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Location Problems on Undirected Flow Networks

Hiroshi TAMURA†, Masakazu SENGOKU††, Shoji SHINODA†††
and Takeo ABE††, *Members*

SUMMARY Location theory on networks is concerned with the problem of selecting the best location in a specified network for facilities. In networks, the distance is an important measure to quantify how strongly related two vertices are. Mereover, the capacity between two vertices is also an important measure. In this paper, we define the location problems called the p-center problem, the r-cover problem and the p-median problem on undirected flow networks. We propose polynomial time algorithms to solve these problems.

1. Introduction

Location theory[1] on networks is concerned with the problem of selecting the best location in a specified network for facilities. Many studies for the theory have been done. Most of these studies treat location problems on networks from the standpoint of measuring the closeness between two vertices by the distance between two vertices. On the other hand, location problems on networks from the standpoint of measuring the closeness between two vertices by the capacity (maximum flow value) between two vertices have not been studied yet.

This paper concerns location problems on undirected flow networks. We define the location problems called the p-center problem, the r-cover problem and the p-median problem on undirected flow networks. We propose polynomial time algorithms to solve these ploblems.

2. Definitions

Let us consider an undirected flow network $N=(V, E, w_N)$ such that V, E and w_N are the vertex set, the edge set and the function assigning a positive real number $w_N(e)$, called edge-capacity, to each edge $e \in E$, respectively. The capacity of minimum cutset between two vertices x and y in N is called the capacity between x and y, denoted by $g(x, y)$. Especially, let $g(x, x)=\infty$.

For a subset U of V, let

$$g(U, x)=\max\{g(x, y)|y \in U\}$$

and

$$e_N(U)=\min\{g(U, x)|x \in V\}.$$

$e_N(U)$ is called the eccentricity of U which is concerned with capacity (we simply call $e_N(U)$ the eccentricity of U, hereafter).

For $1 \leqq p \leqq |V|$, a subset U of V such that $|U|=p$ and $e_N(U)=\max\{e_N(U')|U' \subset V, |U'|=p\}$ is called a p-center of N. We simply call the problem of finding a p-center the p-center problem.

For a positive real number r, let

$$q_N(r)=\min\{|U|\,|U \subset V, e_N(U) \geqq r\}.$$

A subset U of V such that $e_N(U) \geqq r$ and $|U|=q_N(r)$ is called a r-cover of N. We simply call the problem of finding a r-cover the r-cover problem.

Let

$$t_N(U)=\sum_{x \in V-U} g(U, x).$$

$t_N(U)$ is called the transportation number[2] of U.

For $1 \leqq p \leqq |V|$, a subset U of V such that $|U|=p$ and $t_N(U)=\max\{t_N(U')|U' \subset V, |U'|=p\}$ is called a p-median of N. We simply call the problem of finding a p-median the p-median problem.

In a communication network, vertices represent terminal computers and edges represent links between computers. In this network, consider how we assign some data to files. We assume that the delay time can be ignored in this network, In this case, for each terminal computer pair, the number of links between the computers is the measure representing the closeness between the computers. Location theory on flow networks is applicable to the above case.

For example, let us consider the network N shown in Fig. 1 where the value attached to each edge represents the edge capacity. Let U be $\{x_1, x_3\}$. Since

$$g(U, x_2)=g(x_1, x_2)=6,$$

$$g(U, x_4)=g(x_3, x_4)=5 \text{ and}$$

$$g(U, x_5)=g(x_1, x_5)=6,$$

the eccentricity and the transportation number of U are following,

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† The author is with the Graduate School of Science and Technology, Niigata University, Niigata-shi, 950-21 Japan.
†† The authors are with the Faculty of Engineering, Niigata University, Niigata-shi, 950-21 Japan.
††† The author is with the Faculy of Science and Engineering, Chuo University, Tokyo, 112 Japan.

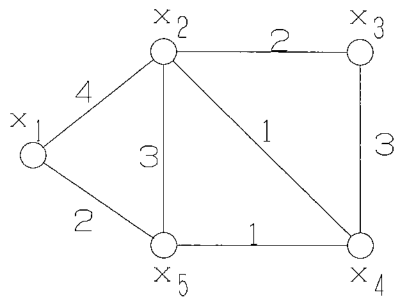
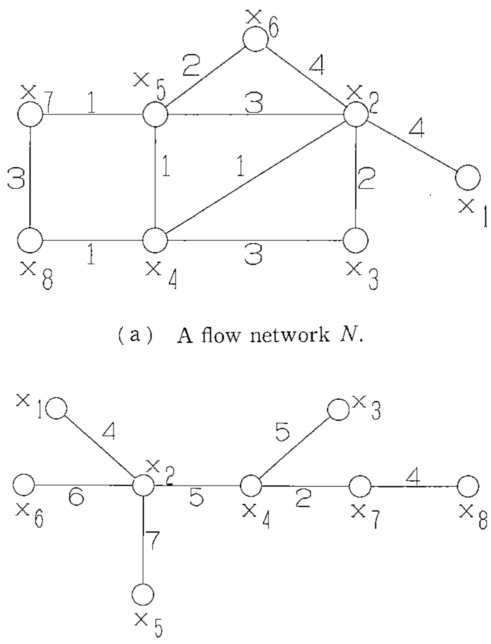


Fig. 1 A flow network N.



(a) A flow network N.

(b) A tree flow network T.

Fig. 2 A flow network and its tree flow network.

$e_N(U)=\min\{6, 5, 6\}=5$ and

$t_N(U)=6+5+6=17$.

Location problems on networks from the standpoint of measuring the closeness between two vertices by the distance between two vertices are well-known and it is also well-known that the p-center, the r-cover and the p-median problems are NP-hard problems[(3)-(5)].

For any undirected flow network N, there exists a tree flow network T such that $g_N(x, y)=g_T(x, y)$ for any vertex pair and T can be obtained by solving $|V|-1$ maximum flow problems[(6)]. The time complexity to construct T from N is $O(|V|s(|V|, |E|))$ where $|V|=|V(N)|$, $|E|=|E(N)|$ and $s(|V|, |E|)$ is the time required to solve a maximum flow problem in N. The best time bound for $s(|V|, |E|)$ known to date is $O(|V|\,|E|\log(|V|^2/|E|))$[(7)]. Therefore $O(|V|s(|V|, |E|))=O(|V|^2\,|E|\log(|V|^2/|E|))$.

For example, let us consider the network N shown in Fig. 2(a). Then, a tree flow network T such that $g_N(x, y)=g_T(x, y)$ for any vertex pair is shown in Fig. 2(b). So we consider location problems on tree flow networks for flow networks, hereafter.

3. Results

3.1 The p-Center Problem and the r-Cover Problem

This section concerns the p-center problem and the r-cover problem.

[Theorem 1] Let T be a tree flow network with $E(T)=\{(x_1, y_1), \cdots, (x_m, y_m)\}$ such that $w(x_1, y_1)\geqq \cdots \geqq w(x_m, y_m)$ and let $T_1, \cdots, T_p$ be all connected components of $T'=T-\{(x_{m-p+2}, y_{m-p+2}), \cdots, (x_m, y_m)\}$. Then, $U=\{z_1, \cdots, z_p\}$ is a p-center of T, where $z_i\in V(T_i)$ for each i, $1\leqq i\leqq p$.

(proof) Let $w(x_{m-p+1}, y_{m-p+1})=a$. For any $x\in V(T)$, $g(U, x)\geqq a$, because $g(x, z_i)\geqq a$ if x belongs to a connected component T_i. Let T_j be a connected component that includes an edge (x_{m-p+1}, y_{m-p+1}). Without loss of generality, we may assume that z_j belongs to a connected component including x_{m-p+1} in $T_j-\{(x_{m-p+1}, y_{m-p+1})\}$. Since $g(U, y_{m-p+1})=g(z_j, y_{m-p+1})=a$, $e_T(U)=a$. The eccentricity of U does not depend on how to choose z_j in each connected component T_j and the value is a.

Let U' be a subset of $V(T)$ and $|U'|=p$. We assume that there exists a connected component T_i such that $V(T_i)\cap U'=\phi$ in T'. Since, for a vertex x in T_i and each element z in U', the path from x to z in T includes an edge (x_{m-p+2}, y_{m-p+2}) or an edge (x_{m-p+3}, y_{m-p+3}) or$\cdots$ or an edge (x_m, y_m), $e_T(U')\leqq a$. Therefore U is a p-center of T. □

For example, let us consider the network T shown in Fig. 2(b) and let $p=3$. The subnetwork T' is obtained by deleting the edge set $\{(x_1, x_2), (x_4, x_7)\}$ (see Fig. 3). Therefore, from Theorem 1, $U=\{x_1, x_2, x_7\}$ is a 3-center of T.

The time complexity of sorting all edges in order of size of edge weights is $O(|V|\log|V|)$, where $|V|=|V(T)|$. We can choose U in $O(|V|)$ time from T' by depth-first search[(8)]. Therefore, a p-center of a tree flow network T can be obtained in $O(|V|\log|V|)$ time.

[Theorem 2] Let T be a tree flow network and let $T_1, \cdots, T_q$ be all connected components of $T'=T-\{(x, y)\in E(T)|w(x, y)<r\}$. Then, $U=\{z_1, \cdots, z_q\}$ is a r-cover of T, where $z_i\in V(T_i)$ for each i, $1\leqq i\leqq q$.

(proof) Clearly, $e_T(U)\geqq r$. Let U' be a subset of $V(T)$ where $|U'|<|U|$. There exists T_i such that $V(T_i)\cap U'=\phi$ in T'. Since, for a vertex x in T_i and each element z of U', the path from x to z in T includes an edge whose weight is less than r, $g(U', x)<r$. Since $e_T(U')<r$, U is a r-cover. □

For example, let us consider the network T shown in Fig. 2(b) and $r=5$, edges whose weights are less than 5 are (x_1, x_2), (x_4, x_7) and (x_7, x_8). So, the

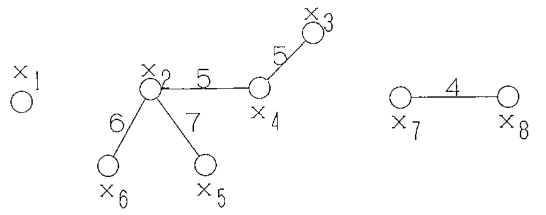


Fig. 3 A network $T-\{(x_1, x_2), (x_4, x_7)\}$.

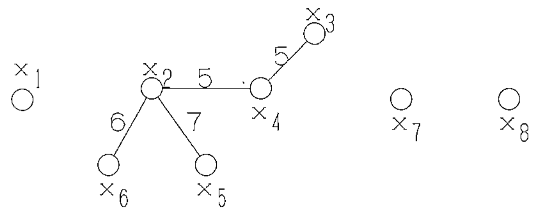


Fig. 4 A network $T-\{(x_1, x_2), (x_4, x_7), (x_7, x_8)\}$.

subnetwork T' is obtained by deleting these edges (see Fig. 4). Therefore, from Theorem 2, $U=\{x_1, x_2, x_7, x_8\}$ is a 5-cover of T.

The time complexity of constructing T' is $O(|V|)$. The time complexity of choosing U is $O(M)$. Therefore, a r-cover of a tree flow network T can be obtained in $O(|V|)$ time.

From Theorem 1 and Theorem 2, a p-center and a r-cover of a flow network, which is not necessary a tree, can be obtained in $O(|V|\, s\,(|V|, |E|))$ time.

3.2 The p-Median Problem

This section concerns the p-median problem. In an undirected flow network N, let

$$s_N(p)=\max\{t_N(U)|U\subset V, |U|=p\}.$$

[Theorem 3] Let T be a tree flow network and (z_1, z_2) be a minimum weight edge of T. In $T-\{(z_1, z_2)\}$, let T_i be the connected component that includes z_i and $U_i\subset V(T_i)$ $(i=1, 2, U_i\neq\phi)$.

Then $t_T(U_1\cup U_2)=t_{T1}(U_1)+t_{T2}(U_2)$.

(proof) From the property of trees, $g(x, y)$ is the minimum value $w(e)$ such that the edge e belongs to the path P from x to y. Since (z_1, z_2) is a minimum weight edge of T, $g(x, y)\geqq w(z_1, z_2)$.

Let $V(T)-(U_1\cup U_2)=\{x_1,\cdots, x_k\}$.

$$t_T(U_1\cup U_2)=\sum_{i=1,\cdots,k} g(U_1\cup U_2, x_i)$$
$$=\sum \max\{g(U_1, x_i), g(U_2, x_i)\}.$$

If $x_i\in V(T_1)$ then $g(U_2, x_i)=w(z_1, z_2)$. Hence $g(U_1, x_i)\geqq g(U_2, x_i)$.

If $x_i\in V(T_2)$ then $g(U_1, x_i)\leqq g(U_2, x_i)$. Therefore

$$\sum \max\{g(U_1, x_i), g(U_2, x_i)\}$$
$$=\sum_{x_i\in V(T_1)} g(U_1, x_i)+\sum_{x_j\in V(T_2)} g(U_2, x_j)$$
$$=t_{T1}(U_1)+t_{T2}(U_2). \qquad \square$$

From Theorem 3, the transportation number of a subset of $V(T)$ is equal to the sum of the transportation numbers in each connected component in $T-\{(z_1, z_2)\}$. Therefore, in the case of given $s_{T1}(1), s_{T1}(2),\cdots, s_{T1}(t_1)$, $s_{T2}(1),\cdots$ and $s_{T2}(t_2)$ where $t_i=\min\{p, |V(T_i)|\}$ $(i=1, 2)$, $s_T(p)$ is given by the following expression.

$$\begin{aligned} s_T(p)=\max\{&s_{T1}(t_1)+s_{T2}(p-t_1),\\ &s_{T1}(t_1+1)+s_{T2}(p-t_1-1),\\ &\cdots,\\ &s_{T1}(p-t_2)+s_{T2}(t_2)\}, \end{aligned} \qquad (1)$$

where $s_{Ti}(0)=w(z_1, z_2)|V(T_i)|$ $(i=1, 2)$.

Hence, if i-medians of T_1 $(i=1,\cdots, t_1)$ and j-medians of T_2 $(j=1,\cdots, t_2)$ are given, then we can obtain a p-median of T.

For example, let us consider the network N shown in Fig. 2(b) and let $p=3$. The minimum weight edge of T is (x_4, x_7) and its value is 2. In $T-\{(x_4, x_7)\}$, let T_1 be the connected component that includes x_4 and T_2 be the connected component that includes x_7 (see Fig. 5). $t_1=\min\{p, |V(T_1)|\}=\min\{3, 6\}=3$ and $t_2=\min\{3, 2\}=2$. s_{T1} and s_{T2} are given as follows.

$s_{T1}(0)=w(x_4, x_7)|V(T_1)|=2\times 6=12$,

$s_{T1}(1)=27$ $(t_{T1}(\{x_2\})=27)$,

$s_{T1}(2)=23$ $(t_{T1}(\{x_1, x_2\})=23)$,

$s_{T1}(3)=18$ $(t_{T1}(\{x_1, x_2, x_3\})=18)$,

$s_{T2}(0)=w(x_4, x_7)|V(T_2)|=2\times 2=4$,

$s_{T2}(1)=4$ $(t_{T2}(\{x_7\})=4)$,

$s_{T2}(2)=0$ $(t_{T2}(\{x_7, x_8\})=0)$.

From above values, We can obtain $s_T(3)$.

$$\begin{aligned} s_T(3)=\max\{&s_{T1}(3)+s_{T2}(0),\\ &s_{T1}(2)+s_{T2}(1), s_{T1}(1)+s_{T2}(2)\}\\ =\max\{&18+4, 23+4, 27+0\} \end{aligned}$$

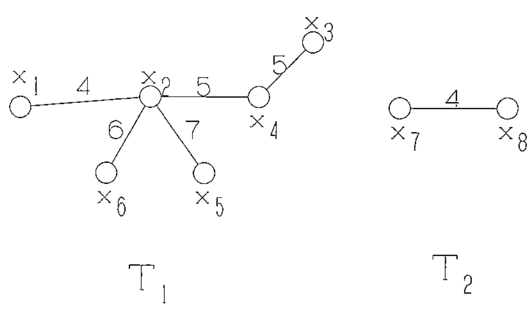


Fig. 5 A network $T-\{(x_4, x_7)\}$.

$=27.$

Therefore, $\{x_1, x_2\} \cup \{x_7\}$ is a 3-median of $T(\{x_2\} \cup \{x_7, x_8\}$ is also a 3-median of T).

The following algorithm SUB-MEDIAN $(p, T_1, T_2, (z_1, z_2))$, where $p \leqq |V(T_1) \cup V(T_2)|$, z_i belongs to T_i ($i=1, 2$) and (z_1, z_2) is a minimum weight edge of T, is the algorithm to obtain a p-median of T $(=T_1 \cup T_2 + \{(z_1, z_2)\})$, when each j-median $\mathrm{Set}_{Ti}(j)$ and its transportation number $s_{Ti}(j)$ of T_i are given.

procedure SUB-MEDIAN $(p, T_1, T_2, (z_1, z_2))$
begin
S 1 $s_{T1}(0) := w(z_1, z_2)|V(T_1)|$;
S 2 $s_{T2}(0) := w(z_1, z_2)|V(T_2)|$;
S 3 $t_1 := \min\{|V(T_1)|, p\}$; $t_2 := \min\{|V(T_2)|, p\}$;
S 4 $s_0 := 0$; $j_0 := 0$;
S 5 **for** $j = p - t_2$ **to** t_1 **do**
  begin
S 6   **if** $s_0 \leqq s_{T1}(j) + s_{T2}(p-j)$ **then**
    begin
S 7 $s_0 := s_{T1}(j) + s_{T2}(p-j)$;
S 8 $j_0 := j$;
    end
  end
S 9 $s_T(p) := s_0$;
S 10 $\mathrm{Set}_T(p) := \mathrm{Set}_{T1}(j_0) \cup \mathrm{Set}_{T2}(p-j_0)$
  (* $\mathrm{Set}_T(p)$ represents a p-median of T *)
end

Since S5 requires $O(p)$, the time complexity of SUB-MEDIAN is $O(p)$.

Using above algorithm, a p-median of a tree flow network T can be obtained by the following algorithm MEDIAN (T, p).

procedure MEDIAN (T, p)
begin
M 1 sort all the edges in order of size of edge weights; (* let $w(x_1, y_1) \geqq \cdots \geqq w(x_m, y_m)$ *)
M 2 let T_0 be a null network whose vertex set is $V(T)$;
M 3 **for** each $x \in V(T)$ **do**
  begin
M 4   let T_x be the connected component includes x in T_0;
M 5 $s_{Tx}(1) := 0$; $\mathrm{Set}_{Tx}(1) := \{x\}$;
  end
M 6 **for** $i=1$ **to** m **do**
  begin
M 7   let T_{xi} be the connected component includes x_i in T_0;
M 8   let T_{yi} be te connected component includes y_i in T_0;
M 9 $t := \min\{p, |V(T_{xi})| + |V(T_{yi})|\}$;
M 10   **for** $k=1$ **to** t **do**
M 11   SUB-MEDIAN $(k, T_{xi}, T_{yi}, (x_i, y_i))$;
M 12 $T_0 := T_0 + \{(x_i, y_i)\}$
  end
end.

In M1, sorting of edges requires $O(|V|\log|V|)$ where $|V| = |V(T)|$. M6 M10 and M11 require $O(|V|)$, $O(p)$, and $O(p)$, respectively. Therefore the time complexity of MEDIAN is $O(|V|\log|V| + p^2|V|)$. So, a p-median of a flow network N, which is not necessary a tree, can be obtained in $O(|V|s(|V|, |E|))$ time.

4. Conclusion

In this paper, we have given the difinitions of location problems on undirected flow networks and we have proposed the $O(|V|s(|V|, |E|))$ time algorithms to solve the p-cnter problem, the r-cover problem and the p-median problem on an undirected flow network N, where $|V| = |V(N)|$, $|E| = |E(N)|$ and $s(|V|, |E|)$ is the time required to solve a maximum flow problem in N. The algorithms to solve these problems are applicable to the assignment of files in a computer network, where the vertices represent terminal computers and the edges represent links between computers.

For a directed flow network N, there does not always exist a tree flow network T such that $g_N(x, y) = g_T(x, y)$ for any vertex pair. So, the same discussion as in this paper does not apply to directed flow networks. The study of location problems on directed flow networks is a future problem.

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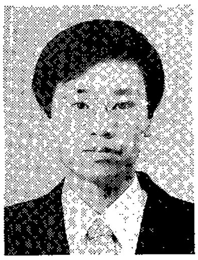

Hiroshi Tamura was born in Saitama prefecture, Japan, November 16, 1959. He received the B. Educ., M. S. and Ph. D. degrees from Niigata University in 1982 1986 and 1990, respectively. In April, 1990, he joined the staff at the Graduate School of Science and Technology, Niigata University as a Research Associate. His research interests are in computational geometry, netwok theory and graph theory. Dr. Tamura is a member of the Matheratical Society of Japan.

Masakazu Sengoku was born in Nagano prefecture, Japan, on October 18, 1944. He received the B. E. degree in electrical engineering from Niigata University, Niigata, Japan, in 1967 and the M. E. and Ph. D. degrees from Hokkaido University in 1969 and 1972, respectively. In 1972, he joined the staff at the Department of Electronic Engineering, Hokkaido University as a Research Associate. In 1978, he was an Associate Professor at the Department of Information Engineering, Niigata university, where he is presently a Professor. His research interests include network theory, graph theory, transmission of information and mobile communications. Dr. Sengoku is a member of IEEE and IPS of Japan.

Shoji Shinoda was born on December 15, 1941 in Hokkaido, Japan. He received the B. E., M. E. and Ph. D. degrees, all in electrical engineering, from Chuo University in 1964, 1966 and 1973, respectively. He joined the Department of Electrical Engineering at Chuo University in 1965 as a Research Assistant, where he was promoted to an Assistant Professor in 1970, to an Associate Professor in 1974 and then to a Professor in 1982. Currently, he is a Professor in the Department of Electrical and Electronic Engineering at the same university. Meanwhile, he was with the Coordinate Science Laboratory, University of Illinois, Urbana-Champaign, as a visiting scholar for a half year starting from March 30, 1981. His research interest has been in fault-diagnosis of analog circuits and discrete mathematics with applications. He has co-authored eight books on basic circuit theory, applied graph theory and appled mathematics,one of which is the book "Foundations of Circuit Theory" (Tokyo: Corona, 1990).

Takeo Abe was born in Niigata, Japan, on March 8, 1926. He received the B. E. and Dr. Eng. degrees from Tokyo Institute of Technology, Tokyo, Japan, in 1949 and 1966, respectively. From 1950 to 1959, he was a Research Scientist at the Electrotechnical Laboratory, Tokyo, Japan. From 1962 to 1966, he was an Associate Professor of Electrical Engineering at Tokyo Institute of Technology. From 1966 to 1978, he was a Professor of Electronic Engineering, Niigata University. Since 1978, he has been a Professor of Information Engineering at Niigata University and is now Dean of the Faculty of Engineering. He has been engaged in research and education in electromagnetic theory, microwave engineering, transmission of information and network theory. Dr. Abe is a member of IEEE, the Japanese Society of Snow and Ice and the Japan Society of Snow Engineering.