). \tag{18}$$

It is easily shown that this filter can not be represented as Eq. (17), and the polyphase filters are given as

$$F(z_0, z_1) = R_{\mathbf{k}_0}(z_0^2, z_1) + z_0 R_{\mathbf{k}_1}(z_0^2, z_1) \tag{19}$$

$$R_{\mathbf{k}_0}(z_0, z_1) = (1 + z_1^{-1}) \tag{20}$$

$$R_{\mathbf{k}_1}(z_0, z_1) = 2z_0^{-1}. \tag{21}$$

Since Eqs. (20) and (21) satisfy Eq. (12), the filter $F'(\mathbf{z})$ does not cause the checkerboard effect. From this example, the Eq. (13) is not a necessary condition. □

Theorem 3: A necessary and sufficient condition for MD filters to avoid the checkerboard effect is given as

$$F(e^{j\boldsymbol{w}}) = 0 \quad \text{at} \quad \boldsymbol{w} = 2\pi \mathbf{M}_U^{-T}\mathbf{k}'_i, \tag{22}$$

where

$$\mathbf{k}'_i \in \mathcal{N}(\mathbf{M}_U^T), \quad \mathbf{k}'_i \neq \mathbf{0}, \tag{23}$$

where $F(e^{j\boldsymbol{w}})$ is the frequency response of $F(\mathbf{z})$ and $\boldsymbol{w}$ is the column vector of angular frequencies, that is $\boldsymbol{w} = [\omega_0, \cdots, \omega_{D-1}]^T$.

Proof: First, let us show that Eq. (22) is a necessary condition. By substituting $\mathbf{z} = e^{j\boldsymbol{w}}|_{\boldsymbol{w}=2\pi\mathbf{M}_U^{-T}\mathbf{k}'_l}$ into $F(\mathbf{z})$, we have

$$F(e^{j2\pi\mathbf{M}_U{}^{-T}\mathbf{k}'_i}) = \sum_{l=0}^{M-1} R_{\mathbf{k}_l}(\mathbf{1}) e^{j2\pi\mathbf{k}'_i{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_l}. \tag{24}$$

Please note that $\mathbf{z}^{\mathbf{M}_U}|_{\mathbf{z}=e^{j\boldsymbol{w}}, \boldsymbol{w}=2\pi\mathbf{M}_U^{-T}\mathbf{k}'_l} = \mathbf{1}$. If the checkerboard effect is not caused, all polyphase filters have the same DC gain. In that case, Eq. (24) can be rewritten as

$$F(e^{j2\pi\mathbf{M}_U{}^{-T}\mathbf{k}'_i}) = R_{\mathbf{k}_0}(\mathbf{1}) \sum_{l=0}^{M-1} e^{j2\pi\mathbf{k}'_i{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_l}. \tag{25}$$

From [11], the following property is satisfied.

$$\sum_{l=0}^{M-1} e^{j2\pi\mathbf{k}'_i{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_l} = 0. \tag{26}$$

By substituting Eq. (26) into Eq. (25), we obtain

$$F(e^{j2\pi\mathbf{M}_U{}^{-T}\mathbf{k}'_i}) = 0. \tag{27}$$

Thus, Eq. (22) is a necessary condition.

Next, let us show that Eq. (22) is a sufficient condition. When Eq. (22) is satisfied, Eq. (24) can be rewritten as

$$\mathbf{M}_{exp} \times \begin{bmatrix} R_{\mathbf{k}_0}(\mathbf{1}) \\ R_{\mathbf{k}_1}(\mathbf{1}) \\ \vdots \\ R_{\mathbf{k}_{M-1}}(\mathbf{1}) \end{bmatrix} = \begin{bmatrix} F(\mathbf{1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{28}$$

where

$$\mathbf{M}_{exp} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{j2\pi\mathbf{k}'_1{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_0} & \cdots & e^{j2\pi\mathbf{k}'_1{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_{M-1}} \\ \vdots & \vdots & \vdots \\ e^{j2\pi\mathbf{k}'_{M-1}{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_0} & \cdots & e^{j2\pi\mathbf{k}'_{M-1}{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_{M-1}} \end{bmatrix}.$$

$F(\mathbf{1})$ is the DC gain of $F(\mathbf{z})$ and corresponds to the case of $\mathbf{k}'_0 = \mathbf{0}$ in Eq. (24). Using the Cramer's rule to Eq. (28), $R_{\mathbf{k}_j}(\mathbf{1})$ is given by

$$R_{\mathbf{k}_j}(\mathbf{1}) = \frac{(-1)^j F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,0)} & \cdots & e^{k(1,j-1)} & e^{k(1,j+1)} & \cdots & e^{k(1,M-1)} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ e^{k(M-1,0)} & \cdots & e^{k(M-1,j-1)} & e^{k(M-1,j+1)} & \cdots & e^{k(M-1,M-1)} \end{vmatrix}, \tag{29}$$

where $|\mathbf{A}|$ is the determinant of the matrix $\mathbf{A}$ and for the convenience, we use $e^{k(i,j)}$ instead of $e^{j2\pi\mathbf{k}'_i{}^T\mathbf{M}_U{}^{-1}\mathbf{k}_j}$. By substituting Eq. (26) into the first column of Eq. (29), subtracting the other columns from the first column and permuting the columns to make the order of index number be from 1 to $M-1$, Eq. (29) can be arranged as

$$R_{\mathbf{k}_j}(\mathbf{1}) = \frac{F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,1)} & \cdots & e^{k(1,M-1)} \\ \vdots & \cdots & \vdots \\ e^{k(M-1,1)} & \cdots & e^{k(M-1,M-1)} \end{vmatrix} \tag{30}$$

(see also Appendix A). Since the right-hand side of Eq. (30) does not depend on the index j, all polyphase filters $R_{\mathbf{k}_j}(\mathbf{z})$, $j = 0, \cdots, M-1$ have the same DC gain. Therefore, Eq. (22) is a sufficient condition for MD multirate filters without checkerboard effect. □

From the above considerations, we have the following conclusions.

- To design all classes of MD multirate filters without checkerboard effect, we can utilize the condition Eq. (12) or the condition Eq. (22).
- From Theorem 2, we may design MD multirate filters without checkerboard effect by the condition Eq. (13). This condition shows that the checkerboard effect can be avoided by 0th order interpolation for arbitrary $\mathbf{M}_U$. Equation (13) allows us to use the standard method for designing $F(\mathbf{z})$. From Eq. (13), the design of $F(\mathbf{z})$ is reduced to $P(\mathbf{z})$ which is done without considering the checkerboard effect.
- From Theorem 2, we can perfectly avoid the checkerboard effect by using the cascade form strucure of the 0th order interpolation and any digital filter, even if the filter has finite word length.

3. MD Multirate Filter Banks without Checkerboard Effect

In this section, we consider the conditions for MD maximally decimated filter banks.

3.1 MD Multirate Filter Banks [11]

Figure 4 shows an MD maximally decimated filter bank with a factor $\mathbf{M}$, where $\mathbf{M}$ is a $D \times D$ nonsingular integer matrix. In Fig. 4, $H_m(\mathbf{z})$ and $F_m(\mathbf{z})$ denote an analysis filter and a synthesis filter respectively. M is the absolute determinant of $\mathbf{M}$ and the channel number $m = 0$ corresponds to the lowpass channel.

The MD filter bank in Fig. 4 can be equivalently represented as shown in Fig. 5, where $\mathbf{E}(\mathbf{z})$ and $\mathbf{R}(\mathbf{z})$ are the $M \times M$ type-I and type-II polyphase matrices respectively and are defined as

$$\mathbf{E}(\mathbf{z}) = \begin{bmatrix} E_{0,0}(\mathbf{z}) & \cdots & E_{0,M-1}(\mathbf{z}) \\ \vdots & \ddots & \vdots \\ E_{M-1,0}(\mathbf{z}) & \cdots & E_{M-1,M-1}(\mathbf{z}) \end{bmatrix} \tag{31}$$

$$\mathbf{R}(\mathbf{z}) = \begin{bmatrix} R_{0,0}(\mathbf{z}) & \cdots & R_{0,M-1}(\mathbf{z}) \\ \vdots & \ddots & \vdots \\ R_{M-1,0}(\mathbf{z}) & \cdots & R_{M-1,M-1}(\mathbf{z}) \end{bmatrix}, \tag{32}$$

where $E_{i,j}(\mathbf{z}^{\mathbf{M}})$ and $R_{i,j}(\mathbf{z}^{\mathbf{M}})$ are the simplified expressions of the $E_{i,\mathbf{k}_j}(\mathbf{z}^{\mathbf{M}})$ and the $R_{\mathbf{k}_i,j}(\mathbf{z}^{\mathbf{M}})$ which are given by

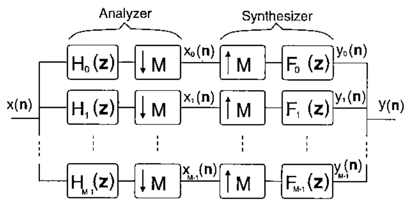


Fig. 4 MD multirate filter bank.

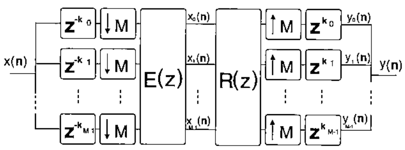


Fig. 5 Polyphase structure.

$$H_m(\mathbf{z}) = \sum_{l=0}^{M-1} \mathbf{z}^{(-\mathbf{k}_l)} E_{m,\mathbf{k}_l}(\mathbf{z}^{(\mathbf{M})}) \tag{33}$$

$$F_m(\mathbf{z}) = \sum_{l=0}^{M-1} \mathbf{z}^{(\mathbf{k}_l)} R_{\mathbf{k}_l,m}(\mathbf{z}^{(\mathbf{M})}) \tag{34}$$

$$\mathbf{k}_l \in \mathcal{N}(\mathbf{M}).$$

By using the polyphase matrices, the perfect reconstruction (PR) condition for MD filter banks is given as

$$\mathbf{R}(\mathbf{z})\mathbf{E}(\mathbf{z}) = \mathbf{z}^{-\mathbf{c}}\mathbf{I}_M, \tag{35}$$

where $\mathbf{c}$ is a $1 \times D$ arbitrary integer vector and $\mathbf{I}_M$ is the $M \times M$ identity matrix. When an MD filter bank satisfies Eq. (35), it is called the MD perfect reconstruction (PR) filter bank.

3.2 Checkerboard Effect in MD Multirate Filter Banks

As shown in Fig. 4, MD filter banks consist of analyzer and synthesizer. The analyzer divides the input signal $x(\mathbf{n})$ into M subband signals $x_m(\mathbf{n})$ and the synthesizer combines $x_m(\mathbf{n})$ into the output signal $y(\mathbf{n})$. Here, each channel of the synthesizer includes an up-sampler and a digital filter. Therefore, the checkerboard effect in MD filter banks is caused by the synthesizer.

Next, let us consider the steady state value of the step response as well as the case of MD multirate filters. In Fig. 5, when the input signal $x(\mathbf{n})$ is the unit step signal $u(\mathbf{n})$ and the components of the vector $\mathbf{n}$ are large enough, the subband signal $x_m(\mathbf{n})$ becomes the steady state value $a_m(\mathbf{n})$

$$a_m(\mathbf{n}) = H_m(\mathbf{1}), \tag{36}$$

and the signal $y_m(\mathbf{n})$ for the mth channel in the synthesizer becomes the steady state value $s_m(\mathbf{n})$

$$s_m(\mathbf{n}) = a_m(\mathbf{n}) \times \sum_{\mathbf{v}\in\mathcal{N}} r_{l,m}(\mathbf{v}), \quad \mathbf{n} = \mathbf{Mt} - \mathbf{k}_l$$

$$= H_m(\mathbf{1}) \times \begin{cases} R_{0,m}(\mathbf{1}), & \mathbf{n} = \mathbf{Mt} - \mathbf{k}_0 \\ \vdots & \vdots \\ R_{M-1,m}(\mathbf{1}), & \mathbf{n} = \mathbf{Mt} - \mathbf{k}_{M-1} \end{cases}, \quad (37)$$

where $r_{l,m}(\mathbf{n})$ denotes the impulse response of the $R_{l,m}(\mathbf{z})$.

From Eq. (37), $s_m(\mathbf{n})$ is not constant and has the period M. If the PR condition Eq. (35) is satisfied, the periodicity of $s_m(\mathbf{n})$ is able to be canceled each other and thus the output signal $y(\mathbf{n})$ is constant. However, even if an MD filter bank is designed under the PR condition, it is broken in some practical applications such as subband coding. In this case, the periodicity of $s_m(\mathbf{n})$ can not be canceled each other. As a result, $y(\mathbf{n})$ is not constant and the checkerboard effect is generated.

3.3 Conditions for MD Multirate Filter Banks without Checkerboard Effect

As shown in the previous subsection, the checkerboard effect is caused by the synthesizer. Therefore, to avoid the checkerboard effect, we have to consider the synthesis filters or the subband signals $x_m(\mathbf{n})$.

In the following, we derive the conditions for MD maximally decimated PR filter banks.

Theorem 4: A necessary and sufficient condition for MD maximally decimated PR filter banks to avoid the checkerboard effect is given as

$$R_{0,0}(\mathbf{1}) = \cdots = R_{M-1,0}(\mathbf{1}), \quad (38)$$

where $R_{l,0}(\mathbf{z})$ denotes the DC gain of a polyphase filter of the lowpass filter $F_0(\mathbf{z})$ in the synthesizer.

In addition, the condition Eq. (38) is equal to

$$H_m(\mathbf{1}) = \sum_{l=0}^{M-1} E_{m,l}(\mathbf{1}) = 0, \quad m = 1, \cdots, M-1, \quad (39)$$

where $H_m(\mathbf{1})$ denotes the DC gain of the analysis filters except for the lowpass channel.

Proof: First, let us show that Eq. (38) is a necessary and sufficient condition.

In general, the DC component of $x(\mathbf{n})$ is not zero. Thus, for the lowpass channel ($m = 0$), Eq. (38) is a necessary and sufficient condition, because Eq. (38) is the condition given by Theorem 1. Therefore, it is clear that Eq. (38) is a necessary condition for MD filter banks without the checkerboard effect.

Next, let us show that Eq. (38) is a sufficient condition. Substituting $\mathbf{z} = e^{j\boldsymbol{w}}|_{\omega_0=0,\cdots,\omega_{D-1}=0}$ into Eq. (35) yields

$$\mathbf{R}(\mathbf{1})\mathbf{E}(\mathbf{1}) = \mathbf{I}_M. \quad (40)$$

Since $\mathbf{E}(1)$ and $\mathbf{R}(1)$ are $M \times M$ square matrices, Eq. (40) can be represented as

$$\mathbf{E}(\mathbf{1})\mathbf{R}(\mathbf{1}) = \mathbf{I}_M. \quad (41)$$

From Eq. (41), we obtain

$$\sum_{l=0}^{M-1} E_{m,l}(\mathbf{1})R_{l,0}(\mathbf{1}) = 0, \quad m = 1, \cdots, M-1. \quad (42)$$

When Eq. (38) is satisfied, Eq. (42) can be rewritten as

$$R_{0,0}(\mathbf{1}) \sum_{l=0}^{M-1} E_{m,l}(\mathbf{1}) = 0, \quad m = 1, \cdots, M-1. \quad (43)$$

Since $R_{0,0}(\mathbf{1}) \neq 0$ from Eq. (41), Eq. (43) is reduced to

$$\sum_{l=0}^{M-1} E_{m,l}(\mathbf{1}) = H_m(\mathbf{1}) = 0, \quad m = 1, \cdots M-1. \quad (44)$$

This means the subband signal $a_m(\mathbf{n}) = 0$, $m = 1, \cdots, M-1$. That is, the DC component of $x_m(\mathbf{n})$ in Fig. 4 is zero value. By substituting Eq. (44) into Eq. (37), $s_m(\mathbf{n})$ except for $m = 0$ becomes

$$s_m(\mathbf{n}) = 0, \quad m = 1, \cdots, M-1. \quad (45)$$

Thus, $s_m(\mathbf{n})$ except for the lowpass channel has zero value. Therefore, Eq. (38) is a sufficient condition.

Next, we show that the condition Eq. (39) is equal to Eq. (38).

From the above consideration, it is clear that if Eq. (38) is satisfied, Eq. (39) is done. Therefore, let us show that if Eq. (39) is satisfied, Eq. (38) is done.

From Eq. (41), we have the following equation

$$\begin{bmatrix} E_{0,0}(\mathbf{1}) & \cdots & E_{0,M-1}(\mathbf{1}) \\ \vdots & \ddots & \vdots \\ E_{M-1,0}(\mathbf{1}) & \cdots & E_{M-1,M-1}(\mathbf{1}) \end{bmatrix} \begin{bmatrix} R_{0,0}(\mathbf{1}) \\ \vdots \\ R_{M-1,0}(\mathbf{1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T. \quad (46)$$

Using the Cramer's rule to the Eq. (46), $R_{i,0}(\mathbf{1})$ is given by

$$\begin{aligned} R_{i,0}(\mathbf{1}) &= \frac{1}{|\mathbf{E}(\mathbf{1})|} \times \begin{vmatrix} E_{0,0} & \cdots & E_{0,i-1} & 1 & E_{0,i+1} & \cdots & E_{0,M-1} \\ E_{1,0} & \cdots & E_{1,i-1} & 0 & E_{1,i+1} & \cdots & E_{1,M-1} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ E_{M-1,0} & \cdots & E_{M-1,i-1} & 0 & E_{M-1,i+1} & \cdots & E_{M-1,M-1} \end{vmatrix} \\ &= \frac{1}{|\mathbf{E}(\mathbf{1})|} \times (-1)^i \times \begin{vmatrix} E_{1,0} & \cdots & E_{1,i-1} & E_{1,i+1} & \cdots & E_{1,M-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ E_{M-1,0} & \cdots & E_{M-1,i-1} & E_{M-1,i+1} & \cdots & E_{M-1,M-1} \end{vmatrix}, \end{aligned} \quad (47)$$

where for the convenience, we use $E_{i,j}$ instead of $E_{i,j}(\mathbf{1})$. By substituting Eq. (39) into Eq. (47), Eq. (47) can be arranged as

$$R_{i,0}(\mathbf{1}) = \frac{1}{|\mathbf{E}(\mathbf{1})|} \begin{vmatrix} E_{1,1} & \cdots & E_{1,M-1} \\ \vdots & \cdots & \vdots \\ E_{M-1,1} & \cdots & E_{M-1,M-1} \end{vmatrix}. \tag{48}$$

Since the right-hand side of Eq. (48) does not depend on the index i, all polyphase filters $R_{i,0}(\mathbf{z}), i = 0, \cdots, M-1$ for the lowpass channel have the same DC gain. Therefore, when Eq. (39) is satisfied, Eq. (38) is also satisfied. Thus, Eq. (38) is equal to Eq. (39). □

In the case of 2D orthogonal filter banks, please note that the condition Eq. (38) is equal to the condition of 0th vanishing moment for 2D orthogonal wavelet [14].

From Theorem 4, to design MD maximally decimated PR filter banks without checkerboard effect, we have to base on Eq. (38) for synthesis filter or Eq. (39) for analysis filter. We have shown that the MD maximally decimated PR filter banks can be easily designed by applying the conditions [9].

4. Examples

In order to verify the significance of the derived theorems, we show some examples. Although the following examples are in 2D, the theorems are applicable to any dimension.

Example 1: In this example, let us show how the conditions are given for

$$\mathbf{M}_U = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{M}_D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{49}$$

From Theorem 1, the condition can be derived as follows. Figure 6 shows the impulse response of $F(z_0, z_1)$. $M = |\mathbf{M}_U| = 2$ and from Eq. (14), the vector $\mathbf{k}_i$ is given as

$$\mathbf{k}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{k}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{50}$$

By substituting Eq. (50) into Eq. (6), the impulse response in Fig. 6 can be decomposed to the two polyphase filters denoted by black and white dots in Fig. 7 (a). Please note that this decomposition does not depend on $\mathbf{M}_D$. From Theorem 1, if all polyphase filters have the same DC gain, the checkerboard effect is not caused. In Fig. 7 (a), it means that each summation of the values in black and in white dots is equal.

We can derive another condition as follows. From Theorem 3, if $F(z_0, z_1)$ has no gain at the frequency point given by Eq. (24), the checkerboard effect is not caused. In this example, from Eq. (23), $\mathbf{k}'_i$ is given as

$$\mathbf{k}'_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{51}$$

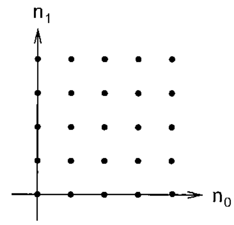
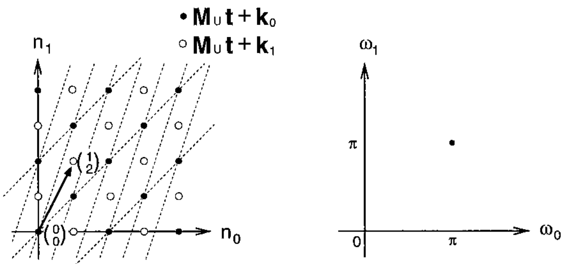


Fig. 6 Impulse response of $F(z_0, z_1)$.



(a) Spatial domain (b) Frequency domain

Fig. 7 Conditions to avoid the checkerboard effects.

and the frequency point is derived as

$$\boldsymbol{w} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}. \tag{52}$$

Thus, if $F(z_0, z_1)$ has no gain at the frequency point in Fig. 7 (b), the checkerboard effect is not generated.

Moreover, from Theorem 2, if a 2D filter $F(z_0, z_1)$ is shown as Eq. (53), the checkerboard effect is not caused.

$$F(z_0, z_1) = (1 + z_0^{-1} z_1^{-2}) P(z_0, z_1) \tag{53}$$

Note that Eq. (53) is clearly non separable filter. Let us consider another 2D filter $F'(z_0, z_1)$ given as Eq. (54)

$$F'(z_0, z_1) = (1 + z_0^{-1}). \tag{54}$$

From Eq. (54), $F'(z_0, z_1)$ can not be represented as Eq. (53), but the polyphase filters are given as

$$F'(z_0, z_1) = R_{\mathbf{k}_0}(\mathbf{z}^{\mathbf{M}_U}) + z_0^1 z_1^2 R_{\mathbf{k}_1}(\mathbf{z}^{\mathbf{M}_U}) \tag{55}$$

$$R_{\mathbf{k}_0}(z_0, z_1) = 1 \tag{56}$$

$$R_{\mathbf{k}_1}(z_0, z_1) = z_0^{-2} \tag{57}$$

and we can see that $F'(z_0, z_1)$ satisfies the condition Eq. (12) to avoid the checkerboard effect. Note that Eq. (55) is a separable filter and it is superior in the sense of simplicity.

Example 2: As processing examples, we convert the resolution of the image in Fig. 8 (a) by the 2D multirate filter, where $\mathbf{M}_U$ and $\mathbf{M}_D$ are

$$\mathbf{M}_U = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{M}_D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{58}$$

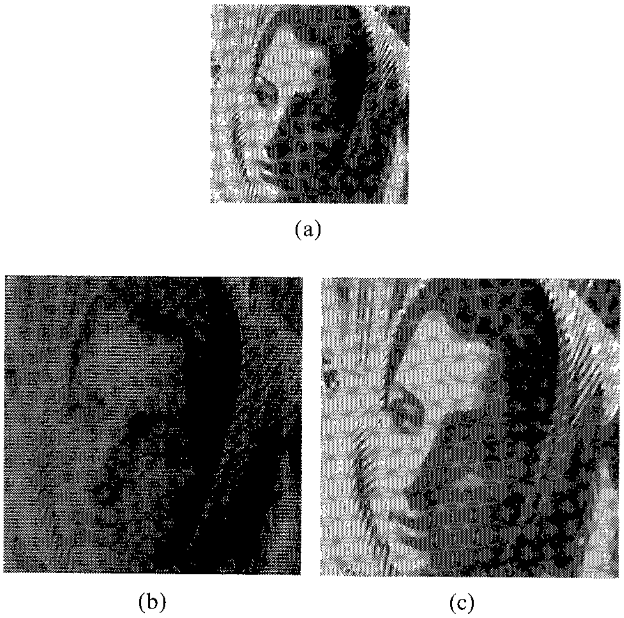


Fig. 8 Processed image.

This means that the size of image is expanded by factor 3/2 in vertical and horizontal directions respectively. The used $F(z_0, z_1)$s have 5×5 taps and they are optimized by minimizing the stopband attenuation. The stopband and the passband are

passband: $\omega_0 < \pi/2$ and $\omega_1 < \pi/2$
stopband: $2\pi/3 < \omega_0$ and $2\pi/3 < \omega_1$

The processed images are shown in Figs. 8 (b), (c). Figure 8 (b) is converted by the filter which does not satisfy Eq. (12) and Fig. 8 (c) is done by the filter which satisfies Eq. (12).

From Fig. 8, we see that the checkerboard effect can be avoided by using the proposed condition.

Example 3: In this example, we show the design examples of MD maximally decimated filter banks which are designed to satisfiy the condition given in Theorem 4 and not to do it. For the design examples shown here, we use the design method of MD linear-phase Paraunitary filter banks with a lattice structure [13] and the object function of the optimization is chosen as the minimum stoppband attenuation.

conditions

- filter tap: 6×6
- stopbannd for each channels

$$F_0(z_0, z_1) : 3\pi/4 < w_0 < \pi,\ 3\pi/4 < w_1 < \pi$$
$$F_1(z_0, z_1) : 0 < w_0 < \pi/4,\ 0 < w_1 < \pi/4$$
$$F_2(z_0, z_1) : 0 < w_0 < \pi/4,\ 3\pi/4 < w_1 < \pi$$
$$F_3(z_0, z_1) : 3\pi/4 < w_0 < \pi,\ 0 < w_1 < \pi/4$$

- $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

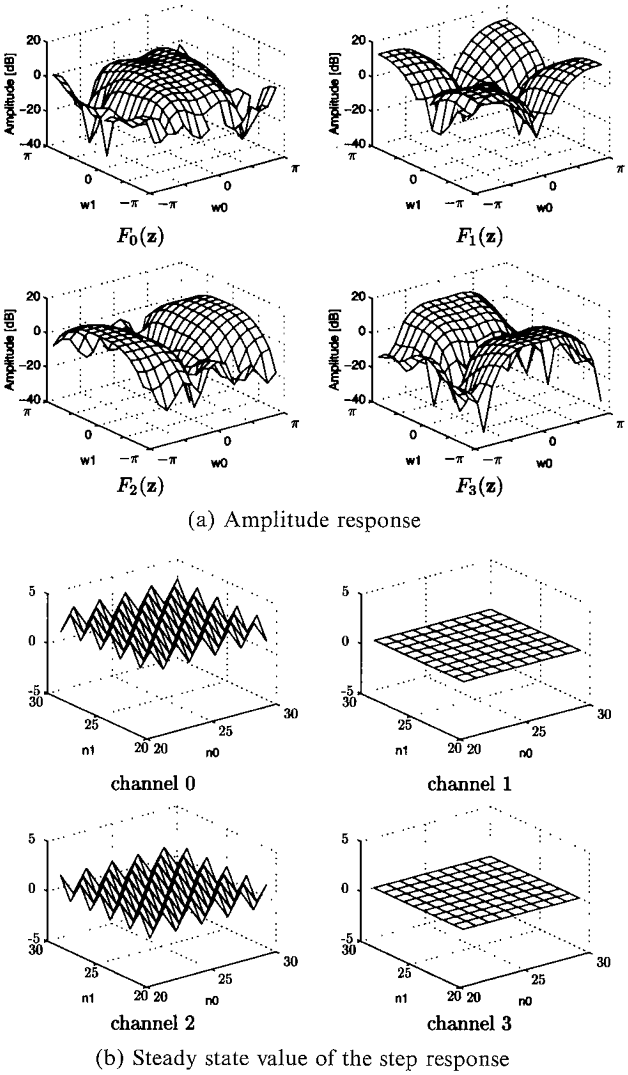


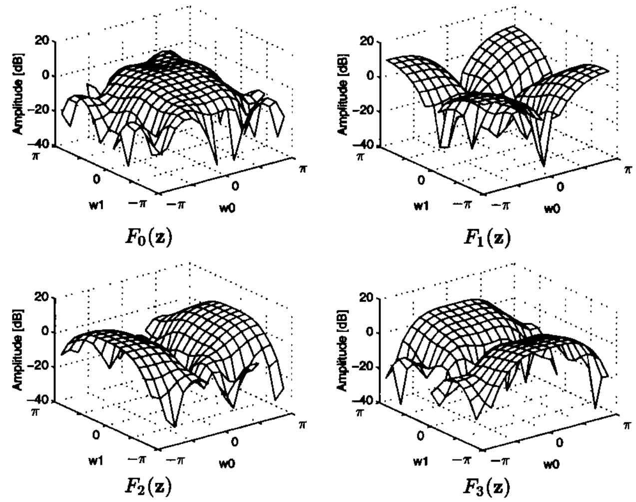
Fig. 9 A design example of 2D filter bank which does not satisfy Eq. (38).

The amplitude response and the steady state value of the step response for each channels are shown in Figs. 9 and 10.

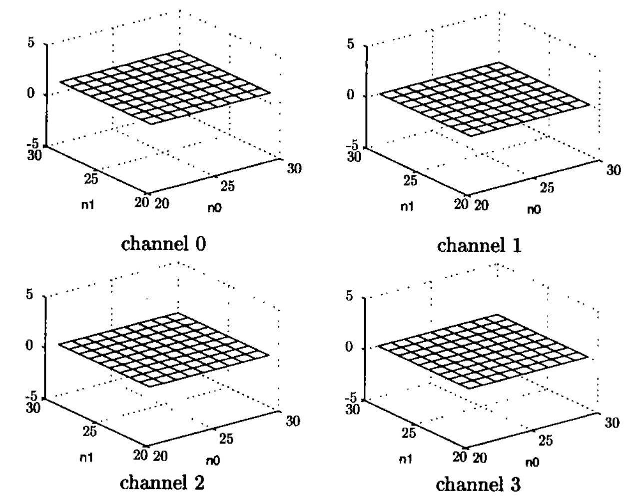
From Fig. 9, we see that the checkerboard effect is caused, when the condition in Theorem 4 is not satisfied. On the other hand, from Fig. 10, we see that the checkerboard effect can be avoided by using the proposed condition in Theorem 4.

5. Conclusion

In this work, we considered the checkerboard effect in MD multirate filters and filter banks. Some theorems about the conditions for the MD multirate filters to avoid the checkerboard effect. In addition, the conditions for MD multirate filter banks to avoid the checkerboard effect are shown. Simulation examples show that the checkerboard effect can be avoided by using the pro-



(a) Amplitude response



(b) Steady state valuae of the step response

Fig. 10 A design example of 2D filter bank which satisfies Eq. (38).

posed conditions. All though we show simulation examples in the case of 2D, the theorems in this paper are applicable to any dimensional applications (c.g. 3D filter bank, video standards conversion). In future, we will consider such 3D applications without checkerboard effect.

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References

[1] G. Strang and T. Nguyen, "Wavelets and Filter Banks," Wellesley-Cambridge Press, 1996.

[2] M. Kato, T. Yoshida, and A. Nishihara, "Design of multirate filters with time-domain constraints," IEICE Technical Report, DSP96-24, Jun. 1996.

[3] Y. Harada, S. Muramatsu, and H. Kiya, "Two-channel QMF bank without checkerboard effect and its lattice structure," IEICE Trans., vol.J80-A, no.11, 1997.

[4] Y. Harada and H. Kiya, "Multirate filter without checkerboard effect and its zero point location," IEICE Technical Report, CAS97-01, Jan. 1997.

[5] M. Kato, T. Yoshida, and A. Nishihara, "Design of filter banks with time domain constraints," IEICE 11th Digital Signal Processing Symposium, no.B3-1, pp.219–224, 1996.

[6] M. Kato, T. Yoshida, and A. Nishihara, "Necessary and sufficient condition for M-channel perfect reconstruction filter banks to avoid checkerboard effects," IEICE Technical Report, CAS96-79, Jan. 1997.

[7] H. Kobayashi and H. Kiya, "Checkerboard effect of oversampled filter bank," IEICE 12th Digital Signal Processing Symposium, no.A1-1, Nov. 1997.

[8] Y. Harada, S. Muramatsu, and H. Kiya, "Checkerboard effects in multidimensional multirate systems," IEICE Technical Report, DSP97-41, June 1997.

[9] Y. Harada, S. Muramatsu, and H. Kiya, "Multidimensional filter banks without checkerboard effects," IEICE Technical Report, DSP97-115, Oct. 1997.

[10] M. Vetterli and J. Kovaĉević, "Wavelet and Subband Coding," Prentice Hall, Englewood Cliffs, 1995.

[11] P.P. Vaidyanthan, "Multirate Systems and Filter Banks," Prentice Hall, Englewood Cliffs, 1993.

[12] A. Ramesh and C. Sidney, "On upsampling, downsampling, and rational sampling rate filter banks," IEEE Trans. Signal Processing, vol.42, no.4, 1994.

[13] S. Muramatsu, A. Yamada, and H. Kiya, "A design method of multidimensional linear-phase paraunitary filter banks with a lattice structure," IEEE TENCON'97, Dec. 1997.

[14] D. Stanhill and Y.Y. Zeevi, "Two-dimensional orthogonal wavelets with vanishing moments," IEEE Trans. Signal Processing, vol.44, no.10, 1996.

Appendix A: Deriving Eq. (30)

Let us derive Eq. (30) from Eq. (29). In this appendix, although we show the case of $M = 4$ and $j = 2$, the derivation can be easily generalized to any M and j.

For $M = 4$ and $j = 2$, Eq. (29) is given by

$$R_{\mathbf{k}_2}(\mathbf{1}) = \frac{(-1)^2 F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,0)} & e^{k(1,1)} & e^{k(1,3)} \\ e^{k(2,0)} & e^{k(2,1)} & e^{k(2,3)} \\ e^{k(3,0)} & e^{k(3,1)} & e^{k(3,3)} \end{vmatrix}. \qquad \text{(A· 1)}$$

From Eq. (26), the following equation is given.

$$e^{k(i,0)} = -(e^{k(i,1)} + e^{k(i,2)} + e^{k(i,3)}) \qquad \text{(A· 2)}$$

By substituting Eq. (A· 2) into the first column of Eq. (A· 1) and eliminating the minus sign of the first column, we have

$$R_{\mathbf{k}_2}(\mathbf{1})$$
$$= \frac{(-1)^3 F(\mathbf{1})}{|\mathbf{M}_{exp}|}$$

$$\times \begin{vmatrix} (e^{k(1,1)} + e^{k(1,2)} + e^{k(1,3)}) & e^{k(1,1)} & e^{k(1,3)} \\ (e^{k(2,1)} + e^{k(2,2)} + e^{k(2,3)}) & e^{k(2,1)} & e^{k(2,3)} \\ (e^{k(3,1)} + e^{k(3,2)} + e^{k(3,3)}) & e^{k(3,1)} & e^{k(3,3)} \end{vmatrix}. \tag{A·3}$$

By subtracting the second column and the third one from the first one, Eq. (A· 3) is rewritten as

$$R_{\mathbf{k}_2}(\mathbf{1}) = \frac{(-1)^3 F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,2)} & e^{k(1,1)} & e^{k(1,3)} \\ e^{k(2,2)} & e^{k(2,1)} & e^{k(2,3)} \\ e^{k(3,2)} & e^{k(3,1)} & e^{k(3,3)} \end{vmatrix}. \tag{A·4}$$

By permuting the first column and the second one, we obtain

$$R_{\mathbf{k}_2}(\mathbf{1}) = \frac{(-1)^4 F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,1)} & e^{k(1,2)} & e^{k(1,3)} \\ e^{k(2,1)} & e^{k(2,2)} & e^{k(2,3)} \\ e^{k(3,1)} & e^{k(3,2)} & e^{k(3,3)} \end{vmatrix}$$

$$= \frac{F(\mathbf{1})}{|\mathbf{M}_{exp}|} \times \begin{vmatrix} e^{k(1,1)} & e^{k(1,2)} & e^{k(1,3)} \\ e^{k(2,1)} & e^{k(2,2)} & e^{k(2,3)} \\ e^{k(3,1)} & e^{k(3,2)} & e^{k(3,3)} \end{vmatrix}. \tag{A·5}$$

Therefore, Eq. (30) is derived from Eq. (29).

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